# ASYMPTOTIC DIMENSION OF GROUPS

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ABSTRACT. These notes are based on the paper [2].

## Contents



## 1. Ultrametric spaces

**Definition 1.1.** A metric space $(X, d)$ is called *ultrametric* if for all $x, y, z \in X$ we have $d(x, z) \leq \max\{d(x, y), d(y, z)\}$.

An ultrametric space $X$ can be characterized by any of the following properties:

- **Isosceles triangles**. If a triangle in a space $X$ has sides (distances between vertices) $a \leq b \leq c$, then $b = c$.
- **Radius $\geq$ diameter**. For any ball its radius is greater or equal to its diameter.
- **Disjoint balls**. Two balls of radius $D$ are either $D$-disjoint or identical.

**Proposition 1.2.** *Let $(X, d)$ be a metric space. The metric $d$ is an ultrametric if and only if $f(d)$ is an ultrametric for every nondecreasing function $f\colon \mathbb{R}_+ \to \mathbb{R}_+$.*

*Proof.* If $d$ is ultrametric and $a \leq b = c$ are sides of a triangle in $(X, d)$ then $f(a) \leq f(b) = f(c)$ are sides of the corresponding triangle in $(X, f(d))$ and therefore $f(d)$ is an ultrametric.

If $d$ is not an ultrametric then there is a triangle in $(X, d)$ with sides $a \leq b < c$. Consider the function

$$f(t) = \begin{cases} t \text{ if } t \leq b \\ \frac{2b}{c-b}t + \frac{bc-3b^2}{c-b} \text{ if } t \geq b \end{cases}$$

The sides of the corresponding triangle in $(X, f(d))$ are $f(a) \leq f(b) = b < 3b = f(c)$ which contradicts the triangle inequality. □

**Exercise 1.3.** Let $d$ be the standard metric on the real line $\mathbb{R}$ and $q$ be a positive real number.

Is $d^q$ a metric on $\mathbb{R}$? [Not if $q > 1$]

If yes, what is the length of the interval $[0, 1]$ in $(\mathbb{R}, d^q)$?

**Definition 1.4.** A metric is said to be $10^n$*-valued* if the only positive values assumed by the metric are $10^n$, $n \in \mathbb{Z}$.

**Exercise 1.5.** Any $10^n$-valued metric is an ultrametric.

**Exercise 1.6.** Any $x^n$-valued metric is an ultrametric if $x > 2$.

**Lemma 1.7.** *Any ultrametric space can be equipped with a $10^n$-valued ultrametric.*

*Proof.* Apply Proposition 1.2 with the following function $f$:

$$f(d) = 10^n \qquad \text{if} \qquad 10^{n-1} < d \leq 10^n.$$

□

**Exercise 1.8.** Let $(X, d)$ be a metric space and $f$ is a nondecreasing function $f\colon \mathbb{R}_+ \to \mathbb{R}_+$. Consider the identity map $F\colon (X, d) \to (X, f(d))$. When is $F$ continuous? [Answer: when $\lim_{t\to 0} f(t) = 0$] When is $F^{-1}$ continuous? [Always].

## 2. Universal ultrametric spaces

Let us introduce a new metric on the set $\mathbb{R}$ of real numbers. Given two numbers $a, b \in \mathbb{R}$, write them using decimal presentation:

$$a = a_k a_{k-1} \dots a_1 a_0 . a_{-1} a_{-2} \dots$$

$$b = b_n b_{n-1} \dots b_1 b_0 . b_{-1} b_{-2} \dots$$

Then look at these numbers from left to right and find the first index $m_{ab}$ where the presentations are different. Put $\mu(a, b) = 10^{m_{ab}}$.

**Exercise 2.1.** Prove that $(\mathbb{R}, \mu)$ is an ultrametric space.

Let us describe an ultrametric space $(L_\omega, \mu)$ which is universal for all separable ultrametric spaces with $10^n$-valued metrics. This space appeared naturally in different areas of mathematics (see for example [3] and references therein). Let us fix a countable set $S$ with a distinguished element $s_0 \in S$. The set $L_\omega$ is a subset of the set of infinite sequences $\bar{x} = \{x_n\}_{n\in\mathbb{Z}}$ with all elements $x_n$ from the set $S$. A sequence $\bar{x}$ belongs to $L_\omega$ if there exists an index $k \in \mathbb{Z}$ such that $x_n = s_0$ for all $n > k$. The metric $\mu$ is defined as $\mu(\bar{x}, \bar{y}) = 10^m$ where $m \in \mathbb{Z}$ is the maximal index such that $x_m \neq y_m$. Clearly, the space $L_\omega$ is a complete separable $10^m$-valued ultrametric space.

