1. Let $\Omega = \mathbb{R}^2 \setminus \{(0, 0)\}$. Consider the first-order p.d.e.

$$u_x^2 + u_y^2 = u^2 \quad \text{on } \Omega$$

satisfying $u = 1$ on $x^2 + y^2 = 1$. Prove that there exist exactly two solutions $u \in C^1(\Omega)$. Also find $\lim_{r \to 0} u(x, y)$, $r = (x^2 + y^2)^{1/2}$.

2. Let $0 < R_1 < R_2$, $\Omega = \{x = (x_1, x_2) \in \mathbb{R}^2 : R_1 < |x| < R_2\}$, $|x|^2 = x_1^2 + x_2^2$. Suppose $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfies $\Delta u \geq 0$ on $\Omega$. Denote $M(r) = \sup_{|x|=r} u$ for $R_1 \leq r \leq R_2$. Prove

$$M(r) \leq \left[M(R_1) \ln(R_2/r) + M(R_2) \ln(r/R_1)\right] (\ln(R_2/R_1))^{-1}$$

for $r \in [R_1, R_2]$.

Hint: Consider an auxiliary harmonic function $v(r)$.

3. Suppose $\Omega \subset \mathbb{R}^n$ is open and bounded. Assume $b_1, ..., b_n \in C^1(\overline{\Omega})$ and let $Lu = \Delta u + \sum_{i=1}^n b_i(x)u_{x_i}$. Suppose $u \in C^3(\overline{\Omega})$ satisfies $Lu = 0$ on $\Omega$. Define $v = u^2$, $w = |Du|^2 = \sum_{k=1}^n u_{x_k}^2$ on $\overline{\Omega}$.

Prove

(a) $Lv = 2|Du|^2$ on $\Omega$.
(b) For some $M > 0$, $Lw \geq 2|H|^2 - M|Du|^2$ on $\Omega$; here the Hessian $H = [u_{x_ix_j}]$, $|H|^2 = \sum_{i,k=1}^n u_{x_k}^2$.
(c) For some $\lambda > 0$, $L(\lambda v + w) \geq 0$ on $\Omega$, and for some $C > 0$

$$||Du||_{L^\infty(\Omega)} \leq C(||Du||_{L^\infty(\partial\Omega)} + ||u||_{L^\infty(\partial\Omega)}).$$

4. Let $\Omega \subset \mathbb{R}^n$ be open and bounded, $u_0 \in C^0(\overline{\Omega})$, $g \in C^0(\mathbb{R})$, $a(x, t) \in C^1(\overline{\Omega} \times [0, T])$, $a \geq 0$ on $\overline{\Omega} \times [0, T]$. Assume $u \in C^2(\overline{\Omega} \times [0, T])$ solves

$$u_t = \text{div}(a(x, t)\nabla u) + g(u)|\nabla u| \quad \text{on } \Omega \times [0, T]$$

with initial condition $u(x, 0) = u_0(x)$ for $x \in \Omega$, and boundary condition $u(x, t) = 0$ for $(x, t) \in \partial\Omega \times [0, T]$. Prove that $|u(x, t)| \leq \max_{\overline{\Omega}} |u_0|$ for all $(x, t) \in \overline{\Omega} \times [0, T]$.

5. Let $u$ be the bounded solution to the initial value problem

$$u_t = \Delta u \quad \text{on } \mathbb{R}^n \times [0, \infty)$$

1
with initial condition \( u(\cdot, 0) = u_0 \) where \( u_0 \) is bounded on \( \mathbb{R}^n \) and satisfies, for some \( \alpha \in (0, 1) \) and \( C > 0 \), \( |u_0(x) - u_0(y)| \leq C|x-y|^\alpha, \ x, y \in \mathbb{R}^n \). Prove that there exists a constant \( C_1 > 0 \) such that \( |u(x, t) - u(x, s)| \leq C_1|t^\alpha/2 - s^\alpha/2| \) for all \( x \in \mathbb{R}^n, \ s, t \geq 0 \).

6. Let \( f : \mathbb{R}^2 \to \mathbb{R} \) be a function such that for every \( R > 0 \) there exists \( N = N(R) > 0 \) such that

\[
|f(s, t)| \leq N(|s| + |t|) \quad \text{for all} \ (s, t) \in \mathbb{R}^2, \ |s| + |t| \leq R.
\]

Let \( u \) be a smooth compactly supported solution of the nonlinear wave equation

\[
u_{tt} - \Delta u + f(u, u_t) = 0 \quad \text{on} \ \mathbb{R}^3 \times (0, \infty).
\]

Assume that there is \( x_0 \in \mathbb{R}^3 \) and \( t_0 > 0 \) such that

\[
u(x, 0) = u_t(x, 0) = 0 \quad \text{for all} \ x \in B(x_0, t_0)
\]

(\( B(x_0, t_0) \) is the open ball in \( \mathbb{R}^3 \) with radius \( t_0 \) and centered at \( x_0 \)). Prove that \( u = 0 \) in the cone \( K(x_0, t_0) \) defined by

\[
K(x_0, t_0) = \{(x, t) \in \mathbb{R}^4 : 0 \leq t \leq t_0, \ |x - x_0| \leq t_0 - t\}.
\]

Hint: One may consider the energy function \( e(t) = \frac{1}{2} \int_{B(x_0, t_0 - t)} (u_t^2 + |\nabla u|^2 + u^2) \, dx \).

7. Let \( g : \mathbb{R} \to \mathbb{R} \) be defined by \( g(x) = 1 \) if \( |x| < 1 \), \( g(x) = 0 \) if \( |x| \geq 1 \). Use d’Alembert’s formula to find the solution \( u \) of the wave equation

\[
u_{tt} - u_{xx} = 0 \quad \text{on} \ \mathbb{R} \times (0, \infty)
\]

with \( u(x, 0) = x^2 \) and \( u_t(x, 0) = g(x), \ x \in \mathbb{R} \). Show that \( u \) is not differentiable with respect to the variable \( t \) at \( (x_0, t_0) = (0, 1) \).
1. Let $\Omega = \{(x,t) : x > 0, \ t > 0\}$. Assume $f \in C^\infty(\overline{\Omega})$, $f$ has bounded support and $f = 0$ on \{t = 0\}. Suppose $u \in C^2(\overline{\Omega})$ is a solution of

$$u_t + u_x + u = f(x,t) \quad \text{on } \Omega,$$

$$u = 0 \quad \text{on } \{x = 0\} \cup \{t = 0\}.$$

(a) For each $t > 0$, prove that $u(\cdot,t)$ has bounded support.
(b) For each $t > 0$, prove

$$\int_0^\infty u_t^2 \, dx \leq \int_0^t e^{s-t} \int_0^\infty f_t^2(x,s) \, dx \, ds.$$

(c) Prove there exists $K > 0$ such that $\int_0^\infty u_t^2 \, dx \leq Ke^{-t}$ for all $t > 0$.

2. Let $a > 0$, $\Omega = (-1,1) \times (-a,a) \subset \mathbb{R}^2$. Suppose $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ is a solution of

$$\Delta u = -1 \quad \text{on } \Omega, \quad u = 0 \quad \text{on } \partial \Omega.$$

Using the functions $v(x,y) = (1 - x^2)(a^2 - y^2)$, $w(x,y) = 2 - x^2 - \frac{y^2}{a^2}$ (or constant multiples of them), find positive bounds $C_1(a)$ and $C_2(a)$ such that

$$C_1(a) \leq u(0,0) \leq C_2(a).$$

3. Suppose $\Omega \subset \mathbb{R}^n \ (n \geq 3)$ is open, bounded with $C^\infty$-smooth boundary $\partial \Omega$. Let $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ be a solution of

$$-\Delta(u^3) = u \quad \text{on } \Omega, \quad u = 0 \quad \text{on } \partial \Omega.$$

(a) Using the Green’s function show there exists a constant $C > 0$ depending only on $\Omega$, but not on the solution, such that $\int_\Omega |u(x)|^3 \, dx \leq C$, and sup $|u| \leq C$.
(b) Show that, if $u \geq 0$ on $\Omega$, then either, $u \equiv 0$ on $\Omega$ or $u > 0$ on $\Omega$.
(c) Let $v$ be the eigenfunction corresponding to the first (least) eigenvalue $\lambda$ of $-\Delta v = \lambda v$ on $\Omega, \ v = 0$ on $\partial \Omega$ (recall $v > 0$ on $\Omega$). Show that, if $u \geq v$, then $u^3 \geq \frac{1}{\lambda}v$. 

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(d) Assuming also $u^3 \in C^1(\Omega)$, prove $\int_{\Omega} |\nabla (u^2)|^2 \, dx = C_1 \int_{\Omega} u^2 \, dx \leq C_2$ where $C_1, C_2$ depend only on $\Omega$, not on $u$.

4. Let $u_0 : \mathbb{R}^n \to \mathbb{R}$ be smooth and compactly supported, and

$$m = \int_{\mathbb{R}^n} u_0(y) \, dy.$$ 

Let $u$ be a solution of the Cauchy problem

$$u_t - \Delta u = 0 \quad \text{on} \quad \mathbb{R}^n \times (0, \infty),$$

$$u(x,0) = u_0(x) \quad x \in \mathbb{R}^n,$$

with $|u(x,t)| \leq Ae^{a|x|^2}$ for some fixed $A, a > 0$ and all $(x,t) \in \mathbb{R}^n \times (0, \infty)$.

Prove that there is a constant $N$ depending only on $n$ such that

$$\sup_{x \in \mathbb{R}^n} |u(x,t) - m \Phi(x,t)| \leq \frac{N}{t^{n/2}} \int_{\mathbb{R}^n} |y| |u_0(y)| \, dy, \quad \text{for all} \quad t > 0,$$

where $\Phi(x,t) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}$.

5. Let $u$ be a smooth function on $\overline{B}_1 \times [0,1]$ that satisfies the equation

$$a_0 \, u_t - b_0 \, \Delta u + u = 1 \quad \text{on} \quad B_1 \times (0,1),$$

$$u = 1 \quad \text{on} \quad \partial B_1 \times (0,1),$$

$$u(x,0) = 1 \quad x \in B_1,$$

where $a_0, b_0 : \overline{B}_1 \times [0,1] \to [0, \infty)$ are given continuous functions ($B_1$ = unit ball in $\mathbb{R}^n$). Prove that $u \leq 1$ on $\overline{B}_1 \times [0,1]$.

6. Assume that $\Omega \subset \mathbb{R}^n$ is an open, bounded set with $C^\infty$-smooth boundary $\partial \Omega$. Let $T > 0$, $\Omega_T = \Omega \times (0,T]$. Suppose $a \in C^1(\overline{\Omega})$, $a > 0$ on $\overline{\Omega}$, $\phi, \psi \in C^2(\overline{\Omega})$. Suppose $u \in C^2(\Omega_T)$ is a solution of

$$u_{tt} - a(x) \Delta u = u^3 \quad \text{on} \quad \Omega_T,$$

$$\frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial \Omega \times [0,T],$$

$$u = \phi, \quad u_t = \psi \quad \text{on} \quad \Omega \times \{t = 0\}.$$ 

Prove that $u$ is unique.

7. Assume $\phi \in C^2(\mathbb{R})$ and $h, \psi \in C^1(\mathbb{R})$. Consider the initial-value problem with $u \in C^2(\mathbb{R} \times [0, \infty))$

$$u_{tt} - u_{xx} = h(x-t) \quad \text{on} \quad \mathbb{R} \times [0, \infty), \quad (1)$$
\[ u = \phi(x), \quad u_t = \psi(x) \text{ at } t = 0, \quad x \in \mathbb{R}. \]  \hspace{1cm} (2)

(a) Find a solution of the p.d.e. in (1).
(b) Find a solution of (1) and (2).
Question 1: Let \( g : \mathbb{R} \to \mathbb{R} \) be a smooth function. Find solutions of the following initial-value problem in \( \mathbb{R}^2 \)

\[
    u_x + (1 + x^2)u_y - u = 0 \quad \text{with} \quad u(x, \frac{1}{3} x^3) = g(x).
\]

Question 2: Let \( h : \mathbb{R} \to \mathbb{R} \) be a smooth function. Consider the following equation in \( \mathbb{R}^2 \)

\[
    xu_x + yu_y = 2u \quad \text{with} \quad u(x, 0) = h(x).
\]

(a) Check that the line \( \{ y = 0 \} \) is characteristic at each point and find all \( h \) satisfying the compatibility condition on \( \{ y = 0 \} \).

(b) For \( h \) as compatible in (a), solve the PDE.

Question 3: Let \( \phi \) be smooth, compactly supported function defined in the unit ball \( B_1 \subset \mathbb{R}^n \) such that \( \phi = 1 \) on \( B_1/2 \), where \( B_1/2 \subset \mathbb{R}^n \) is the ball of radius 1/2 centered at the origin. Suppose that \( u \) is harmonic in \( B_1 \).