To prove that any separable ultrametric space with $10^n$-valued metric embeds isometrically into $(L_\omega, \mu)$ we follow the idea of P.S. Urysohn [4] and show that the space $L_\omega$ is *finitely injective*:

**Lemma 2.2.** *Let $(X, d)$ be a finite metric space with $10^n$-valued metric $d$. For any subspace $A \subset X$, any isometric map $f\colon A \to L_\omega$ admits an isometric extension $\tilde{f}\colon X \to L_\omega$.*

*Proof.* It is sufficient to prove Lemma in case $X \setminus A$ consists of one point $x$. In such case we have to find a point $\bar{z} \in L_\omega$ such that $\mu(\bar{z}, f(a)) = d(x,a)$ for every point $a \in A$. Let $A_x = \{a \in A \mid d(x,a) = d(x,A)\}$ be the set of all points in $A$ closest to $x$ and let $d(x,A) = 10^n$. Fix a point $b \in A_x$ and define $\bar{z} = \{z_n\}_{n\in\mathbb{Z}}$ as follows: $z_m = f(b)_m$ if $m > n$; $z_m = s_0$ if $m < n$; $z_n$ is any element of the set $S$ other than $f(c)_n$ for any point $c \in A_x$.

Clearly, $\mu(\bar{z}, f(c)) = 10^n = d(x,c)$ for any point $c \in A_x$. For any point $a \in A\setminus A_x$ we have $d(a,x) = d(a,b) = 10^m > 10^n$ which means that $f(a)_m \neq f(b)_m = z_m$ and therefore $\mu(\bar{z}, f(a)) = 10^m = d(x,a)$. □

**Exercise 2.3.** Is the space $L_\omega$ finitely homogeneous (i.e. any isometry $f\colon A \to B$ of finite subsets $A, B \subset L_\omega$ extends to a surjective isometry of the whole space $L_\omega$)?

**Problem 2.4.** Is the space $L_\omega$ countably homogeneous (i.e. any isometry $f\colon A \to B$ of countable subsets $A, B \subset L_\omega$ extends to a surjective isometry of the whole space $L_\omega$)?

**Definition 2.5.** A subset $S$ of a metric space $X$ is called *dense* if for any point $x \in X$ and any number $\varepsilon > 0$ there is a point $s_x \in S$ with $d(s_x, x) < \varepsilon$.

A metric space is called *separable* if it contains a dense countable subset.

**Exercise 2.6.** Prove that the space $L_\omega$ is separable.

**Exercise 2.7.** If $X$ is a separable metric space and $f\colon X \to Y$ is a continuous surjective map, then $Y$ is separable.

**Theorem 2.8.** *Any separable metric space $(X,d)$ equipped with $10^n$-valued metric $d$ embeds isometrically into the space $(L_\omega, \mu)$.*

*Proof.* Since $X$ is separable, it is sufficient to embed isometrically a countable dense subspace $A$ of $X$. One can embed such a subspace by induction using Lemma 2.2. □

**Exercise 2.9.** Let $A, A', B, B'$ be four points in an ultrametric space $X$. If $d(A,A') < d(A,B)$ and $d(B,B') < d(A,B)$, then $d(A',B') = d(A,B)$.

**Proposition 2.10.** *If $(X,d)$ is a separable ultrametric space, then $d$ assumes only countably many different values.*

*Proof.* If a metric space $X$ is countable, then $d$ assumes only countably many different values. Since $X$ is separable, it contains a dense countable subset $A$. Since $A$ is countable, then $d$ assumes only countably many different values on pairs of points from $A$.

Now we show that for any points $x_1, x_2 \in X$ there are points $a_1, a_2 \in A$ such that $d(x_1,x_2) = d(a_1,a_2)$. Since $A$ is dense in $X$, there are points $a_1, a_2 \in A$ such that $d(x_1,a_1) < d(x_1,x_2)$ and $d(x_2,a_2) < d(x_1,x_2)$. Then by Exercise 2.9 we have $d(x_1,x_2) = d(a_1,a_2)$. □

**Definition 2.11.** Let $M$ be a countable subset of positive reals. A metric is said to be *$M$-valued* if the only positive values assumed by the metric are those from $M$.

**Problem 2.12.** Let $M$ be a countable subset of positive reals.

- Does there exist a universal separable $M$-valued ultrametric space?
- Is this space finitely homogeneous?
- Is this space countably homogeneous?

## 3. Ultrametric spaces and Lipschitz maps

**Definition 3.1.** A map $f\colon X \to Y$ is called a *non-expansive* if $d_Y(f(x), f(z)) \leq d_X(x, z)$ for any points $x, z \in X$.

**Theorem 3.2.** *A metric space $X$ is ultrametric if and only if every non-expansive map $f$ of a closed subset $A$ of $X$ to a 0-dimensional sphere $S^0$ can be extended to a non-expansive map of $X$ to $S^0$.*

*Proof.* Denote by $B$ and $C$ the preimages of points of $S^0$ under the map $f$. Clearly, $A = B \cup C$. If one of the preimages, say $B$, is empty, then $f$ is a constant map which has a constant extension to $X$.

Consider a point $x \in X \setminus A$ and compare the distances $\operatorname{dist}(x, B)$ and $\operatorname{dist}(x, C)$. Define an extension $\tilde{f}$ on $x$ as follows:

$$\tilde{f}(x) = \begin{cases} f(B) \text{ if } \operatorname{dist}(x, B) \leq \operatorname{dist}(x, C) \\ f(C) \text{ if } \operatorname{dist}(x, B) > \operatorname{dist}(x, C) \end{cases}$$

Let us check that the map $\tilde{f}$ is non-expansive. Consider two points $x, z \in X$. If $\tilde{f}(x) = \tilde{f}(z)$ then clearly $d_{S^0}(\tilde{f}(x), \tilde{f}(z)) = 0 \leq d_X(x, z)$. Suppose that $\tilde{f}(x) = f(B)$ and $\tilde{f}(z) = f(C)$. Then $\operatorname{dist}(x, B) \leq \operatorname{dist}(x, C)$ and $\operatorname{dist}(z, B) > \operatorname{dist}(z, C)$ by definition of $\tilde{f}$.

There is a point $c_z \in C$ such that $d(z, c_z) < \operatorname{dist}(z, B)$. Then for any point $b \in B$ the triangle $c_z, z, b$ is isosceles with $d(z, c_z) < d(z, b) = d(c_z, b) \geq \operatorname{dist}(B, C) \geq D$ where $D$ denotes the diameter of the 0-sphere $d_{S^0}(f(B), f(C))$. Therefore, $d(z, b) \geq D$. If $d(x, z) < D$ then $d(x, c_z) < D$ (look at the triangle $x, z, c_z$) and $\operatorname{dist}(x, B) \leq \operatorname{dist}(x, C) \leq d(x, c_z) < D$. Choose a point $b \in B$ such that $d(x, b) < D$ and consider the triangle $x, c_z, b$. Since $d(x, c_z) < D$ and $d(x, b) < D$, then $d(c_z, b) < D$ which contradicts $\operatorname{dist}(B, C) \geq D$. □

**Exercise 3.3.** Prove the "if" part of Theorem 3.2.

**Definition 3.4.** A map $r\colon X \to X$ is called a *retraction* if $r(x) = x$ for every point $x \in r(X)$.

A subspace $A \subset X$ is called a *retract of* $X$ if there exists a retraction of $X$ onto $A$.

It would be nice to have the following

**Non-Theorem.** A metric space $X$ is ultrametric if and only if every closed subset $A$ of $X$ is a non-expansive retract of $X$.

**Exercise 3.5.** Give a simple proof of Theorem 3.2 using Non-Theorem.

**Exercise 3.6.** Prove the "if" part of Non-Theorem.

**Example 3.7.** Let $X = \{x_n\}_{n=1}^{\infty}$ be a sequence of points. Define $d(x_1, x_n) = 1 + \frac{1}{n}$ and $d(x_m, x_n) = \max\{1 + \frac{1}{m}, 1 + \frac{1}{n}\}$ for any $m, n > 1$.

**Exercise 3.8.** Show that $d$ is an ultrametric on $X$.

**Exercise 3.9.** Show that there is no non-expansive retraction of $X$ onto $A = \{x_n\}_{n=2}^{\infty}$.

**Definition 3.10.** A subspace $A \subset X$ is called an *almost non-expansive retract of* $X$ if for any $\lambda > 1$ there exists a retraction $r_\lambda$ of $X$ onto $A$ such that $d(r_\lambda(x), r_\lambda(z)) \leq \lambda \cdot d(x, z)$ for any points $x, z \in X$.

**Theorem 3.11.** *A metric space $X$ is ultrametric if and only if every closed subset $A$ of $X$ is an almost non-expansive retract of $X$.*

*Proof.* Suppose that $X$ is an ultrametric space and $A \subset X$ is a closed subspace. If $\lambda > 1$ is given, choose a number $\delta > 1$ such that $\delta^2 < \lambda$.

Let us fix an arbitrary well-order $\prec$ on $X$. We define a retraction $r\colon X \to A$ as follows. For a point $x \in X$ we look at the nonempty set

$$A_x = \{a \in A \mid d(x,a) \le \delta \cdot \operatorname{dist}(x,A)\}$$

and put $r(x)$ to be the minimal point in the set $A_x$ with respect to the order $\prec$.

Let us show that the retraction $r$ is $\lambda$-Lipschitz. Assume that for some points $x, y \in X$ we have $d(r(x), r(y)) > \lambda \cdot d(x,y)$. Without loss of generality we may assume that $r(x) \prec r(y)$.

If $d(y, r(x)) \le d(y, r(y))$, then $r(x) \in A_y$ and $r(x) \prec r(y)$ contradicts the choice of $r(y)$ to be the minimal point in the set $A_y$.