(a) Prove that there is \( \alpha > 0 \) depending only on \( n \) and \( \sup |\Delta \phi| \) and \( \sup |\nabla \phi| \) such that

\[
    \Delta (\phi^2 |\nabla u|^2 + \alpha u^2) \geq 0 \quad \text{in} \quad B_1.
\]

(b) Use part (a) and the maximum principle to conclude that there is a constant \( C > 0 \) depending only on \( n, \phi \) such that

\[
    \sup_{B_1/2} |\nabla u| \leq C \sup_{\partial B_1} |u|.
\]

Question 4: Let \( B_1 \subset \mathbb{R}^2 \) be the unit ball with boundary \( \partial B_1 \). Let \( f, c \in C(\overline{B_1}) \) and \( g \in C(\partial B_1) \). Assume that \( c(x, y) > 0 \) for all \( (x, y) \in B_1 \). Prove that there exists at most one \( C^2 \)-solution to the following equation

\[
    \left\{ \begin{array}{ll}
        -x^2 u_{xx} - y^2 u_{yy} + c(x, y)u & = f \\
        u & = g \quad \text{in} \quad B_1 \\
        u(x, 0) & = g_0(x) \quad x \in \mathbb{R}^n.
\end{array} \right.
\]

Question 5: Let \( a_0 \) be a smooth and compactly supported function defined on \( \mathbb{R}^n \) and \( p_0 \in (1, \infty) \). Consider the following Cauchy problem

\[
    \left\{ \begin{array}{ll}
        u_t - \Delta u & = |u|^{p_0-1}u \\
        u(x, 0) & = a_0(x) \quad x \in \mathbb{R}^n.
\end{array} \right.
\]

Define the scaling

\[
    u_\lambda(x, t) = \lambda^\beta u(\lambda x, \lambda^2 t), \quad \lambda > 0.
\]

(a) Find \( \beta \) (possibly depending on \( n, p_0 \)) so that if \( u \) is a solution of (1), then \( u_\lambda \) is also a solution (1) (with appropriate scaled initial data \( a_\lambda^0 \)).

(b) Recall that the \( L^p \)-norm is defined by

\[
    \|u(\cdot, t)\|_{L^p(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} |u(x, t)|^p dx \right)^{1/p}, \quad p \in [1, \infty).
\]

For \( \beta \) found in a), find \( p \) so that if \( u \) is a solution of (1) then

\[
    \|u(\cdot, \lambda^2 t)\|_{L^p(\mathbb{R}^n)} = \|u_\lambda(\cdot, t)\|_{L^p(\mathbb{R}^n)}
\]

for all \( \lambda > 0 \) and for all \( t > 0 \).
**Question 6:** Let us denote $\mathbb{R}^2_+ = \mathbb{R} \times (0, \infty)$ and $B^+_1 = B_1 \cap \mathbb{R}^2_+$, where $B_1$ is the unit ball in $\mathbb{R}^2$. Assume that $u = u(x, y, t)$ is a smooth function defined on $B^+_1 \times [0, 1]$ and satisfying

$$u_t - y^\alpha [u_{xx} + u_{yy}] + u_y + u \leq 0 \quad \text{for} \quad (x, y) \in B^+_1 \quad \text{and} \quad t \in (0, 1),$$

where $\alpha > 0$ is a given number. Assume that $u(x, y, 0) \leq 0$, and that $u \leq 0$ on $(\partial B_1 \cap \mathbb{R}^2_+) \times (0, 1)$, where $\partial B_1$ denotes the boundary of $B_1$. Prove that $u \leq 0$ on $B^+_1 \times [0, 1]$.

**Note:** We are not given any information on the boundary data on the part of the boundary where $y = 0$.

**Question 7:** Let $u_1(x)$ and $u_2(x)$ be smooth functions whose supports are in the unit ball $B_1 \subset \mathbb{R}^n$. For each $x_0 \in \mathbb{R}^n$ and each $t_0 > 0$, let $C(x_0, t_0)$ be the cone defined by

$$C(x_0, t_0) = \{(x, t) : 0 \leq t \leq t_0, \quad |x - x_0| \leq t_0 - t\}.$$

Assume that $u \in C^2$ is the solution of the equation

$$u_{tt} - \Delta u = 0 \quad \text{in} \quad \mathbb{R}^n \times (0, \infty)$$

with given initial data $u(x, 0) = u_1(x)$ and $u_t(x, 0) = u_2(x)$.

Give the proof for the finite propagation speed result for the wave equation, namely $u = 0$ on $C(x_0, t_0)$ for all $x_0 \in \mathbb{R}^n$ with $|x_0| > 1$ and $t_0 = |x_0| - 1$.

**Question 8:** Let $u$ be a smooth solution of the equation

$$u_{tt} - \Delta u = f \quad \text{on} \quad \mathbb{R}^3 \times (0, \infty)$$

with $u(\cdot, 0) = u_t(\cdot, 0) = 0$. Also, let $v$ be a smooth solution of the equation

$$v_{tt} - \Delta v = g \quad \text{on} \quad \mathbb{R}^3 \times (0, \infty)$$

with $v(\cdot, 0) = v_t(\cdot, 0) = 0$. Assume that $|f|^2 \leq g$. Prove that $2u(x, t)^2 \leq t^2 v(x, t)$ for all $x \in \mathbb{R}^3$ and $t > 0$. 
Question 1: Solve the Cauchy problem
\[
\begin{cases}
  xu_x - yu_y = u - y, & x > 0, y > 0, \\
  u(y^2, y) = y, & y > 0.
\end{cases}
\]

Question 2: Let \(a, R\) be positive numbers and consider the equation
\[
\begin{cases}
  u_t + au_x = f(x, t), & 0 < x < R, \quad t > 0, \\
  u(0, t) = 0, & t > 0, \\
  u(x, 0) = 0, & 0 < x < R.
\end{cases}
\]
Prove that for each solution \(u(x, t) \in C^1((0, R) \times (0, \infty))\) we have
\[
\int_0^R u^2(x, t)dx \leq e^t \int_0^t \int_0^R f^2(x, s)dxds, \quad \forall t > 0.
\]

Question 3: Let \(r > 0\) and let \(f, g\) be continuous functions defined on \(B_r(0)\). Let \(u\) be in \(C^2(B_r(0)) \cap C(\overline{B_r(0)})\) be the solution of the equation
\[
\begin{cases}
  -\Delta u = f, & B_r(0), \\
  u = g, & \partial B_r(0).
\end{cases}
\]
Prove that
\[
u(0) = \int_{\partial B_r(0)} g(x)dS(x) + \frac{1}{n(n-2)\alpha(n)} \int_{B_r(0)} \left[ \frac{1}{|x|^{n-2}} - \frac{1}{r^{n-2}} \right] f(x)dx.
\]
Hint: Consider
\[
\phi(s) = \int_{\partial B_s(0)} u(y)dS, \quad 0 < s \leq r.
\]
Compute \(\phi'(s)\) and then find \(\phi(0)\).

Question 4: Let \(R > 0\) and we denote \(B_R\) the ball of radius \(R\) centered at the origin in \(\mathbb{R}^n\). Let \(c, f\) be continuous functions on \(\overline{B}_R\). Assume that \(c \leq 0\) on \(\overline{B}_R\), and also assume that \(u \in C^2(B_R) \cap C(\overline{B_R})\) satisifies
\[
\begin{cases}
  \Delta u + cu = f \quad \text{in} \quad B_R, \\
  u = 0 \quad \text{on} \quad \partial B_R.
\end{cases}
\]
Prove that
\[
\sup_{B_R} |u| \leq \frac{R^2}{2n} \sup_{B_R} |f|
\]
Hint: Let \(A = \sup_{B_R} |f|\) and
\[
v(x) = \frac{AR^2}{2n}(R^2 - |x|^2)
\]
Use the maximum principle to prove that \(|u(x)| \leq v(x)\) on \(B_R\).

Question 5: Let \(u_0\) be the smooth and compactly supported function defined on \(\mathbb{R}^n\). Assume that \(u\) is a solution of the Cauchy problem
\[
\begin{cases}
  u_t - \Delta u = 0 & \text{in} \quad \mathbb{R}^n \times (0, \infty) \\
  u(x, 0) = u_0(x) & x \in \mathbb{R}^n.
\end{cases}
\]
Let $p, q \in (1, \infty)$ with $p \geq q$ and consider the inequality

$$\|u(\cdot, t)\|_{L^p(\mathbb{R}^n)} \leq \frac{N}{t^\alpha} \|u_0\|_{L^q(\mathbb{R}^n)}, \quad t > 0$$

with $N = N(n, p, q)$ and $\alpha = \alpha(n, p, q)$, where we denote

$$\|u(\cdot, t)\|_{L^p(\mathbb{R}^n)} = \left(\int_{\mathbb{R}^n} |u(x, t)|^p \, dx\right)^{\frac{1}{p}}$$

and similar notation is also used for $\|u_0\|_{L^q(\mathbb{R}^n)}$.

Use the scaling property of the heat equation to find the number $\alpha$ (certainly, show all of the work).

**Question 6:** Assume that $u$ is a smooth, bounded solution of the equation

$$\begin{cases}
    u_t - \Delta u &= u(1 - u) \quad \text{in } B_1 \times (0, 1] \\
    u &= 0 \quad \text{on } \partial B_1 \times (0, 1] \\
    u &= \frac{1}{2} \quad \text{on } B_1 \times \{0\}.
\end{cases}$$

Prove that $0 \leq u \leq 1$.

**Question 7:** Let $\varphi$ be a smooth, compactly supported function on $\mathbb{R}^2$. Assume that $u$ is a smooth solution of

$$\begin{cases}
    u_{tt} - \Delta u &= 0 \quad \text{in } \mathbb{R}^2 \times (0, \infty), \\
    u(\cdot, 0) &= 0 \quad \text{on } \mathbb{R}^2, \\
    u_t(\cdot, 0) &= \varphi \quad \text{on } \mathbb{R}^2.
\end{cases}$$

Prove that

$$|u(x, t)| \leq \frac{1}{2\sqrt{t}} \left(\|\varphi\|_{L^1(\mathbb{R}^2)} + \|\nabla \varphi\|_{L^1(\mathbb{R}^2)}\right), \quad \forall t > 1.$$ 

**Question 8:** Assume that $u \in C^2(\mathbb{R}^n \times [0, \infty))$ is a solution of the wave equation

$$u_{tt} = \Delta u \quad \text{in } \mathbb{R}^n \times (0, \infty).$$

Let

$$E(t) = \frac{1}{2} \int_{B_{1-t}} \left[|u_t(x, t)|^2 + |\nabla u(x, t)|^2\right] \, dx \quad \text{for } t \in (0, 1),$$

where $\nabla u = (u_{x_1}, u_{x_2}, \ldots, u_{x_n})$ and $B_r$ denotes the ball in $\mathbb{R}^n$ with radius $r > 0$ and centered at the origin.

(a) Prove that

$$E'(t) = \int_{B_{1-t}} \left[u_t(x, t)u_{tt}(x, t) + \sum_{i=1}^n u_{x_i} u_{x_i t}\right] \, dx$$

$$- \frac{1}{2} \int_{\partial B_{1-t}} \left[u_t^2(x, t) + |\nabla u(x, t)|^2\right] \, dS(x).$$

(b) Use the note that

$$[u_{x_i} u_{t}]_{x_i} = u_{x_i} u_{x_i t} + u_{x_i x_i} u_{tt}.$$

to prove that $E'(t) \leq 0$. Then, conclude also that $u = 0$ on $\{(x, t) : |x| \leq 1 - t, \, 0 \leq t \leq 1\}$ if $u(x, 0) = u_t(x, 0) = 0$ for $x \in B_1$. 
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Question 1: For $x > 0$, consider the equation:
\begin{align*}
\begin{cases}
u u_x + 2x u_y = 0 \text{ in } \mathbb{R}^2 \\ u(x,0) = \frac{1}{x} \text{ for } x > 0.
\end{cases}
\end{align*}

For $t_0, t_1 > 0$ with $t_0 \neq t_1$, let $C_0$ be the characteristic passing through the point $(t_0,0,1/t_0)$ and let $C_1$ be the characteristic passing through $(t_1,0,1/t_1)$. Determine whether the projections of $C_0$ and $C_1$ onto the $x$-$y$ plane intersect for some $y > 0$ (i.e., whether a shock develops), and if they do, find the point $(x,y)$ of intersection.

Question 2: Given a bounded domain $\Omega$ in $\mathbb{R}^n$, let $h$ be the solution to
\[ \Delta h = -1 \text{ in } \Omega, \quad h = 0 \text{ on } \partial \Omega. \]

Let $a > 0$ be a constant.

Prove: If there exists a function $u > 0$ that satisfies the equation
\[ \Delta u = \frac{1}{u} \text{ in } \Omega, \quad u \equiv a \text{ on } \partial \Omega, \]

then $a \geq \sqrt{\max_{\Omega} h}$.

Hint: Prove $u \leq a$. Then prove a better upper bound for $u$.

Question 3:
(a) Suppose $f : \mathbb{R} \to \mathbb{R}$ is continuous, bounded, and even (that is, $f(-x) = f(x)$ for all $x \in \mathbb{R}$). Suppose $u = u(x,t) \in C^2_t(\mathbb{R}_+^2) \cap C(\mathbb{R}^2_+)$ satisfies
\begin{align*}
\begin{cases}
u u_t = u_{xx} & \text{for } x \in \mathbb{R}, 0 < t < \infty, \\ u(x,0) = f(x) & \text{for } x \in \mathbb{R}, \\ |u(x,t)| \leq K e^{a|x|^2} & \text{for } x \in \mathbb{R}, 0 < t < \infty,
\end{cases}
\end{align*}

for some positive constants $K$ and $a$. Prove that for each $t > 0$, $u(x,t)$ is an even function of $x$: i.e., $u(-x,t) = u(x,t)$ for all $t > 0$.