In case $d(y, r(x)) > d(y, r(y))$ we denote by $D$ the distance between $r(x)$ and $r(y)$ and notice that $d(y, r(x)) = d(r(x), r(y)) = D$ in the isosceles triangle $\{y, r(x), r(y)\}$. Since $D > d(x,y)$, we have $d(x, r(x)) = d(y, r(x)) = D$ in the isosceles triangle $\{x, y, r(x)\}$.

$$d(x, r(y)) \ge \operatorname{dist}(x, A) \ge \frac{1}{\delta} \cdot d(x, r(x)) = \frac{D}{\delta} > \frac{D}{\lambda} > d(x,y)$$

Therefore $d(x, r(y)) = d(y, r(y))$ in the isosceles triangle $\{x, y, r(y)\}$. The point $r(x)$ does not belong to $A_y$ since $r(x) \prec r(y)$, thus $d(y, r(x)) = D > \delta \cdot \operatorname{dist}(y, A)$. Then there exists a point $z \in A$ with $d(y,z) < \frac{D}{\delta}$.

$$d(y,z) \ge \operatorname{dist}(y,A) \ge \frac{d(y, r(y))}{\delta} = \frac{d(x, r(y))}{\delta} \ge \frac{D}{\delta^2} > \frac{D}{\lambda} > d(x,y)$$

Then $d(x,z) = d(y,z)$ in the isosceles triangle $\{x,y,z\}$. Since $d(x,z) < d(x, r(x))$, we have $z \in A_x$, but $d(x,z) < \frac{D}{\delta} = \frac{d(x,r(x))}{\delta}$ contradicts the definition of $A_x$ (two points $a, a' \in A_x$ cannot satisfy $d(x,a) < \frac{d(x,a')}{\delta}$). □

**Exercise 3.12.** Prove the "if" part of Theorem 3.11.

**Definition 3.13.** A map $f\colon X \to Y$ of metric spaces is called *Lipschitz* if there is a constant $\lambda > 0$ such that the inequality $d_Y(f(x), f(x')) \le \lambda \cdot d_X(x, x')$ holds for all points $x, x' \in X$. $f$ is called *$\lambda$-Lipschitz* if we need to specify the constant $\lambda$. $f$ is called *$\lambda$-bi-Lipschitz* if both $f$ and $f^{-1}$ are $\lambda$-Lipschitz.

For any Lipschitz map $f$ we denote

$$\operatorname{Lip}(f) = \inf\{\lambda \mid f \text{ is } \lambda\text{-Lipschitz}\}$$

Notice that a Lipschitz map $f$ is $\operatorname{Lip}(f)$-Lipschitz.

**Exercise 3.14.** Prove or disprove: if $f\colon X \to Y$ and $g\colon Y \to Z$ are Lipschitz maps, then $\operatorname{Lip}(g \circ f) = \operatorname{Lip}(f) \cdot \operatorname{Lip}(g)$.

Notice that the identity map $\mathrm{id}\colon (X,d) \to (X, f(d))$ in Lemma 1.7 is expanding and 10-Lipschitz, thus it is 10-bi-Lipschitz.

**Corollary 3.15.** *Any separable ultrametric space admits 10-bi-Lipschitz embedding into the space $(L_\omega, \mu)$.*

*Proof.* Combine Lemma 1.7 and Theorem 2.8. □

## 4. Assouad-Nagata dimension

**Definition 4.1.** Let $X$ be a metric space, $A$ be a subspace of $X$, and $S$ be a positive number.

$A$ is *$S$-bounded* if for any points $x, x' \in A$ we have $d_X(x, x') \leq S$.

An *$S$-chain* in $A$ is a sequence of points $x_1, \dots, x_k$ in $A$ such that for every $i < k$ the set $\{x_i, x_{i+1}\}$ is $S$-bounded.

$A$ is *$S$-connected* if for any points $x, x' \in A$ can be connected in $A$ by an $S$-chain.

Notice that any subset $A$ of $X$ is a union of its *$S$-components* (the maximal $S$-connected subsets of $A$). If $B$ and $B'$ are two $S$-components of the set $A$ then $B$ and $B'$ are $S$-disjoint. Intuitively, a metric space $X$ has dimension 0 at scale $S > 0$ if all $S$-components of $X$ are uniformly bounded.

**Definition 4.2.** A metric space $X$ has Assouad-Nagata dimension zero (notation $dim_{AN}(X) \leq 0$) if there exists a constant $m \geq 1$, such that for any $S > 0$ all $S$-components of $X$ are $mS$-bounded.

**Exercise 4.3.** Let $\mathbb{R}$ be the real line with the standard metric. Let $X = \{x_n\}_{n=1}^{\infty}$ be a sequence of points in $\mathbb{R}$ such that $x_{n+1} > x_n$ and $x_{n+1} - x_n > x_n - x_{n-1}$. What does it mean for the space $X$ to have Assouad-Nagata dimension zero?

**Exercise 4.4.** Bi-Lipschitz maps preserve Assouad-Nagata dimension zero.

Ultrametric spaces are the best examples of metric spaces of Assouad-Nagata dimension zero. Indeed, for any positive number $D$ any $D$-component of an ultrametric space is a $D$-ball and therefore is $D$-bounded. Let us characterize spaces of Assouad-Nagata dimension 0 using ultrametrics.