(b) Assume $f : [0,\infty) \to \mathbb{R}$ is continuous and bounded. For $x \geq 0$ and $t \geq 0$, suppose $u = u(x,t) \in C^2([0,\infty) \times [0,\infty))$ satisfies
\begin{align*}
\begin{cases}
u u_t = u_{xx} & \text{for } 0 < x < \infty, 0 < t < \infty, \\ u(x,0) = f(x) & \text{for } 0 \leq x < \infty, \\ u_x(0,t) = 0 & \text{for } 0 < t < \infty \\ |u(x,t)| \leq Ke^{a|x|^2} & \text{for } x \in \mathbb{R}_+, 0 < t < \infty,
\end{cases}
\end{align*}

for some positive constants $K$ and $a$. Here $u_x(0,t)$ is interpreted as the $x$-derivative of $u$ from the right at $(0,t)$. Find a function $H = H(x,y,t)$ such that
\[ u(x,t) = \int_0^\infty H(x,y,t) f(y) \, dy, \]

and justify your answer.
Question 4: Consider the nonlinear PDE

\[ u_{tt} - \Delta u + u^3 = 0, \quad x \in \mathbb{R}^3, \quad t \in \mathbb{R}. \]

1. Assume that \( u \) is smooth and has compact support in \( x \) for each \( t \). What is the energy expression

\[ E(t) = \int_{\mathbb{R}^3} q(u, u_t, \nabla u) \, dx \]

which is conserved, i.e., \( E'(t) = 0 \)?

2. For any \( \alpha > 0 \), and \( x_0 \in \mathbb{R}^3 \), denote by

\[ E_{\alpha}(t) = \int_{B_{\alpha}(x_0)} q(u, u_t, \nabla u) \, dx \]

the energy contained in the ball of radius \( \alpha > 0 \) centered at \( x_0 \). Show that for any \( T > 0 \) and \( a > 0 \),

\[ E_{\alpha}(T) \leq E_{\alpha+t}(0) \]

Hint: Work with the 'energy'

\[ \tilde{E}(t) := \int_{B_{T+a}(x_0)} q(u, u_t, \nabla u) \, dx \]

3. Given \( a > 0 \), show that if \( u(x,0) = u_t(x,0) = 0 \) for \( |x| > a \), then \( u(x,t) = 0 \) for all \( |x| \geq a+t, \ t \geq 0 \).

Question 5: Let \( B \) be the unit ball in \( \mathbb{R}^n \) and let \( u \in C^\infty(\overline{B} \times [0,\infty)) \) satisfy

\[
\begin{align*}
u_t - \Delta u + u^{1/2} &= 0 \quad &\text{on } B \times (0,\infty) \\
0 \leq u &= &\text{on } B \times (0,\infty) \\
u &= 0 &= &\text{on } \partial B \times (0,\infty).
\end{align*}
\]

(a) Show that, if \( u|_{t=t_0} \equiv 0 \), then \( u \equiv 0 \) for \( t > t_0 \) as well.

(b) Prove that there is a number \( T \) depending only on \( M := \max u|_{t=0} \) such that \( u \equiv 0 \) on \( B \times (T,\infty) \).

Hint: Let \( v \) be the solution of the IVP,

\[
\frac{dv}{dt} + v^{1/2} = 0, \quad v(0) = M,
\]

and consider the function \( w = v - u \).

Question 6:

(a) Find a \( C^1 \) solution in \( \mathbb{R}^+ \times \mathbb{R} \ni (x,y) \) to:

\[
x^2 u_x - y^2 u_y = u^2 \quad \text{for } x > 0, y \in \mathbb{R}, \quad u(1,y) = \frac{1}{1+y^2}
\]

(b) Explain why this solution is not unique as a solution in \( C^1(\mathbb{R}^+ \times \mathbb{R}) \), but its restriction to some appropriate open set \( U \) containing the initial curve \( \{1\} \times \mathbb{R} \) is unique in \( C^1(U) \).
**Question 7:** Suppose $f, g \in C^\infty(\mathbb{R}^n)$. Suppose $u \in C^2(\mathbb{R}^n \times [0, \infty))$ satisfies

\begin{align*}
    u_{tt} &= \Delta u, \quad (x, t) \in \mathbb{R}^n \times (0, \infty), \\
    u(x, 0) &= f(x), \quad x \in \mathbb{R}^n, \\
    u_t(x, 0) &= g(x), \quad x \in \mathbb{R}^n.
\end{align*}

Prove that

$$
\int_{\mathbb{R}^n} u(x, t) \, dx = C_1 t + C_2,
$$

for all $t > 0$, where $C_1 = \int_{\mathbb{R}^n} g(x) \, dx$ and $C_2 = \int_{\mathbb{R}^n} f(x) \, dx$, under either of the two conditions:

(i) $n = 3$, $\int_{\mathbb{R}^n} |f(x)| \, dx < \infty$, $\int_{\mathbb{R}^n} |\nabla f(x)| \, dx < \infty$, and $\int_{\mathbb{R}^n} |g(x)| \, dx < \infty$; or

(ii) $n \in \mathbb{N}$, and $f$ and $g$ have compact support.

**Question 8:** Let $u \in C^2(\mathbb{R}^n)$ be a subharmonic function and consider the spherical averages

$$
v(r) := \int_{\partial B_r(0)} u(x) \, dS(x).
$$

(a) Show that the function $x \mapsto v(|x|)$ is also subharmonic in $\mathbb{R}^n$, and that $r \mapsto r^{n-1}v'(r)$ is monotonic.

(b) Now let $n = 2$. Prove that, if $u$ is also bounded, then $u$ is a constant.
PDE Preliminary Exam, January 2018

Instruction:

Solve all eight problems. Begin your answer to each question on a separate sheet. Explain all your steps.

1. A smooth function $u$ defined in the first quadrant on the $xy$-plane satisfies

$$-y \frac{\partial u}{\partial x} + x \frac{\partial u}{\partial y} = -2u, \quad u(x, 0) = x.$$ Determine $u(0, y)$.

2. Suppose that $u(x, t)$ is a smooth solution of

$$\begin{cases} 
    u_t + uu_x = 0 & \text{for } x \in \mathbb{R}, \ t > 0 \\
    u(x, 0) = f(x), & \text{for } x \in \mathbb{R} 
\end{cases}$$

Assume that $f$ is a $C^1$ function such that

$$f(x) = \begin{cases} 
    0 & \text{for } x < -1 \\
    1 & \text{for } x > 1 
\end{cases} \quad \text{and } f'(x) > 0, \ \text{for } |x| < 1.
$$

(a) Sketch the characteristics emanating from $(x_0, 0)$ for several values of $x_0 < -1, x_0 \in (-1, 1)$, and $x_0 > 1$.

(b) Show that for $t > 0$,

$$\lim_{r \to \infty} u(rx, rt) = \begin{cases} 
    0 & \text{for } x < 0 \\
    x/t & \text{for } 0 < x < t \\
    1 & \text{for } x > t 
\end{cases}$$

3. Suppose that for all $r > 2$, there exists a function $u_r : \mathbb{R}^3 \to \mathbb{R}$ that is continuous and satisfies

$$\begin{cases} 
    \Delta u = 0 & \text{in } B_r(0) \setminus \overline{B_1(0)} \\
    u(x) = 0 & \text{for } |x| \geq r \\
    u(x) = 1, & \text{for } x \in \overline{B_1(0)}. 
\end{cases}$$

(a) Show that for all $x \in \mathbb{R}^3$, if $2 < r_1 \leq r_2$, then

$$0 \leq u_{r_1}(x) \leq u_{r_2}(x) \leq 1.$$
(b) Show that

i. \( u(x) = \lim_{r \to \infty} u_r(x) \) is harmonic on \( \mathbb{R}^3 \setminus \overline{B_1(0)} \)

ii. \( \lim_{|x| \to \infty} u(x) = 0. \)

[Hint: noting that \( \frac{1}{|x|} \) is harmonic, study \( u_r(x) - \frac{1}{|x|} \) over an annulus.]

4. Denote by \( \mathbb{R}^n_+ = \{ x = (x', x_n) : x_n > 0 \} \), \( \Sigma = \{ x = (x', x_n) : x_n = 0 \} \).

Suppose that \( u \) is harmonic in \( \mathbb{R}^n_+ \), continuous on \( \mathbb{R}^n_+ \cup \Sigma \), and \( u = 0 \) on \( \Sigma \). Define

\[
\overline{u}(x', x_n) = \begin{cases} 
  u(x', x_n) & \text{for } x_n \geq 0, \\
  -u(x', -x_n) & \text{for } x_n < 0.
\end{cases}
\]

Then show that \( \overline{u} \) is harmonic in \( \mathbb{R}^n \).

5. Let \( \Omega \subseteq \mathbb{R}^n \) be a \( C^\infty \) bounded domain. Assume that \( u_0 \in C^\infty(\overline{\Omega}) \), \( a \in C([0, \infty)) \), and \( \lim_{t \to \infty} a(t) \leq 0 \). Suppose also \( u \in C^2(\overline{\Omega} \times [0, \infty)) \) satisfies

\[
\begin{cases}
u_t = \Delta u + a(t)u & \text{on } \Omega \times (0, \infty), \\
u = 0 & \text{on } \partial \Omega \times (0, \infty), \\
u = u_0 & \Omega \times \{t = 0\}.
\end{cases}
\]

Prove that

\[
\lim_{t \to \infty} \int_{\Omega} u^2(x, t) dx = 0
\]

(Hint: Use the Energy method. You may apply Poincaré's inequality.)

6. Let \( \Omega \subseteq \mathbb{R}^n \) be a \( C^\infty \) bounded domain, \( T > 0 \), and \( a \in \mathbb{R}^n \) is a given vector. Suppose \( u \in C^2(\overline{\Omega} \times [0, T]) \) satisfies

\[
\begin{cases}
u_t = \Delta u + a \cdot \nabla u + u^2 & \text{on } \Omega \times (0, T), \\
u = 0 & \text{on } \partial \Omega \times (0, T), \\
u = 0 & \Omega \times \{t = 0\}.
\end{cases}
\]

Prove that

(a) \( u \geq 0 \), on \( \Omega \times (0, T] \),

(b) \( u_t \geq 0 \) on \( \Omega \times (0, T] \).

(Hint: What equation does \( u_t \) solve?)
7. Let \( \Omega \subseteq \mathbb{R}^n \) be a \( C^\infty \) bounded domain and let \( T > 0 \). Suppose \( V = V(x) \) and \( h = h(x) \) are continuous functions on \( \bar{\Omega} \), with \( V(x) \geq 0 \). Suppose \( u = u(x, t) \in C^2(\bar{\Omega} \times [0, T]) \), where \( x \in \Omega \) and \( t \in [0, T] \), and \( u \) satisfies

\[
\begin{align*}
    u_t - \Delta u + V(x)u &= h(x) \quad \text{on } \Omega \times (0, T); \\
    u(x, 0) &= 0 \quad \text{on } \Omega; \\
    u_t(x, 0) &= 0 \quad \text{on } \Omega; \\
    u &= -D_n u \quad \text{on } \partial\Omega \times (0, T),
\end{align*}
\]

where \( D_n u \) is the outward normal derivative of \( u \) on \( \partial\Omega \).

(a) Prove that \( \int_\Omega h(x)u(x, t) \, dx \geq 0 \) for all \( t \geq 0 \).

Hint: Consider

\[
E(t) = \frac{1}{2} \int_\Omega u_t^2 + |\nabla u|^2 + Vu^2 - 2hu \, dx + \frac{1}{2} \int_{\partial\Omega} u^2 \, d\sigma,
\]

where \( d\sigma \) is surface measure on \( \partial\Omega \).

(b) Suppose in addition that \( V(x) \geq A \) and \( |h(x)| \leq B \), for all \( x \in \Omega \), for some constants \( A > 0 \) and \( B > 0 \). Prove that

\[
\int_\Omega |u(x, t)| \, dx \leq \frac{2B|\Omega|}{A},
\]

for all \( t \geq 0 \), where \( |\Omega| = \int_\Omega dx \) is the measure of \( \Omega \).

Hint: Start by writing \( \int_\Omega |u| \, dx = \int_\Omega \frac{\sqrt{V}|u|}{\sqrt{V}} \, dx \), and apply Cauchy Schwartz.

8. Suppose \( u \in C^2(\mathbb{R}^n \times [0, \infty)) \) is a solution of

\[
\begin{align*}
    u_t &= \Delta u \quad \text{on } \mathbb{R}^n \times (0, \infty); \\
    u(x, 0) &= f(x) \quad \text{on } \mathbb{R}^n; \\
    u_t(x, 0) &= g(x) \quad \text{on } \mathbb{R}^n,
\end{align*}
\]

where \( f, g \in C^\infty(\mathbb{R}^n) \) have compact support: there exists \( R > 0 \) such that \( f(x) = 0 \) and \( g(x) = 0 \) if \( |x| > R \). Consider the statement:

(S): For all such \( f, g \) and \( R \), and all \( x_0 \in \mathbb{R}^n \), there exists \( T = T(x_0, R) > 0 \) such that \( u(x_0, t) = 0 \) for all \( t > T \).

(a) Is (S) true if \( n = 1 \)? Either prove (S) or give an example showing that (S) fails.

(b) Is (S) true if \( n = 3 \)? Either prove (S) or give an example showing that (S) fails.
1. For a given continuous function $f$, solve the initial-boundary value problem

$$\begin{cases} u_t + (x + 1)^2u_x = x, & \text{for } x > 0, t > 0 \\
u(x, 0) = f(x), & x > 0 \\
u(0, t) = -1 + t, & t > 0. \end{cases}$$

Find a condition on $f$ so that the solution $u(x, t)$ is continuous on the first quadrant of $\mathbb{R}^2$, i.e. the region $\{(x, t) \in \mathbb{R}^2 : x > 0, t > 0\}$.

2. Determine an integral (weak) solution to the Burgers equation

$$u_t + \left(\frac{1}{2}u^2\right)_x = 0, \quad (x, t) \in \mathbb{R} \times (0, \infty)$$

with initial data

$$u(x, 0) = \begin{cases} 1 & \text{if } x < 0 \\
1 - x & \text{if } 0 < x < 1 \\
0 & \text{if } x > 1. \end{cases}$$

3. Let $n \geq 2$, and let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain with $C^\infty$-smooth boundary. Suppose $p$ and $q$ are non-negative continuous functions defined on $\Omega$, satisfying $p(x) + q(x) > 0$ (strict inequality) for all $x \in \Omega$. Find all functions $u \in C^2(\bar{\Omega})$ satisfying

$$\begin{cases} \Delta u = pu^3 + qu & \text{on } \Omega, \\
\frac{\partial u}{\partial n} = 0 & \text{on } \partial \Omega, \end{cases}$$

where $n(x)$ is the outward unit normal to $\Omega$ at $x \in \partial \Omega$.

4. Suppose $u$ is harmonic on a $C^\infty$ domain $\Omega \subseteq \mathbb{R}^n$, and let $u(x) = 0$ for $x \not\in \Omega$. Suppose $\varphi$ is a $C^\infty$ function on $\mathbb{R}^n$ such that $\varphi(x) = 0$ if $|x| \geq 1$, and $\varphi$ is radial: there exists a function $\varphi_0 : [0, \infty) \to \mathbb{R}$ such that $\varphi(x) = \varphi_0(|x|)$. For $\epsilon > 0$, let

$$\varphi_\epsilon(x) = \frac{1}{\epsilon^n} \varphi\left(\frac{x}{\epsilon}\right).$$

Let

$$A = \int_{\mathbb{R}^n} \varphi(x) \, dx.$$ 

Fix $x_0 \in \Omega$ and let $R > 0$ be such that $x \in \Omega$ if $|x - x_0| < R$. For $0 < \epsilon < R$, prove that

$$\varphi_\epsilon * u(x_0) = Au(x_0),$$

where $*$ denotes convolution: by definition, $f * g(x) = \int_{\mathbb{R}^n} f(x - y)g(y) \, dy$. 


5. Suppose that \( \mathbf{b} \in \mathbb{R}^n \), and \( \beta \in \mathbb{R} \) are given. Consider the Cauchy problem

\[
\begin{align*}
(*): \quad & \begin{cases}
  u_t + \mathbf{b} \cdot \nabla u + \beta u = \Delta u, & \text{in } \mathbb{R}^n \times (0, \infty) \\
  u(x, 0) = f(x), & \text{on } \mathbb{R}^n.
\end{cases}
\end{align*}
\]

(a) Determine \( \mathbf{a} \in \mathbb{R}^n \) and \( \alpha \in \mathbb{R} \) such that if \( u \) is a smooth solution to (*), then \( v(x, t) = e^{-(\mathbf{a} \cdot x + \alpha t)} u(x, t) \) solves the Cauchy problem

\[
\begin{align*}
\{ v_t &= \Delta v, & \text{in } \mathbb{R}^n \times (0, \infty) \\
 v(x, 0) &= e^{-(\frac{1}{4} \beta) x} f(x), & \text{on } \mathbb{R}^n.
\end{align*}
\]

(b) Write down an explicit formula for a solution \( u(x, t) \) to (*).

6. Let \( \Omega \subset \mathbb{R}^n \) a bounded domain with smooth boundary, and \( T > 0 \). Denote the cylinder \( \Omega_T = \Omega \times (0, T] \) and its parabolic boundary \( \partial_p \Omega_T = (\partial \Omega \times [0, T]) \cup (\Omega \times \{0\}) \).

(a) Prove the following version of the maximum principle. Suppose that \( u \) and \( v \) are two functions in \( C^2(\overline{\Omega_T}) \) such that

\[
\begin{align*}
  u_t - \Delta u &\leq v_t - \Delta v \quad \text{in } \Omega_T \\
  u &\leq v \quad \text{on } \partial_p \Omega_T.
\end{align*}
\]

Then \( u \leq v \) in \( \Omega_T \).

(b) Suppose that \( f(x, t), u_0(x) \) and \( \phi(x, t) \) are continuous functions in their respective domains. Let \( u \in C^2(\overline{\Omega_T}) \) satisfy

\[
\begin{align*}
\{ u_t - \Delta u &= e^{-u} - f(x, t), & \text{in } \Omega_T \\
 u|_{t=0} &= u_0, & \text{in } \Omega \\
 u|_{\partial \Omega \times (0, T)} &= \phi.
\end{align*}
\]

Let \( a = \|f\|_{L^\infty} \) and \( b = \sup\{\|u_0\|_{L^\infty}, \|\phi\|_{L^\infty}\} \).

i. Show that \( -(aT + b) \leq u(x, t) \), for all \( (x, t) \in \overline{\Omega_T} \).

\textit{Hint: Introduce } \( v(x, t) = -(at + b) \) \text{ and use part a).}

ii. Prove \( u(x, t) \leq Te^{aT+b} + aT + b \), for all \( (x, t) \in \overline{\Omega_T} \).
7. Suppose that \( f \in C^2(\mathbb{R}) \) is odd and 2-periodic (i.e. \( f(x + 2) = f(x) \) for all \( x \in \mathbb{R} \)). Let \( u \in C^2([0,1] \times \mathbb{R}) \) solve
\[
\begin{align*}
\begin{cases}
    u_{tt} - u_{xx} = \sin(\pi x) & \text {in } (0,1) \times \mathbb{R} \\
    u(x,0) = f(x), & \text{ in } (0,1) \times \mathbb{R} \\
    u_t(x,0) = 0, & x \in [0,1] \\
    u(0,t) = 0 = u(1,t), & t \in \mathbb{R}.
\end{cases}
\end{align*}
\]
(a) Prove uniqueness of the solution \( u \in C^2([0,1] \times \mathbb{R}) \).

(b) Find the solution \( u \), and show that it satisfies \( u(x,t+2) = u(x,t) \), and \( u(x,-t) = u(x,t) \) for all \( (x,t) \in [0,1] \times \mathbb{R} \).

8. Assume that \( \Omega \subset \mathbb{R}^n \) is open, bounded with \( C^\infty \)-smooth boundary \( \partial \Omega \). Let \( T > 0 \), and denote \( \Omega_T = \Omega \times (0,T] \). Suppose also that \( f \in C^1(\mathbb{R}^{n+2}) \), \( \phi, \psi \in C^2(\overline{\Omega}) \), and \( u \in C^2(\overline{\Omega_T}) \) is a solution of
\[
\begin{align*}
\begin{cases}
    u_{tt} - \Delta u = f(u,u_t,\nabla u), & \text {in } \Omega_T \\
    u = \phi, & \text{ on } \Omega \times \{t = 0\}, \\
    u_t = \psi, & \text{ on } \Omega \times \{t = 0\}, \\
    \frac{\partial u}{\partial n} = 0, & \text{ on } \partial \Omega \times [0,T].
\end{cases}
\end{align*}
\]
Prove that \( u \) is unique.

*Hint: You may use an energy function of the form*
\[
E(t) = \frac{1}{2} \int_{\Omega} \left( w_t^2 + |\nabla w|^2 + w^2 \right) dx.
\]
1.) Consider the PDE, for \( x \in \mathbb{R} \) and \( y \in \mathbb{R} \):

\[
\begin{cases}
2yu_x + u_y = u^4, \\
u(x, 0) = f(x),
\end{cases}
\]

for some \( C^2 \) function \( f \).

(a) Show that (\#) has a solution that exists for all \( x \in \mathbb{R} \) and all \( y > 0 \) if and only if \( f(t) \leq 0 \) for all \( t \in \mathbb{R} \).

(b) Show that if (\#) has a solution for all \((x, y) \in \mathbb{R}^2\), then \( f(t) = 0 \) for all \( t \) and \( u \) is identically 0.

2.) Suppose \( n \geq 2 \), \( R > 0 \), \( B(0, R) \subseteq \mathbb{R}^n \), and \( u : \overline{B(0, R)} \to \mathbb{R} \) satisfies \( u \in C(\overline{B(0, R)}) \), \( u \) is harmonic on \( B(0, R) \), and \( u \geq 0 \) on \( B(0, R) \).

(a) Prove that

\[
\frac{(R - |x|) R^{n-2}}{(R + |x|)^{n-1}} u(0) \leq u(x) \leq \frac{(R + |x|) R^{n-2}}{(R - |x|)^{n-1}} u(0),
\]

for all \( x \in B(0, R) \).

(b) Prove that

\[
|u_{x_j}(x)| \leq \frac{(2n + 2) R^{n-1}}{(R - |x|)^n} u(0),
\]

for \( x \in B(0, R) \) and \( j = 1, 2, \ldots, n \).

3.) Suppose \( n \geq 3 \), and \( \Omega \subseteq \mathbb{R}^n \) is a \( C^\infty \) bounded domain. Let

\[
\Gamma(x) = \frac{1}{(2 - n) \omega_n |x|^{n-2}},
\]

for \( x \in \mathbb{R}^n \setminus \{0\} \), be the fundamental solution for the Laplacian on \( \mathbb{R}^n \). Let \( G(x, y) \) be the Green's function for the Laplacian on \( \Omega \) (i.e., \( G(x, y) = h(x, y) + \Gamma(x - y) \)), where, for each \( x \in \Omega \), \( h(x, y) \) is a harmonic function of \( y \) on \( \Omega \), and \( h(x, y) = -\Gamma(x - y) \) for \( x \in \Omega \) and \( y \in \partial \Omega \). You can assume that \( G \in C^2(\overline{\Omega} \times \overline{\Omega} \setminus \{(x, y) \in \overline{\Omega} \times \overline{\Omega} : x = y\}) \). Prove that \( \Gamma(x - y) < G(x, y) < 0 \), for \((x, y) \in \Omega \times \Omega \) with \( x \neq y \).
4.) Let $\Omega \subseteq \mathbb{R}^n$ be a bounded $C^1$ domain and suppose $T > 0$. Let $\Omega_T = \Omega \times (0, T]$. Suppose $u \in C^2_1(\Omega_T) \cap C(\overline{\Omega_T})$ satisfies
\[
\begin{cases}
  u_t = \Delta u + |\nabla u|^2 - u(u - 1)(u - 2), & \text{for } (x, t) \in \Omega_T, \\
  u(x, t) = e^{-t}[1 + \sin(|x|^2)], & \text{for } (x, t) \in \partial \Omega \times [0, T], \\
  u(x, 0) = 1 + \sin(|x|^2), & \text{for } x \in \Omega.
\end{cases}
\]
Prove that $0 \leq u \leq 2$ on $\overline{\Omega_T}$.

5.) Suppose $g = g(x, t) \in C^2_1(\mathbb{R}^{n+1}_+)$, where $x \in \mathbb{R}^n$ and $t \geq 0$, and suppose $g$ has compact support. Suppose $u \in C^2_1(\mathbb{R}^{n+1}_+ \cap C(\mathbb{R}^{n+1}_+)$ satisfies, for some positive constants $K$ and $a$,
\[
\begin{cases}
  u_t - \Delta u = g(x, t) & \text{for } x \in \mathbb{R}^n, t \in (0, \infty), \\
  u(x, 0) = 0 & \text{for } x \in \mathbb{R}^n, \\
  |u(x, t)| \leq Ke^{at} |x|^2 & \text{for } x \in \mathbb{R}^n, t \in [0, \infty).
\end{cases}
\]
Suppose $p > n/2$ and $M = \max_{x \geq 0} \int_{\mathbb{R}^n} |g(x, t)|^p \, dx$. Prove that there exists a constant $C$, depending only on $n$ and $p$, such that
\[
|u(x, t)| \leq CM^{1/p} t^{1 - \frac{n}{2p}},
\]
for all $(x, t) \in \mathbb{R}^{n+1}_+$.

6.) Suppose $f : \mathbb{R}^3 \to \mathbb{R}$ is harmonic, and $g : \mathbb{R}^3 \to \mathbb{R}$ is $C^\infty$. Suppose $u \in C^2(\mathbb{R}^3 \times [0, \infty))$ satisfies
\[
\begin{cases}
  u_{tt} = \Delta u, & x \in \mathbb{R}^3, \ t > 0 \\
  u(x, 0) = f(x), & x \in \mathbb{R}^3, \\
  u_t(x, 0) = g(x), & x \in \Omega.
\end{cases}
\]
(a) Prove that
\[
|u(x, t)| \leq |f(x)| + \sup_{y \in B(0, 1)} |g(y)|
\]
for $x \in \mathbb{R}^3$ and $0 < t < 1$.