**Theorem 4.5.** *If a metric space $(X, d)$ has Assouad-Nagata dimension $\dim_{AN}(X) \leq 0$, then there is an ultrametric $\rho$ on $X$ such that the identity map $id\colon (X, d) \to (X, \rho)$ is bi-Lipschitz.*

*Proof.* Suppose that for a number $m > 1$, all $S$-components of $X$ are $mS$-bounded. Consider two points $x, z \in X$ and put

$$S = \frac{d(x, z)}{2m}.$$

Then the points $x$ and $z$ belong to different $S$-components of $X$. Thus for any chain $x = x_0, x_1, \dots, x_{k-1}, x_k = z$ we have

$$d(x, z) \leq 2m \cdot \max_{0 \leq i < k} \{d(x_i, x_{i+1})\}.$$

Now define $\rho(x, z)$ to be the infimum of $\max_{0 \leq i < k}\{d(x_i, x_{i+1})\}$ over all finite chains $x_0, x_1, \dots, x_{k-1}, x_k$ with $x = x_0$ and $x_k = z$. Clearly

$$\frac{1}{2m} \cdot d(x, z) \leq \rho(x, z) \leq d(x, z).$$

To see that $\rho$ is an ultrametric, take three points $x, y, z$ in $X$ and let $s$ be the infimum of all positive numbers $S$ such that all three points belong to one $S$-component of $X$. If all three points belong to one $s$-component or all three belong to different $s$-components, then $\rho(x, y) = \rho(x, z) = \rho(y, z) = s$. If the points $x$ and $y$ belong to one $s$-component which does not contain $z$, then $\rho(x, y) \leq s = \rho(x, z) = \rho(y, z)$. □

**Theorem 4.6.** *Any separable metric space of Assouad-Nagata dimension 0 admits a bi-Lipschitz embedding into the space $(L_\omega, \mu)$.*

*Proof.* Apply Theorem 4.5 and Theorem 3.15. □

**Definition 4.7.** A metric space $Y$ is called a *Lipschitz extensor for a metric space* $X$ if there exists a constant $m > 0$ such that for any closed subspace $A \subset X$ any Lipschitz map $f \colon A \to Y$ extends to a Lipschitz map $F \colon X \to Y$ with $Lip(F) \leq m \times Lip(f)$. We call the space $Y$ an *$m\times$-Lipschitz extensor for $X$* if we need to specify the constant $m$.

**Theorem 4.8.** *The following conditions are equivalent:*

1. $\dim_{AN}(X) \leq 0$*;*
2. *there exists a number $\lambda$ such that every closed subset of $X$ is a $\lambda$-Lipschitz retract of $X$;*
3. *there exists a number $\lambda$ such that every metric space is a $\lambda\times$-Lipschitz extensor for $X$;*
4. *the unit 0-sphere $S^0$ is a Lipschitz extensor for $X$.*

*Proof.* (1) $\implies$ (2) Theorem 4.5 allows us to find an ultrametric $\rho$ on $X$ which is bi-Lipschitz equivalent to $d$. Application of Theorem 3.11 completes the proof.

(2) $\implies$ (3) Given a closed subspace $A \subset X$ and a Lipschitz map $f \colon A \to Y$ to some metric space $Y$ we fix a $\lambda$-Lipschitz retraction $r \colon X \to A$. Then the composition $f \circ r \colon X \to K$ has the Lipschitz constant bounded by $\lambda \cdot \mathrm{Lip}(f)$.

(3) $\implies$ (4) Obvious.

(4) $\implies$ (1) Let $m \geq 1$ be a number such that any $\lambda$-Lipschitz map from any closed subspace $A \subset X$ to $S^0$ can be extended to $m\lambda$-Lipschitz map of $X$. If an $S$-component of $X$ is not $mS$-bounded, there are points $z_0$ and $z_1$ with $d(z_0, z_1) > mS$ and an $S$-chain of points $z_0 = x_0, x_1, \ldots, x_k = z_1$. Notice that the map $f \colon \{z_0\} \cup \{z_1\} \to S^0$ defined as $f(z_0) = 0$ and $f(z_1) = 1$ is $\frac{1}{d(z_0,z_1)}$-Lipschitz but any extension of this map to the chain is at least $\frac{1}{S}$-Lipschitz and cannot be $\frac{m}{d(z_0,z_1)}$-Lipschitz (since $\frac{1}{S} > \frac{m}{d(z_0,z_1)}$). □

**Exercise 4.9.** Prove the implication (2) $\implies$ (1) directly.

## 5. Locally finite countable groups

All groups considered in this Section are countable.

**Definition 5.1.** A metric $d$ on a group $G$ is called *left invariant* if $d(x, y) = d(g \cdot x, g \cdot y)$ for any $x, y, g \in G$. In particular, $d(x, y) = d(e, x^{-1} \cdot y)$.

A left invariant metric $d$ on a countable group $G$ is *proper* if and only if every bounded subset of $(G, d)$ is finite. Thus a left invariant proper metric $d$ on $G$ is bounded from below.

What can one say about two different left invariant proper metrics $d$ and $\rho$ on the same group $G$?