(b) Prove that
\[
|u(x, t)| \leq |f(x)| + \frac{3}{4\pi t^2} \int_{B(x, t)} |g(y)| \, dy + \frac{1}{4\pi t} \int_{B(x, t)} |\nabla g(y)| \, dy,
\]
for $x \in \mathbb{R}^3$ and $t \geq 1$. 
7.) Let \( n \geq 2 \), let \( \Omega \subseteq \mathbb{R}^n \) be a \( C^\infty \) bounded domain, and let \( T > 0 \). Suppose \( \vec{h} = (h_1, h_2, \ldots, h_n) \), where each component \( h_j = h_j(x, t) : \overline{\Omega} \times [0, T] \rightarrow \mathbb{R} \) satisfies \( h_j \in C(\overline{\Omega} \times [0, T]) \). Suppose \( f, g : \overline{\Omega} \rightarrow \mathbb{R} \) are continuous. Show that there is at most one function \( u = u(x, t) \in C^2(\overline{\Omega} \times [0, T]) \) satisfying

\[
\begin{align*}
\frac{\partial u}{\partial t} &= \Delta u + \nabla u \cdot \vec{h}, & x \in \Omega, \ 0 < t < T \\
u &= 0, & x \in \partial \Omega, \ 0 \leq t \leq T, \\
u(x, 0) &= f(x), & x \in \Omega, \\
u_t(x, 0) &= g(x), & x \in \Omega.
\end{align*}
\]
In the following, unless otherwise stated, \( \Omega \subset \mathbb{R}^n \) is an open, bounded set with \( C^\infty \)-smooth boundary \( \partial \Omega \). Denote \( \Omega_T = \Omega \times (0,T) \).

1. Let \( \Omega = \{(x,t) : x \in \mathbb{R}, t > 0\} \) and assume \( u_0, v_0 \in C^1(\mathbb{R}) \). Suppose \( u, v \in C^1(\overline{\Omega}) \) solve the system

\[
\begin{align*}
  u_t + u_x &= u \quad \text{on } \overline{\Omega}, \\
  v_t + v_x &= -v + u \quad \text{on } \overline{\Omega}, \\
  u(x,0) &= u_0(x), \\
  v(x,0) &= v_0(x) \quad x \in \mathbb{R}.
\end{align*}
\]

Find \( u(x,t), v(x,t) \) in terms of \( u_0, v_0 \).

2. Let \( R > 0 \). Assume \( u \in C^2(\overline{B_R(0)}) \) is nonnegative and satisfies \( u(0) = 0 \),

\[
0 \leq \Delta u \leq 1 \quad \text{on } \ B_R(0).
\]

Let \( u_1, u_2 \) be the solutions of the following problems

\[
\begin{align*}
  \Delta u_1 &= \Delta u \quad \text{on } \ B_R(0), \\
  u_1 &= 0 \quad \text{on } \partial B_R(0).
\end{align*}
\]

\[
\begin{align*}
  \Delta u_2 &= 0 \quad \text{on } \ B_R(0), \\
  u_2 &= u \quad \text{on } \partial B_R(0).
\end{align*}
\]

(a) Prove that \( u = u_1 + u_2 \) on \( B_R(0) \) and \( u_1 \leq 0, u_2 \geq 0 \) on \( B_R(0) \).

(b) Prove that \( |u_1(x)| \leq \frac{R^2}{2n} \) for all \( x \in B_R(0) \). Hint: Compare \( u_1 \) with \( \phi(x) = \frac{1}{2n}(R^2 - |x|^2) \).

(c) Prove that \( u_2(x) \leq \frac{2^{n-1}}{n} R^2 \) for all \( x \in B_{R/2}(0) \). Conclude \( |u(x)| \leq \frac{1+2^n}{2n} R^2 \) for all \( x \in B_{R/2}(0) \).

3. Let \( n \geq 3, f \in C_0^\infty(\mathbb{R}^n) \). Assume \( u \in C^\infty(\mathbb{R}^n) \) is a solution of

\[
-\Delta u = f \quad \text{on } \mathbb{R}^n
\]

and \( u(x) \to 0 \) as \( |x| \to \infty \). Prove there exists \( C > 0 \) such that

\[
|u(x)| \leq \frac{C}{|x|^{n-2}}
\]
for all $x \in \mathbb{R}^n, x \neq 0$.

4. Let $T > 0$ and assume $\phi, h, f, g$ are $C^\infty$-smooth functions. Suppose $u, v \in C^2(\overline{\Omega_T})$ satisfy

$$
\begin{align*}
&u_t - \Delta u = \phi \text{ on } \Omega_T, \\
&u = h \text{ on } \partial \Omega \times (0, T], \\
&u = f \text{ on } \Omega \times \{t = 0\},
\end{align*}
$$

$$
\begin{align*}
&v_t - \Delta v = \phi \text{ on } \Omega_T, \\
&v = h \text{ on } \partial \Omega \times (0, T], \\
&v = g \text{ on } \Omega \times \{t = 0\}.
\end{align*}
$$

Prove that $\int_{\Omega} |u(x, t) - v(x, t)|^2 dx \leq \int_{\Omega} |f(x) - g(x)|^2 dx$ for all $t \in [0, T]$.

5. Suppose $f : \mathbb{R}^n \rightarrow \mathbb{R}$ is continuous, bounded and $\int_{\mathbb{R}^n} |f| dx < \infty$. Show there exists a unique solution $u \in C^\infty(\mathbb{R}^n \times (0, \infty)) \cap C^0(\mathbb{R}^n \times [0, \infty))$ of

$$
\begin{align*}
&u_t = \Delta u - 2u \quad \text{on } \mathbb{R}^n \times (0, \infty), \\
u = f \quad \text{on } \mathbb{R}^n \times \{t = 0\}, \\
|u(x, t)| \leq Ce^{-2t}(1 + t)^{-\frac{n}{2}} \quad \text{for } (x, t) \in \mathbb{R}^n \times [0, \infty),
\end{align*}
$$

for some constant $C$ depending on $f, n$ but not on $x, t$.

6. Let $f \in C^1(\mathbb{R})$ with $f'$ bounded on $\mathbb{R}$ and $f(0) = 0$. Suppose $\phi, \psi \in C^2(\overline{\Omega})$ and $u \in C^2(\overline{\Omega_T})$ is a solution of

$$
\begin{align*}
&u_{tt} - \Delta u = f(u) \text{ on } \Omega_T, \\
&u_t = 0 \text{ on } \partial \Omega \times (0, T], \\
u = \phi, \ u_t = \psi \text{ on } \Omega \times \{t = 0\}.
\end{align*}
$$

(a) Denoting $E(t) = \frac{1}{2} \int_{\Omega} (u_t^2 + |\nabla u|^2 + u^2) dx$, prove $E(t) \leq E(0)e^{Ct}$ for all $t \in [0, T]$, and for some constant $C > 0$.

(b) Prove the solution $u$ is unique.

7. Let $p > n/2$. Suppose $\phi, \psi \in C^\infty_0(\mathbb{R}^n)$ and $u \in C^2(\mathbb{R}^n \times [0, \infty))$ is a solution of

$$
\begin{align*}
&u_{tt} - \Delta u = 0 \quad \text{on } \mathbb{R}^n \times [0, \infty), \\
u = \phi, \ u_t = \psi \quad \text{on } \mathbb{R}^n \times \{t = 0\}.
\end{align*}
$$

Prove that there exists $C > 0$ such that

$$
\int_{\mathbb{R}^n} \frac{|u_t| + |\nabla u|}{(1 + |x| + t)^p} dx \leq \frac{C}{(1 + t)^{p-n/2}}
$$

for all $t \geq 0$. 2
In the following, unless otherwise stated, $\Omega \subset \mathbb{R}^n$ is an open, bounded set with $C^\infty$-smooth boundary $\partial \Omega$. Denote $\Omega_T = \Omega \times (0, T]$.

1. Let $\Omega = \{(x, t) : x \in \mathbb{R}, t > 0\}$, $b \in \mathbb{R}$ and assume $a \in C^1(\overline{\Omega})$, $\phi \in C^1(\mathbb{R})$ are bounded. Suppose $u \in C^1(\overline{\Omega})$ is a solution of

$$u_t + a(x, t)u_x + bu = 0 \quad \text{on} \; \Omega,$$

$$u(x, 0) = \phi(x), \; x \in \mathbb{R}.$$

(a) Prove $\sup_{x \in \mathbb{R}} |u(x, t)| \leq e^{-bt} \sup_{x \in \mathbb{R}} |\phi|$ for all $t \geq 0$.

(b) Find the solution when $a = a(t)$.

2. Let $\Omega \subset \mathbb{R}^2$ and suppose $g \in C^0(\partial \Omega)$. Show that there exists at most one solution $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ satisfying

$$\Delta u + u_x - u_y = u^3 \quad \text{on} \; \Omega,$$

$$u = g \quad \text{on} \; \partial \Omega.$$

3. Let $\Omega \subset \mathbb{R}^n$. A function $v \in C^0(\Omega)$ is subharmonic on $\Omega$ if for every $x \in \Omega$, there exists $r(x) > 0$ such that $v$ satisfies the mean-value property:

$$v(x) \leq \frac{1}{\omega_n r^{n-1}} \int_{\partial B(x, r)} v(\xi) dS(\xi)$$

for all $r \in (0, r(x)]$, where $\omega_n$ is the surface area of the unit sphere in $\mathbb{R}^n$.

(a) Suppose $u, v \in C^0(\Omega)$, $u$ is harmonic on $\Omega$, $v$ is subharmonic on $\Omega$, $v \leq u$ on $\partial \Omega$. Prove $v \leq u$ on $\Omega$. You can assume the maximum principle for subharmonic functions.

(b) Let $v \in C^0(\Omega)$ be subharmonic on $\Omega$ and $B(x_0, R) \subset \Omega$. For $r \in (0, R)$ define

$$g(r) = \frac{1}{\omega_n r^{n-1}} \int_{\partial B(x_0, r)} v(\xi) dS(\xi).$$

Prove $g$ is nondecreasing on $(0, R)$. Deduce the mean-value property

$$v(x_0) \leq \frac{1}{\omega_n r^{n-1}} \int_{\partial B(x_0, r)} v(\xi) dS(\xi)$$
holds for any \( \overline{B(x_0, r)} \subset \Omega \) (note, in the definition of subharmonic function, this is assumed only for sufficiently small \( r \)). Hint: for \( r_1 < r_2 \) use the Poisson Integral Formula on \( B(x_0, r_2) \) to get a harmonic function.

4. Let \( m > 0 \), \( T > 0 \) and assume \( u_0 \in C^0(\overline{\Omega}) \) is nonnegative on \( \Omega \). Suppose \( u \in C^{2,1}(\Omega_T) \cap C^0(\overline{\Omega} \times [0, T]) \) is a solution of

\[
\begin{align*}
    u_t &= \Delta u + |\nabla u|^2 + u(m - u) \quad \text{on} \quad \Omega_T, \\
    u &= 0 \quad \text{on} \quad \partial \Omega \times [0, T], \\
    u &= u_0 \quad \text{on} \quad \Omega \times \{t = 0\}.
\end{align*}
\]

Prove \( 0 \leq u \leq \max \{m, \sup_{\Omega} u_0\} \) on \( \overline{\Omega} \times [0, T] \).

5. Let \( 1 < p < \infty \), \( u_0 \in C^0(\overline{\Omega}) \). Consider

\[
\begin{align*}
    u_t &= \Delta u + |u|^{p-1} u \quad \text{on} \quad \Omega_T, \\
    u &= 0 \quad \text{on} \quad \partial \Omega \times [0, T], \\
    u &= u_0 \quad \text{on} \quad \Omega \times \{t = 0\}.
\end{align*}
\]

For each \( u_0 \), let \( T_{\text{max}} = T_{\text{max}}(u_0) \in (0, \infty) \) be the maximal time such that the problem above has a solution \( u \in C^{2,1}(\overline{\Omega} \times [0, T_{\text{max}}]) \). Let \( E(t) = \frac{1}{2} \int_{\Omega} |\nabla u|^2 \, dx - \frac{1}{p+1} \int_{\Omega} |u|^{p+1} \, dx \), \( y(t) = \int_{\Omega} u^2 \, dx \) for \( t \in [0, T_{\text{max}}) \).

(a) Prove \( \frac{d}{dt} E(t) = -\int_{\Omega} u_t^2 \, dx \), \( t \in (0, T_{\text{max}}) \).

(b) With \( c = \frac{2(p-1)}{p+1} \), prove \( \frac{d}{dt} y(t) \geq -4E(t) + cy(t)^{\frac{p+1}{2}} \), \( t \in (0, T_{\text{max}}) \).

(c) Assume \( u_0 \) is nontrivial, \( E(0) < 0 \) and prove \( T_{\text{max}}(u_0) < \infty \).

6. Consider the initial-boundary value problem

\[
\begin{align*}
    u_{tt} - u_{xx} &= -2 + \sin x \quad \text{on} \quad (0, \pi) \times (0, \infty), \\
    u &= x^2 - \pi x, \quad u_t = 0 \quad \text{at} \quad t = 0, \\
    u &= 0 \quad \text{at} \quad x = 0, \pi.
\end{align*}
\]

(a) Find the steady state solution \( u = f(x) \) of the differential equation and boundary conditions.

(b) Find the solution of the entire problem.

7. Suppose \( a \in C^0(\mathbb{R}^n), a \geq 1 \) on \( \mathbb{R}^n \) and \( u_0, u_1 \in C^\infty(\mathbb{R}^n) \). Suppose \( u \in C^2(\mathbb{R}^n \times [0, \infty)) \) is a solution of the problem

\[
\begin{align*}
    u_{tt} - \Delta u + a(x)u_t &= 0 \quad \text{on} \quad \mathbb{R}^n \times (0, \infty),
\end{align*}
\]
$u(x, 0) = u_0(x), \ x \in \mathbb{R}^n,$

$u_t(x, 0) = u_1(x), \ x \in \mathbb{R}^n.$

Let $E(t) = \int_\Omega (u_t^2 + |\nabla u|^2)dx, \ K(t) = \int_\Omega (uu_t + \frac{1}{2}au^2)dx, \ t \in [0, \infty).$

(a) Prove $\frac{d}{dt}E \leq 0, \ \frac{d}{dt}(K + E) \leq -E,$ and $K + E \geq 0$ for all $t \geq 0.$ You may assume finite speed of propagation of solutions (the support of $u(\cdot, t)$ is bounded in $\mathbb{R}^n$ for each $t \geq 0$).

(b) Prove $E(t) \leq Ct^{-1}$ for all $t > 0.$ Hint: Integrate an inequality in (a).
1. In the region $R := \{(x, t) : x > 0, t > 0\}$, solve the PDE

$$u_t + t^2 u_x = 4u, \quad \text{with,} \quad u(0, t) = h(t), \quad u(x, 0) = 1.$$ 

Find the conditions on $h$ so that the solution is continuous on $R$.

2. Solve the following PDE (also state the domain of the solution)

$$x^2 u_x + yu_y = u^3, \quad \text{and} \quad u = 1, \quad \text{on the curve} \quad y = x^2.$$ 

3. Let $a > 0$ and $D = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < a^2\}$. Consider the equation

$$\begin{cases} 
\Delta u = 0, & \text{in} \quad D, \\
u = 1 + x^2 + 3xy, & \text{on} \quad \partial D.
\end{cases}$$ 

without solving the equation, find $u(0,0)$, $\max_D u$, and $\min_D u$.

4. Let $B_1 = \{x \in \mathbb{R}^n : |x| < 1\}$ for $n > 2$. Let $u$ be defined on $\overline{B_1 \setminus \{0\}}$. Assume that $u \in C(\overline{B_1 \setminus \{0\}}) \cap C^2(B_1 \setminus \{0\})$, $u$ is harmonic in $B_1 \setminus \{0\}$, and

$$\lim_{|x| \to 0} \frac{u(x)}{|x|^{2-n}} = 0.$$ 

Prove that $u$ can be extended to 0 so that $u \in C^2(B_1)$.

**Hint:** By using the maximum principle on $B_1 \setminus B_r$ for $0 < r < 1$, one proves that $u = v$ in $B_1 \setminus \{0\}$, where $v$ is the solution of the equation

$$\begin{cases} 
\Delta v = 0, & \text{in} \quad B_1, \\
v = u, & \text{on} \quad \partial B_1.
\end{cases}$$

5. Let $\Omega$ be a non-empty, smooth bounded domain in $\mathbb{R}^n$. Let $f : \mathbb{R} \to \mathbb{R}$ be a $C^1$ function such that $|f'|$ is bounded. Consider the reaction-diffusion equation

$$\begin{cases} 
u_t - \Delta u + f(u) = 0, & \text{in} \quad \Omega \times (0, \infty), \\
u = 0, & \text{on} \quad \partial \Omega \times (0, \infty), \\
u(x, 0) = u_0(x), & x \in \Omega.
\end{cases}$$ 

Prove that $C^2$ solutions to the problem are unique.
6. Let \( u_0 \in C^\infty_c(\Omega) \) for some non-empty, open, smooth bounded domain \( \Omega \subset \mathbb{R}^n \) with \( n > 2 \). Assume also that \( u_0 \geq 0 \). Let \( u \in C^\infty(\Omega \times [0,\infty)) \) be a solution of the equation
\[
\begin{cases}
  u_t = \Delta u, & \text{in } \Omega \times (0,\infty), \\
  u(\cdot, t) = 0, & \text{on } \partial \Omega \times (0,\infty), \\
  u(\cdot, 0) = u_0(\cdot), & \text{on } \Omega.
\end{cases}
\]

(a) Prove that for all \( t > 0 \),
\[
\|u(\cdot, t)\|_{L^1(\Omega)} \leq \|u_0\|_{L^1(\Omega)}, \quad \text{and} \quad \|u(\cdot, t)\|_{L^2(\Omega)} \leq \|u_0\|_{L^1(\Omega)}^\alpha \|u(\cdot, t)\|_{L^2(\Omega)}^{1-\alpha},
\]
where
\[
\alpha = \frac{2^* - 2}{2(2^* - 1)}, \quad \text{for} \quad 2^* = \frac{2n}{n - 2}.
\]

(b) Prove that there is \( C > 0 \) depending on \( n, \Omega \) such that
\[
\frac{d}{dt} \int_\Omega u^2(x, t)dx \leq -C \|u_0\|_{L^1(\Omega)}^{-\frac{2\alpha}{1-\alpha}} \left\{ \int_\Omega u^2(x, t)dx \right\}^{\frac{1}{1-\alpha}}.
\]

(c) Prove that (for some new \( C = C(n, \Omega) > 0 \))
\[
\|u(\cdot, t)\|_{L^2(\Omega)} \leq C \|u_0\|_{L^2(\Omega)}(1 + t)^{-\frac{\alpha}{2}}, \quad t \geq 0.
\]

Remark: The following inequalities maybe useful

(i) Hölder’s inequality:
\[
\|f\|_{L^p(\Omega)} \leq \|f\|_{L^{p_1}(\Omega)}^{\theta_1} \|f\|_{L^{p_2}(\Omega)}^{\theta_2},
\]
with
\[
\frac{1}{p} = \frac{\theta_1}{p_1} + \frac{\theta_2}{p_2}, \quad \theta_1 + \theta_2 = 1, \quad p, p_1, p_2 \in (1, \infty), \quad \theta_1, \theta_2 \in (0, 1).
\]

(ii) Sobolev - Poincaré inequality:
\[
\|\varphi\|_{L^2(\Omega)} \leq C(n, \Omega) \|\nabla \varphi\|_{L^2(\Omega)}, \quad \forall \varphi \in C^\infty(\Omega), \quad \varphi|_{\partial \Omega} = 0.
\]

7. Let \( c > 0 \) be a fixed number. Solve the following wave equation
\[
\begin{cases}
  u_{tt} = c^2 u_{xx} + \cos(ct) \cos(x), & -\infty < x < \infty, \quad t > 0, \\
  u(x, 0) = x, \quad u_t(x, 0) = \sin(x), & -\infty < x < \infty.
\end{cases}
\]

8. Let \( u(x, t) \) be a \( C^2 \), compactly supported solution to the equation
\[
u_{tt} - \Delta u = 0, \quad u(x, 0) = 0, \quad u_t(x, 0) = g(x), \quad x \in \mathbb{R}^3, \quad t > 0.
\]
Assume that \( \int_{\mathbb{R}^3} g(x)^2dx < \infty \). Show that
\[
\int_0^\infty u(0, t)^2dt \leq \frac{1}{4\pi} \int_{\mathbb{R}^3} g(x)^2dx.
\]
1. Let $g$ be a given smooth function on $\mathbb{R}$. Solve the PDE
\[
\begin{aligned}
&\begin{cases}
    u_x + u_y = u^2, & \text{on } \{(x,y) \in \mathbb{R}^2, \ y > 0\}, \\
    u(x,0) = g(x), & x \in \mathbb{R}.
\end{cases}
\end{aligned}
\]

2. Let $\Omega \subset \mathbb{R}^n$ be an open bounded domain with smooth boundary $\partial \Omega$ for $n \in \mathbb{N}$. Let $u$ be a harmonic function in $\Omega$ and $x_0 \in \Omega$. Prove that
\[
\left| \frac{\partial u(x_0)}{\partial x_i} \right| \leq \frac{n}{d} \sup_{x \in \Omega} \left| u(x) - u(x_0) \right|, \quad \text{where} \quad d = \text{dist}(x_0, \partial \Omega), \quad \forall \ i = 1, 2, \cdots, n.
\]
Assume in addition that $u \geq 0$ in $\Omega$, show that
\[
\left| \frac{\partial u(x_0)}{\partial x_i} \right| \leq \frac{n}{d} u(x_0), \quad \forall \ i = 1, 2, \cdots, n.
\]

3. Let $\Omega = \mathbb{R}^3 \setminus \overline{B_1(0)}$, where $B_1(0)$ is an open unit ball in $\Omega$. Let $u$ be a harmonic function in $\Omega$ such that $u(x) \to 0$ as $|x| \to \infty$. Prove that there exist $r_0 > 1$ and $M > 0$ such that
\[
|u(x)| \leq \frac{M}{|x|}, \quad |u_{xk}(x)| \leq \frac{M}{|x|^2}, \quad \forall |x| \geq r_0, \quad \forall k = 1, 2, 3.
\]

4. Let $T \in (0, \infty)$ and $\Omega \subset \mathbb{R}^n$ be an open bounded domain with smooth boundary $\partial \Omega$ for $n \in \mathbb{N}$. Let $\Omega_T = \Omega \times (0, T]$ and $u \in C^2(\Omega_T)$ be a solution of the equation
\[
\begin{aligned}
&\begin{cases}
    u_t - \Delta u + c(x,t) u = u^2(1 - u), & \text{in } \Omega_T, \\
    u + \frac{\partial u}{\partial \nu} = 0, & \partial \Omega \times (0, T], \\
    u(x,0) = g(x), & x \in \Omega,
\end{cases}
\end{aligned}
\]
with some given function $c(x,t)$ and $g(x)$. Assume that $c > 0$ on $\Omega_T$ and $0 \leq g \leq 1$ on $\Omega$. Prove that $0 \leq u \leq 1$ on $\Omega_T$.

5. Consider $\Omega = [0, a] \times [0, b] \subset \mathbb{R}^2$ for some fixed $a > 0, b > 0$.
   (a) Use separation of variables to find the first (i.e. the smallest) eigenvalue $\lambda_1$ and eigenfunction $\phi_1$ of the eigenvalue problem
\[
\begin{aligned}
&\begin{cases}
    -\Delta \phi = \lambda_1 \phi, & \Omega, \\
    \phi = 0, & \partial \Omega.
\end{cases}
\end{aligned}
\]
   \textbf{Remark}: Eigenfunctions must be non-trivial.

   (b) Let $g$ be a smooth function on $\overline{\Omega}$ and $g$ vanishes on $\partial \Omega$. Also, let $\kappa < \lambda_1$. Assume that $u$ is a solution of the heat equation
\[
\begin{aligned}
&\begin{cases}
    u_t = \Delta u + \kappa u, & x \in \Omega, \ t > 0, \\
    u(x,t) = 0, & x \in \partial \Omega, \ t > 0, \\
    u(x,0) = g(x), & x \in \Omega.
\end{cases}
\end{aligned}
\]
prove that $u(x, t) \to 0$ uniformly in $x$ as $t \to \infty$. 
6. Let $T \in (0, \infty)$ and $\Omega \subset \mathbb{R}^n$ be an open bounded domain with smooth boundary $\partial \Omega$ for $n \in \mathbb{N}$. Let us denote $\Omega_T = \Omega \times (0, T)$ and $\Gamma_T$ the parabolic boundary of $\Omega_T$. Suppose that $u \in C(\overline{\Omega_T}) \cap C^2(\Omega_T)$ satisfies the PDE

$$u_t - \Delta u = c(x, t)u, \quad (x, t) \in \Omega_T$$

for some $c \in C(\overline{\Omega_T})$ and $c \leq 0$. Show that if $u \geq 0$ on $\Gamma_T$, then

$$\max_{(x, t) \in \overline{\Omega_T}} u(x, t) = \max_{(x, t) \in \Gamma_T} u(x, t).$$

Give a counter example showing that the conclusion does not hold if the condition $u \geq 0$ on $\Gamma_T$ is violated.