**Definition 5.2.** We call a map $f \colon X \to Y$ of metric spaces *uniform* if there is a function $\delta_f \colon \mathbb{R}_+ \to \mathbb{R}_+$ with $\lim_{t\to 0} \delta_f(t) = 0$ such that $d_Y(f(x), f(x')) \leq \delta_f(d_X(x, x'))$ for all points $x, x' \in X$. To specify the function $\delta_f$ we sometimes say that the map $f$ is $\delta_f$-uniform. A map $f$ is called *bi-uniform* if both $f$ and $f^{-1}$ are uniform.

**Theorem 5.3.** *If $d$ and $\rho$ are two different left invariant proper metrics on the same group $G$, then the metric spaces $(G, d)$ and $(G, \rho)$ are bi-uniformly equivalent.*

*Proof.* □

**Definition 5.4.** Let $X$ be a metric space. We say that $X$ has *asymptotic dimension zero* (notation $\operatorname{asdim}(X) \leq 0$) if there is a function $D_X^0 \colon \mathbb{R}_+ \to \mathbb{R}_+$ (called 0-dimensional control function) such that for any $S > 0$ every $S$-component of $X$ is $D_X^0(S)$-bounded.

**Definition 5.5.** A group $G$ is called *locally finite* if every finitely generated subgroup of $G$ is finite.

**Example 5.6.** Direct sum of finite groups is locally finite.

**Theorem 5.7.** *A countable group $G$ equipped with a proper left invariant metric has asymptotic dimension zero if and only if $G$ is locally finite*

*Proof.* □

**Exercise 5.8.** Bi-uniform maps preserve asymptotic dimension zero.

Let $G$ be a locally finite countable group. Let us describe a particularly simple way to define a proper left-invariant metric on $G$. Consider a filtration $\mathcal{L}$ of $G$ by finite subgroups $\mathcal{L} = \{1 \subset G_1 \subset G_2 \subset G_3 \dots\}$ and define the metric $d_{\mathcal{L}}$ associated to this filtration as:

$$d_{\mathcal{L}}(x, y) = \min\{i \mid x^{-1}y \in G_i\}.$$

**Exercise 5.9.** Show that $d_{\mathcal{L}}$ is an ultrametric.

**Lemma 5.10.** *Suppose two groups $G$ and $H$ have filtrations by finite subgroups: $\mathcal{L} = \{1 \subset G_1 \subset G_2 \subset G_3 \dots\}$ of $G$ and $\mathcal{K} = \{1 \subset H_1 \subset H_2 \subset H_3 \dots\}$ of $H$. If the index $[G_i : G_{i-1}]$ is less than or equal to the index $[H_i : H_{i-1}]$ for all $i$, then $(G, d_{\mathcal{L}})$ admits an isometric embedding into $(H, d_{\mathcal{H}})$. Moreover, if $[G_i : G_{i-1}] = [H_i : H_{i-1}]$ for all $i$ (equivalently, the cardinality of $G_i$ equals cardinality of $H_i$ for all $i$), then the groups $(G, d_{\mathcal{L}})$ and $(H, d_{\mathcal{H}})$ are isometric.*

*Proof.* Put $a_i = [G_i : G_{i-1}]$ and $b_i = [H_i : H_{i-1}]$. Fix an injection $f_1 \colon G_1 \to H_1$ and assume injections $f_k \colon G_k \to H_k$ are known for $k \leq n$ such that the following two properties hold:

(1) $f_i(x) = f_j(x)$ for $i < j$ and $x \in G_i$,
(2) the injection $f_k \colon G_k \to H_k$ is isometric.

Pick an injection of the set of cosets $\{x \cdot G_n\}$ of $G_n$ in $G_{n+1}$ into the set of cosets $\{y \cdot H_n\}$ of $H_n$ in $H_{n+1}$. That amounts to picking representatives 1, $x_1, \dots, x_m$ $(m = a_{n+1} - 1)$ of cosets of $G_n$ in $G_{n+1}$ and picking representatives 1, $y_1, \dots, y_l$ $(l = b_{n+1} - 1)$ of cosets of $H_n$ in $H_{n+1}$. Make sure the injection takes $\{1 \cdot G_n\}$ to $\{1 \cdot H_n\}$. Now we extend $f_n$ to $f_{n+1} \colon G_{n+1} \to H_{n+1}$ as follows: if $x \in G_{n+1} \setminus G_n$, we represent $x$ as $x_k \cdot x'$ for some unique $k \leq m$ and we define $f_{n+1}(x)$ as $y_k \cdot f_n(x')$.

If $x$ and $z$ belong to different cosets of $G_n$ in $G_{n+1}$, then $f_{n+1}(x)$ and $f_{n+1}(z)$ belong to different cosets of $H_n$ in $H_{n+1}$ and $d_{\mathcal{L}}(x, z) = n+1 = d_{\mathcal{H}}(f_{n+1}(x), f_{n+1}(z))$. If $x$ and $z$ belong to the same coset $x_k \cdot G_n$ of $G_n$ in $G_{n+1}$, then $x = x_k \cdot x'$, $z = x_k \cdot z'$. Since $f_{n+1}(x) = y_k \cdot f_n(x')$, $f_{n+1}(z) = y_k \cdot f_n(z')$, and the map $f_n$ is isometry, then $d_{\mathcal{L}}(x, z) = d_{\mathcal{L}}(x', z') d_{\mathcal{H}}(f_n(x'), f_n(z')) d_{\mathcal{H}}(f_{n+1}(x), f_{n+1}(z))$.