7. Let $T \in (0, \infty)$ and $\Omega \subset \mathbb{R}^n$ be an open bounded domain with smooth boundary $\partial \Omega$ for $n \in \mathbb{N}$. Suppose that $u \in C^2(\overline{\Omega} \times [0, T])$ is a classical solution of the equation

$$\begin{cases}
  u_{tt} - \Delta u = f(x, t), & \Omega \times (0, T), \\
  u(x, t) = 0, & (x, t) \in \partial \Omega \times (0, T).
\end{cases}$$

Let

$$E(t) = \frac{1}{2} \int_{\Omega} \left[ u_t^2(x, t) + |\nabla u|^2(x, t) \right] dx.$$

(a) Prove that

$$E(t) \leq e^{T} \left[ E(0) + \frac{1}{2} \int_0^T \int_{\Omega} f^2(x, s) dx ds \right], \quad \forall t \in [0, T].$$

(b) Use the energy estimate to prove the uniqueness of the classical solution of the initial value problem

$$\begin{cases}
  u_{tt} - \Delta u = f(x, t), & \Omega \times (0, T), \\
  u(x, 0) = g(x), & x \in \Omega, \\
  u_t(x, 0) = h(x), & x \in \Omega.
\end{cases}$$

8. Let $f \in C^1(\mathbb{R}^3)$ with compact support. Suppose that $u \in C^2(\mathbb{R}^3 \times (0, \infty))$ and $u$ solves the Cauchy problem

$$\begin{cases}
  u_{tt} - \Delta u = 0, & \mathbb{R}^3 \times (0, \infty), \\
  u(x, 0) = 0, & x \in \mathbb{R}^3, \\
  u_t(x, 0) = f(x), & x \in \mathbb{R}^3.
\end{cases}$$

Prove that there is $M > 0$ such that

$$|u(x, t)| \leq \frac{M}{1 + t} \left[ \|f\|_{L^\infty(\mathbb{R}^3)} + \|f\|_{L^1(\mathbb{R}^3)} + \|\nabla f\|_{L^1(\mathbb{R}^3)} \right], \quad \forall t \geq 0.$$
1.) (a) Solve the following Cauchy problem on $\mathbb{R}^2$:

\[
\begin{aligned}
& u_x + u_y = x + y \\
& u = x^3 \text{ on the line } y = -x.
\end{aligned}
\]

(b) For what $C^1$ function or functions $f(x)$ does the Cauchy problem on $\mathbb{R}^2$:

\[
\begin{aligned}
& u_x + u_y = 3u \\
& u = f(x) \text{ on the line } y = x
\end{aligned}
\]

have a solution? Prove your answer.

2.) Consider Burger’s equation

\[
(\ast)
\begin{aligned}
& uu_x + u_y = 0, \text{ for } x \in \mathbb{R}, y > 0 \\
& u(x, 0) = f(x), \text{ for } x \in \mathbb{R},
\end{aligned}
\]

with initial data

\[
f(x) = \begin{cases} 
4, & \text{for } x < 0, \\
4 - \frac{x}{2}, & \text{for } 0 \leq x \leq 2, \\
3, & \text{for } x > 2.
\end{cases}
\]

(a) Find, with proof, the smallest $y^* > 0$ such that a shock occurs at $(x, y^*)$ for some $x \in \mathbb{R}$.

(b) Find $u(x, y)$ satisfying $(\ast)$ for $x \in \mathbb{R}$ and $0 \leq y < y^*$, except on two line segments where the partial derivatives of $u$ may not exist.

(c) Find the integral, or weak, solution $u(x, y)$ of $(\ast)$ for $y \geq 0$.

3.) (a) Suppose $f \in C^\infty(\mathbb{R}^n)$ satisfies $f(x) > 0$ for all $x \in \mathbb{R}^n$. Suppose $u \in C^2(\mathbb{R}^n)$ satisfies

\[
\Delta u - f(x)u = 0
\]

on $\mathbb{R}^n$, and $u(x) \to 0$ uniformly as $|x| \to \infty$. Prove that $u$ is identically 0.

(b) Find a non-trivial solution of $\Delta u + u = 0$ in $\mathbb{R}^3$ such that $u(x) \to 0$ uniformly as $|x| \to \infty$. Hint: look for a radial solution $u(x, y, z) = v(r)$ where $r = \sqrt{x^2 + y^2 + z^2}$ and note that $rv'' + 2v' = (rv)''$. 
4.) Let $\Omega \subseteq \mathbb{R}^n$ be a bounded open set. Suppose that $\{u_n\}_{n=1}^\infty$ is a sequence of harmonic functions on $\Omega$ such that

$$
\int_{\Omega} |u_n(x) - u_m(x)|^2 \, dx \rightarrow 0
$$

as $\max\{n, m\} \rightarrow \infty$. Prove that $u_n$ converges to a harmonic function on $\Omega$.

5.) Suppose $u = u(x, t) \in C^2([0, 1] \times [0, T])$ satisfies

$$
\begin{cases}
  u_t = u_{xx} + tu_x, & x \in [0, 1], t \in [0, T] \\
  u_x(0, t) = u_x(1, t) = 0, & t \in [0, T].
\end{cases}
$$

Prove that

$$
\max_{[0,1] \times [0,T]} u(x, t) = \max_{[0,1]} u(x, 0).
$$

If you use a major theorem in PDE in your solution, provide the proof of that theorem.

6.) (a) Suppose $u = u(x, t) \in \mathcal{C}(\mathbb{R}^n \times [0, \infty)) \cap \mathcal{C}^2(\mathbb{R}^n \times (0, \infty))$ satisfies

$$
\begin{cases}
  u_t = \Delta u, & \text{for } x \in \mathbb{R}^n, t > 0, \\
  u(x, 0) = f(x), & \text{for } x \in \mathbb{R}^n,
\end{cases}
$$

where $f(x) \geq 0$ is a $C^\infty$, bounded function satisfying $\int_{\mathbb{R}^n} f(x) \, dx = 2$. Suppose $u$ satisfies

$$
|u(x, t)| \leq Ae^{\alpha|x|^2},
$$

for some positive constants $\alpha$ and $A$. Prove that $\lim_{t \rightarrow \infty} u(x, t) = 0$ and $\int_{\mathbb{R}^n} u(x, t) \, dx = 2$ for all $t > 0$.

(b) Does there exist a bounded solution $u(x, t) \in C(\mathbb{R}^n \times [0, \infty)) \cap \mathcal{C}^2(\mathbb{R}^n \times (0, \infty))$ of the initial value problem

$$
\begin{cases}
  u_t = \Delta u + \frac{\cos(|x|^2+1)}{1+|x|^2}, & \text{for } x \in \mathbb{R}^n, t > 0, \\
  u(x, 0) = 0, & \text{for } x \in \mathbb{R}^n?
\end{cases}
$$

Justify your answer.
7.) Suppose $u = u(x, t) \in C^2(\mathbb{R} \times [0, \infty))$ satisfies
\[
\begin{align*}
    u_{tt} - u_{xx} + u &= 0, \quad \text{for } x \in \mathbb{R}, t > 0, \\
    u(x, 0) &= f(x), \quad \text{for } x \in \mathbb{R}, \\
    u_t(x, 0) &= g(x), \quad \text{for } x \in \mathbb{R},
\end{align*}
\]
where $f$ and $g$ are $C^\infty$ and have compact support.

(a) For any $(x_0, t_0) \in \mathbb{R} \times (0, \infty)$ and $0 \leq t \leq t_0$, let $I(t)$ be the interval
\[
I(t) = [x_0 - t_0 + t, x_0 + t_0 - t].
\]
Define
\[
e(t) = \int_{I(t)} [u^2 + u_t^2 + u_x^2](x, t) \, dx,
\]
for $0 \leq t \leq t_0$. Prove that $e$ is non-increasing on $[0, t_0]$.

(b) Suppose that $f(x) = 0$ and $g(x) = 0$ for $|x| \geq 1$. Prove that $u(x, t) = 0$ for $|x| > t + 1$, for all $t > 0$.

8.) Suppose $u = u(x, t) \in C^2(\mathbb{R} \times [0, \infty))$, is the solution of the wave equation
\[
\begin{align*}
    u_{tt} &= \Delta u, \quad x \in \mathbb{R}, t > 0 \\
    u(x, 0) &= f(x), \quad x \in \mathbb{R}, \\
    u_t(x, 0) &= g(x), \quad x \in \mathbb{R}.
\end{align*}
\]
Suppose $g$ and $h$ are $C^\infty$ with $f(x) = g(x) = 0$ for all $x$ such that $|x| \geq R$, for some $R > 0$. The kinetic energy is
\[
k(t) = \frac{1}{2} \int_{\mathbb{R}} u_t^2(x, t) \, dx
\]
and the potential energy is
\[
p(t) = \frac{1}{2} \int_{\mathbb{R}} u_x^2(x, t) \, dx.
\]

(a) Prove that $k(t) + p(t)$ is constant.

(b) Prove that $k(t) = p(t)$ for all $t > R$. 
1.) Consider the equation
   \[ (*) \quad u_x + 2u_y = u, \]
   for \((x, y) \in \mathbb{R}^2\).
   
   (a) Solve \((*)\) with the Cauchy data \(u(x, x) = e^{3x}\) for all \(x \in \mathbb{R}\).

   (b) Suppose \(u\) satisfies \((*)\) with Cauchy data \(u(x, 2x) = f(x)\). Prove that \(f(x) = Ce^x\) for some constant \(C\).

   (c) For each constant \(C \neq 0\), show that \((*)\) with Cauchy data \(u(x, 2x) = Ce^x\) has infinitely many solutions.

2.) Reduce the following equation on \(\mathbb{R}^2\):
   \[ u_{xx} + 6x^2u_{xy} + 9x^4u_{yy} + 6xu_y + y - x^3 = 0 \]
   to canonical form and find the general solution.

3.) Let \(\Omega \subseteq \mathbb{R}^n\) be a smooth \((C^\infty)\), bounded open set. Consider the problem
   \[ (***) \quad \begin{cases} \Delta u(x) = f(x), & \text{for } x \in \Omega \\ u(x) + \frac{\partial u}{\partial n} = g(x), & \text{for } x \in \partial \Omega. \end{cases} \]
   where \(f \in C(\Omega), g \in C(\partial \Omega),\) and \(\frac{\partial}{\partial n}\) is the outward normal derivative on \(\partial \Omega\).

   (a) Prove that there is at most one \(u \in C^2(\overline{\Omega})\) satisfying \((***)\).

   (b) Suppose \(u \in C^2(\overline{\Omega})\) satisfies \((***)\), with \(f \geq 0\) on \(\Omega\) and \(g \leq 0\) on \(\partial \Omega\). Prove that \(u \leq 0\) on \(\Omega\).

4.) Suppose \(u = u(x, t) \in C([0, 1] \times [0, \infty)) \cap C^2((0, 1) \times (0, \infty))\), and \(u\) satisfies
   \[ \begin{cases} u_t = u_{xx}, & \text{for } 0 < x < 1, t > 0, \\ u(0, t) = u(1, t) = 0, & \text{for } t \geq 0, \\ u(x, 0) = 4x(1-x), & \text{for } 0 \leq x \leq 1. \end{cases} \]
   Prove that
   (a) \(0 < u(x, t) < 1\) for \(0 < x < 1, t > 0\);
   (b) \(u(1-x, t) = u(x, t)\) for \(0 \leq x \leq 1, t > 0\);
   (c) \(-8 < u_{xx}(x, t) < 0\) for \(0 < x < 1, t > 0\);
   (d) \(\int_0^1 u^2(x, t) \, dx\) is a strictly decreasing function of \(t\).
5.) Suppose $u = u(x, t) \in C^2([0, 1] \times [0, \infty))$ satisfies
\[
\begin{aligned}
&\quad u_{tt} - u_{xx} = -\frac{u}{1 + u^2}, \quad \text{for } 0 < x < 1, t > 0 \\
u(0, t) = u(1, t) = 0, \quad &\text{for } t \geq 0, \\
u(x, 0) = g(x), \quad &\text{for } 0 \leq x \leq 1,
\end{aligned}
\]
where $g$ is a given function satisfying $g(0) = g(1) = 0$.

(a) Define
\[
E(t) = \frac{1}{2} \int_0^1 u_t^2 + u_x^2 + \log(1 + u^2) \, dx,
\]
for $t \geq 0$. Prove that $E$ is constant.

(b) Show that there exists $C > 0$ such that $|u(x, t)| \leq C$ for all $x \in [0, 1]$ and $t \geq 0$.

6.) Let $\Omega \subseteq \mathbb{R}^n$ be an open set.

(a) Suppose $u \in C^1(\overline{\Omega})$ and
\[
\int_{\partial B(x, r)} \frac{\partial u}{\partial n} \, dS \geq 0
\]
for every $x \in \mathbb{R}^n$ and $r > 0$ such that $B(x, r) \subseteq \Omega$, where $\frac{\partial}{\partial n}$ is the outward normal derivative on $\partial \Omega$ and $dS$ is surface measure on $\partial \Omega$. Prove that $u$ is subharmonic on $\Omega$. Warning: a subharmonic function is not necessarily $C^2$.

(b) Prove the converse of part (a) under the additional assumption that $u \in C^2(\overline{\Omega})$.

7.) Let $\Omega \subseteq \mathbb{R}^n$ be a smooth bounded open set. Let $h \leq 0$ be a continuous function on $\overline{\Omega} \times [0, \infty)$. Prove that there exists at most one function $u = u(x, t) \in C^2(\overline{\Omega} \times [0, \infty))$ satisfying
\[
\begin{aligned}
u_t &= \Delta u + h(x, t)u, \quad \text{for } x \in \Omega, t \geq 0 \\
u(x, 0) &= f(x), \quad \text{for } x \in \Omega, \\
u(x, t) &= g(x, t), \quad \text{for } x \in \partial \Omega, t \geq 0.
\end{aligned}
\]

8.) Suppose $u = u(x, t) \in C^2(\mathbb{R}^3 \times [0, \infty))$, is the solution of the wave equation
\[
\begin{aligned}
u_{tt} &= \Delta u, \quad \text{for } x \in \mathbb{R}^3, t > 0 \\
u(x, 0) &= 0, \quad \text{for } x \in \mathbb{R}^3, \\
u_t(x, 0) &= g(x), \quad \text{for } x \in \mathbb{R}^3.
\end{aligned}
\]
Suppose $g(x) = 1$ for $|x| > 1$. Prove that
\[
u(x, t) = t
\]
if (i) $|x| > t + 1$ or (ii) $|x| < t - 1$. 
Problem 1. Let \( f : \mathbb{R}^n \to \mathbb{R} \) be a bounded \( C^2 \) function that satisfies
\[
\nabla f = G,
\]
where \( G : \mathbb{R}^n \to \mathbb{R}^n \) satisfies
\[
\int_{\partial B_r(x_0)} G(x) \cdot (x - x_0) dA(x) = 0,
\]
for all \( x_0 \in \mathbb{R}^n \), \( r > 0 \). Prove that \( f \) is constant.