By pasting all $f_n$ we get an isometric injection $f\colon G \to H$. Notice that in case $[G_i : G_{i-1}] = [H_i : H_{i-1}]$ for all $i$, the map $f$ is bijective and establishes an isometry between $(G, d_{\mathcal{L}})$ and $(H, d_{\mathcal{H}})$. □

**Lemma 5.11.** *Given two locally finite groups $G$ and $H$ the following conditions are equivalent:*

(1) *There are left-invariant proper metrics $d_G$ on $G$ and $d_H$ on $H$ such that $(G, d_G)$ is isometric to $(H, d_H)$.*
(2) *There are filtrations by finite subgroups: $\mathcal{L} = \{1 \subset G_1 \subset G_2 \subset G_3 \dots\}$ of $G$ and $\mathcal{K} = \{1 \subset H_1 \subset H_2 \subset H_3 \dots\}$ of $H$ such that the cardinality of $G_i$ equals cardinality of $H_i$ for all $i$.*

*Proof.* In view of 5.10, it suffices to show (1) $\implies$ (2). Obviously, we may pick an isometry $f\colon G \to H$ such that $f(1_G) = 1_H$ (replace any $f$ by $f(1_G)^{-1} \cdot f$). Notice $f$ establishes bijectivity between $m$-component of $G$ containing $1_G$ and the $m$-component of $H$ containing $1_H$. Also, those components are subgroups of $G$ and $H$. Thus, define $G_1$ as 1-component of $G$ containing $1_G$ and, inductively, $G_{i+1}$ as $(\operatorname{diam}(G_i) + i)$-component of $G$ containing $1_G$. □

**Main example**. If $G$ is a direct sum of cyclic groups $\bigoplus_{i=1}^{\infty} \mathbb{Z}_{a_i}$ we consider the metric on $G$ associated to the filtration

$$\mathcal{L} = \{1 \subset \mathbb{Z}_{a_1} \subset \mathbb{Z}_{a_1} \oplus \mathbb{Z}_{a_2} \subset \mathbb{Z}_{a_1} \oplus \mathbb{Z}_{a_2} \oplus \mathbb{Z}_{a_3} \subset \dots\}$$

If we write elements of the group $\bigoplus_{i=1}^{\infty} \mathbb{Z}_{a_i}$ as $p = p_1 p_2 \dots p_n$ where $p_j \in \mathbb{Z}_{a_j}$ and denote $|p| = n$ then the ultrametric $d_{\mathcal{L}}$ can be defined explicitly as

$$d_{\mathcal{L}}(p,q) = \begin{cases} \max\{|p|, |q|\} \text{ if } |p| \neq |q| \\ \max\{i \mid p_i \neq q_i\} \text{ if } |p| = |q| \end{cases}$$

**Theorem 5.12.** *A locally finite countable group $G$ with a proper left invariant metric $d$ is bi-uniformly equivalent to a direct sum of cyclic groups.*

*Proof.* Fix a filtration $\mathcal{L}$ of $G$ by finite subgroups $\mathcal{L} = \{1 \subset G_1 \subset G_2 \subset G_3 \dots\}$. Then $(G, d)$ is bi-uniformly equivalent to $(G, d_{\mathcal{L}})$ by 5.3. By 5.10, $(G, d_{\mathcal{L}})$ is isometric to $\bigoplus_{i=1}^{\infty} \mathbb{Z}_{a_i}$ where $a_i = [G_i : G_{i-1}]$. □

**Definition 5.13.** Let $G$ be a countable locally finite group and $p$ be a prime number. We define a *$p$-Sylow number of $G$* (finite or infinite) as follows:

$$|p\text{-Syl}|(G) = \sup\{p^n \mid p^n \text{ divides } |F|, F \text{ a finite subgroup of } G, n \in \mathbb{Z}\}$$

Notice that if the $p$-Sylow number of $G$ is finite, it is equal to the order of a $p$-Sylow subgroup of some finite subgroup of $G$. For an abelian torsion group $G$ the $p$-Sylow number of $G$ is equal to the order of the $p$-torsion subgroup of $G$.

**Theorem 5.14.** *Two countable locally finite groups $G$ and $H$ with proper left invariant metrics are bi-uniformly equivalent if and only if, for every finite subgroup $F$ of $G$, there exists a finite subgroup $E$ of $H$ such that $|F|$ is a divisor of $|E|$, and, for every finite subgroup $E$ of $H$, there exists a finite subgroup $F$ of $G$ such that $|E|$ is a divisor of $|F|$.*

**Corollary 5.15.** *Let $G$ and $H$ be countable direct sums of finite prime cyclic groups. Let $d_G$ and $d_H$ be proper left invariant metrics on $G$ and $H$. Then the metric spaces $(G, d_G)$ and $(H, d_H)$ are bi-uniformly equivalent if and only if the groups $G$ and $H$ are isomorphic.*

**Theorem 5.16.** *Let $G$ and $H$ be locally finite countable groups with proper left invariant metrics $d_G$ and $d_H$. The metric spaces $(G, d_G)$ and $(H, d_H)$ are bi-uniformly equivalent if and only if for every prime $p$ we have $|p\text{-Syl}|(G) = |p\text{-Syl}|(H)$.*

*Proof.* Assume the metric spaces $(G, d_G)$ and $(H, d_H)$ are bi-uniformly equivalent. Our goal is to show that if $|p\text{-Syl}|(G) \geq p^n$, then $|p\text{-Syl}|(H) \geq p^n$. If there is a finite subgroup $F$ of $G$ such that $p^n$ divides $|F|$, then by 5.14 there is a subgroup $E$ of $H$ such that $p^n$ divides $|E|$. Thus $|p\text{-Syl}|(H) \geq p^n$.

Now suppose $|p\text{-Syl}|(G) = |p\text{-Syl}|(H)$ for every prime $p$. By 5.14, it is enough to show that for every finite subgroup $F$ of $G$, there exists a finite subgroup $E$ of $H$ such that $|F|$ is a divisor of $|E|$. If $|F| = p_1^{\alpha_1} \cdot \ldots \cdot p_k^{\alpha_k}$ then $p_i^{\alpha_i} \leq |p_i\text{-Syl}|(H)$ for every $i$. For every $i$ find a subgroup $E_i$ of $H$ such that $p_i^{\alpha_i}$ divides $|E_i|$. Let $E$ be a finite subgroup of $H$ containing all the groups $E_i$. Clearly, $|F|$ divides $|E|$. □

## 6. Coarse equivalence of groups

**Definition 6.1.** Metric spaces $X$ and $Y$ are said to be *coarsely equivalent* if there are uniform maps $f : X \to Y$ and $g : Y \to X$ and a constant $C > 0$ such that both compositions $f \circ g$ and $g \circ f$ are $C$-close to the identity maps.

Let $G$ and $H$ be countable locally finite groups. Using 5.12 one can show that if

$$\sum_{p\text{-prime}} \big| |p\text{-Syl}|(G) - |p\text{-Syl}|(H) \big| < \infty$$

then the groups $G$ and $H$ are coarsely equivalent. Is the converse true?

**Problem 6.2.** Classify countable abelian torsion groups up to coarse equivalence.

Let us suggest a program to answer 6.2. Notice that any abelian torsion group is coarsely equivalent to a direct sum of groups $\mathbb{Z}_p$ with $p$ being prime. Therefore the following groups are of importance: $\mathbb{Z}_p^\infty$ (the infinite direct sum of copies of $\mathbb{Z}_p$) and $\bigoplus_{p\in\mathcal{P}} \mathbb{Z}_{p^{n(p)}}$, where $n(p) \geq 1$ for each $p \in \mathcal{P}$, $\mathcal{P}$ being a subset of primes.

**Problem 6.3.** Suppose $\bigoplus_{p\in\mathcal{P}} \mathbb{Z}_{p^{n(p)}}$ and $\bigoplus_{q\in\mathcal{Q}} \mathbb{Z}_{q^{m(q)}}$ are coarsely equivalent. Is the symmetric difference of $\mathcal{P}$ and $\mathcal{Q}$ finite? If so, does $n(p)$ equal $m(p)$ for all but finitely many $p$?

**Problem 6.4.** Suppose $\bigoplus_{p\in\mathcal{P}} \mathbb{Z}_p^\infty$ and $\bigoplus_{q\in\mathcal{Q}} \mathbb{Z}_q^\infty$ are coarsely equivalent. Is $\mathcal{P} = \mathcal{Q}$?

Call two countable abelian torsion groups $G$ and $H$ *virtually isometric* if there are subgroups of finite index $G'$ of $G$ and $H'$ of $H$ such that $G'$ is isometric to $H'$ for some choice of proper and invariant metrics on $G'$ and $H'$. Notice virtually isometric groups are coarsely equivalent.

**Problem 6.5.** Suppose two countable abelian torsion groups $G$ and $H$ are coarsely equivalent. Are $G$ and $H$ virtually isometric?

## References


[1] S. A. Bogatyĭ, *Metrically homogeneous spaces*, (Russian) Uspekhi Mat. Nauk **57** (2002), no. 2(344), 3–22; translation in Russian Math. Surveys **57** (2002), no. 2, 221–240.

[2] N. Brodskiy, J. Dydak, J. Higes, A. Mitra, *Dimension zero at all scales*, Topology Appl., **154** (2007), 2729–2740.

[3] A. Lemin, *On ultrametrization of general metric spaces*, Proc. Amer. Math. Soc. **131** (2003), 979–989.

[4] P.S. Urysohn, *Sur un espace métrique universel*, Bull. Sci. Math. **51** (1927), 43–65 and 74–96.