Problem 2. Let \( \Omega = \{(x,t) : 0 < x < 1, 0 < t < \infty \} \). Assume that \( u \in C^{2,1}(\Omega) \cap C^0(\overline{\Omega}) \) satisfies the initial boundary value problem given by the equation
\[
\frac{\partial u}{\partial t}(x,t) = \frac{\partial^2 u}{\partial x^2}(x,t)
\]
in the interior of the region \( \Omega \), together with the boundary conditions
\[
u(x,0) = f(x), \ u(0,t) = \alpha(t), \ u(1,t) = \beta(t),
\]
where \( f(0) = \alpha(0), \ f(1) = \beta(0). \)

(a) Show that \( u(x,t) \) cannot have a maximum where \( \frac{\partial^2 u}{\partial x^2} < 0 \) in the interior of the region in \((x,t)\) space with \( t > 0 \) and \( 0 < x < 1 \).

(b) State the strong maximum/minimum principle for the previous IVBP.

(c) Using a maximum/minimum principle show that if \( f(x) \geq 0, \ \alpha(t) \geq 0, \) and \( \beta(t) \geq 0, \) then \( u(x,t) \geq 0. \)

Problem 3. Suppose \( u : \mathbb{R}^2 \to \mathbb{R} \) is \( C^1 \) and bounded and satisfies the PDE
\[
u(x,y) = a(x,y)u_x(x,y) + b(x,y)u_y(x,y).
\]

(a) Show that if \( a \) and \( b \) are constant functions, then \( u \) is identically 0.

(b) Prove that if \( a = 1 + x^2 \) and \( b = 1 + y^2 \), the above PDE has non-vanishing bounded solutions.

Problem 4. Consider the cube \( \Omega = (1, 2) \times (1, 2) \times (1, 2) \). Suppose \( u \in C^2(\Omega) \cap C^0(\overline{\Omega}) \) satisfies
\[
y u_{xx} + x u_{yy} + x u_{xx} = 1
\]
in \( \Omega \), with \( u = 0 \) on the boundary \( \partial \Omega \). Prove that \( u \geq -\frac{1}{8} \).

Hint. Compare with a function of the type \( v(\vec{x}) = a + b|\vec{x} - \vec{x}_0|^2 \), where \( a, b \in \mathbb{R}, \ \vec{x}_0 \in \mathbb{R}^3 \).
Problem 5. Consider the unbounded domain $\Omega = \{(x, y) : y > x^2\} \subset \mathbb{R}^2$. Suppose $u$ is bounded and harmonic on $\Omega$, and vanishes on $\partial \Omega$. Show $u \equiv 0$.

*Hint.* Test with $u\chi$, where $\chi(y)$ is a cutoff function in the second variable $y$, and is nonconstant only on $y \in [\ell, 2\ell]$.

Problem 6. Suppose $u \in C^2(\mathbb{R}^3 \times [0, \infty))$ is a solution of

$$
\begin{aligned}
& u_{tt} - \Delta u = 0 \quad \text{on } \mathbb{R}^3 \times [0, \infty), \\
& u(x, 0) = 0 \quad x \in \mathbb{R}^3, \\
& u_t(x, 0) = \psi(x) \quad x \in \mathbb{R}^3,
\end{aligned}
$$

where $\psi \in C^\infty(\mathbb{R}^3)$ has compact support. Let $p \in [2, \infty)$. Prove that there exists $C > 0$ such that:

(a) $|\nabla u(x, t)| \leq C(1 + t)^{-1}$ for all $(x, t) \in \mathbb{R}^3 \times [0, \infty)$,

(b) $\int_{\mathbb{R}^3} |\nabla u(x, t)|^p dx \leq C(1 + t)^{2-p}$ for all $t \geq 0$.

Problem 7. Suppose $u \in C^2(\mathbb{R}^n \times [0, \infty))$ is a solution of

$$
\begin{aligned}
& u_{tt} - \Delta u = 0 \quad \text{on } \mathbb{R}^n \times [0, \infty), \\
& u(x, 0) = \phi(x) \quad x \in \mathbb{R}^n, \\
& u_t(x, 0) = \psi(x) \quad x \in \mathbb{R}^n,
\end{aligned}
$$

where $\phi, \psi \in C^\infty(\mathbb{R}^n)$ have compact support. Prove that there exists $C, T > 0$ such that

$$
\int_{\mathbb{R}^n} \frac{(|u_t| + |\nabla u|)^4}{1 + |x| + t} \, dx \geq C t^{n-1}
$$

for all $t \geq T$. 
SOLUTIONS

Q1. $G$ is $C^1$ since $f$ is $C^2$. Using the integral condition and the divergence theorem, we obtain that $\int_B G \cdot n dA = \int_B \text{div} G = 0$ on any ball $B$. Since $G$ is $C^1$ it follows that $\text{div} G = 0$ everywhere. Taking the divergence of the first equation we obtain $\text{div} \nabla f = \Delta f = \text{div} G = 0$, i.e. $f$ is harmonic. Since $f$ is also bounded, it must be constant.

Q2. Will type it soon.

Q3. Along the characteristic curves $\dot{x} = a$, $\dot{y} = b$, the solution $u$ satisfies the equation $\dot{z} = z$, hence $z(t) = z(0)e^t$. For $t \in \mathbb{R}$, this is bounded exactly if $z(0) = 0$. The reasoning with $t \in \mathbb{R}$ applies for $a, b$ constant functions, because then the characteristic curves do exist for all $t$, namely $x(t) = x_0 + at$, $y(t) = y_0 + bt$. [The same reasoning would apply for any locally Lipschitz functions $a(\cdot, \cdot)$, $b(\cdot, \cdot)$ that satisfy (eg) linear bounds $|a(x, y)| + |b(x, y)| \leq C_0(|x| + |y|)$, by some ODE theory that we may not assume known in this generality, and which would guarantee global existence in time for the characteristic curves.]

In contrast, for $\dot{x} = 1 + x^2$, $\dot{y} = 1 + y^2$, we cover the plane with characteristic curves $x(t) = \tan(t + c_0) = \tan(t + \arctan x_0)$, $y(t) = \tan(t + c_1) = \tan(t + \arctan y_0)$ that exist for an interval of finite length $\leq \pi$ only. We do not need $z(0) = 0$ for $z(t) = z(0)e^t$ to be bounded on this interval. Specifically, we can choose initial data $x(0) = s$, $y(0) = -s$, $z(0) = f(s)$ for any bounded function $f$. Then

$$u(t \arctan s, t - \arctan s) = f(s)e^t$$

i.e.,

$$u(x, y) = \exp \left[ \frac{1}{2} \left( \arctan x + \arctan y \right) \right] \int \frac{1}{2} \left( \arctan x - \arctan y \right)$$

Q4. We consider $v(x, y, z) := M + \frac{1}{6} ((x - \frac{1}{2})^2 + (y - \frac{1}{2})^2 + (z - \frac{1}{2})^2)$ where $M$ is yet to be determined. (It will turn out that we want $M = -\frac{1}{6}$.) We want to show, by maximum principle, that $w := u - v \geq 0$.

First we note that on $\Omega$, it holds

$$yw_{xx} + zw_{yy} + xw_{zz} = \frac{3}{2}(x + y + z) > 1.$$  
Therefore $yw_{xx} + zw_{yy} + xw_{zz} < 0$ in $\Omega$. Now $w$ does have a minimum on the compact $\Omega$. If the minimum were in the interior, we'd have $w_{xx} \geq 0$, $w_{yy} \geq 0$, $w_{zz} \geq 0$ there, and thus $yw_{xx} + zw_{yy} + xw_{zz} \geq 0$ in violation of the DE. So min $w$ is taken on at the boundary, where it equals $-\max v = -M - \frac{1}{6} ((\frac{1}{2})^2 + (\frac{1}{2})^2 + (\frac{1}{2})^2) = -M - \frac{1}{6}$, which equals 0 for our choice $M = -\frac{1}{6}$.

So we have $w \geq 0$, i.e., $u \geq v \geq M = -\frac{1}{6}$ on $\Omega$.

Q5. We can design $\chi$ in such a way that $\chi(y) = 1$ for $y \leq \ell$, $\chi(y) = 0$ for $y \geq 2\ell$, $|x'| \leq c/\ell$, $|x''| \leq c/\ell^2$.

Then

$$0 = \int_{\Omega} \Delta u (ux) = - \int_{\Omega} \nabla u \cdot (\nabla (ux)) = - \int_{\Omega} |\nabla u|^2 \chi - \frac{1}{2} \int_{\partial \Omega} \nabla u \cdot \nabla \chi$$

$$= - \int_{\Omega} |\nabla u|^2 \chi + \frac{1}{2} \int_{\Omega} u^2 \Delta \chi - \frac{1}{2} \int_{\partial \Omega} u^2 \partial u \chi dS.$$
The boundary term vanishes; the second term, with $u$ bounded by $M$, can be estimated by $M^2 (c/\ell^2)(c\ell^{3/2})$, hence it goes to 0 as $\ell \to \infty$. Hence we find, in this limit, that $0 = -\int_{\Omega} |\nabla u|^2$, and $u \equiv \text{const.}$ By DBC, $u \equiv 0$.

Q6 & Q7. See Henry's sheet.
In the following, unless otherwise stated, $\Omega \subset \mathbb{R}^n$ is an open, bounded set with $C^\infty$-smooth boundary $\partial \Omega$. Denote $\Omega_T = \Omega \times (0,T]$, $\Gamma_T = \text{parabolic boundary of } \Omega_T = \overline{\Omega_T} \setminus \Omega_T$.

**Problem 1.** Let $Q = \{(x,y) \in \mathbb{R}^2 : x > 0, y \geq 0\}$. Find the solution $u \in C^1(\Omega)$ of the initial-value problem

$$-2xu_x + (x + y)u_y = 0, \quad (x,y) \in Q,$$

$$u(x,0) = x, \quad x > 0.$$

**Problem 2.** Let $\Omega = \{x \in \mathbb{R}^3 : 0 < |x| < 1\}$, $S = \{x \in \mathbb{R}^3 : |x| = 1\}$. Suppose $u \in C^2(\Omega) \cap C^0(\Omega \cup S)$ satisfies $\Delta u \geq 0$ on $\Omega$, $u = 0$ on $S$ and $u$ is bounded on $\Omega$. Prove $u \leq 0$ on $\Omega$.

Hint: Consider $v(x) = u(x) - \epsilon(1/|x| - 1)$ on an appropriate subdomain of $\Omega$.

**Problem 3.** Suppose $\alpha \in \mathbb{R}, T > 0$ and $f \in C^0(\Omega)$ with $f > 0$ on $\Omega$. Let $u \in C^{2,1}(\Omega_T) \cap C^0(\overline{\Omega_T})$ be a solution of

$$u_t = \Delta u + f(x) + \alpha u \quad \text{on } \Omega_T,$$

$$u = 0 \quad \text{on } \Gamma_T.$$

Prove $u \geq 0$ and $u_t \geq 0$ on $\Omega \times [0,T]$.

**Problem 4.** Let $a, b \in \mathbb{R}, T > 0$. Suppose $\phi, \psi \in C^\infty(\overline{\Omega})$ and $u \in C^2(\Omega_T) \cap C^0(\overline{\Omega_T})$ is a solution of

$$u_{tt} - \Delta u + au_{x_1} + bu = 0 \quad \text{on } \Omega_T,$$

$$u = 0 \quad \text{on } \partial \Omega \times (0,T],$$

$$u = \phi \quad \text{on } \Omega \times \{t = 0\},$$

$$u_t = \psi \quad \text{on } \Omega \times \{t = 0\}.$$

Denoting the energy $E(t) = \frac{1}{2} \int_{\Omega} (u_t^2 + |\nabla u|^2) dx$, prove $E(t) \leq E(0)e^{kt}$ for all $t \in [0,T]$, for some constant $k > 0$. Here $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. 

1
Problem 5. Let \( Q = \{(x, t) : x > 0, t > 0\} \). Find the solution \( u \in C^2(Q) \cap C^1(\overline{Q}) \) of

\[
\begin{align*}
  u_{tt} - u_{xx} &= 0, \quad (x, t) \in Q, \\
  u(x, 0) &= x, \quad x > 0, \\
  u_t(x, 0) &= -1, \quad x > 0, \\
  u_x(0, t) + fu(0, t) &= 1, \quad t > 0.
\end{align*}
\]

Problem 6. Consider the heat equation

\[
u_t = \Delta u \quad \text{on} \quad \Omega_T
\]

and define \( E(t) = \int_\Omega u(x, t)^2 dx, t \in [0, T] \). With Dirichlet boundary conditions \( u = 0 \) on \( \partial \Omega \times (0, T] \), in order to prove backward uniqueness of solutions, it is sufficient to establish \( E^2(t) \leq EE' \) on \([0, T]\). Prove the same inequality for Robin boundary conditions \( \partial u / \partial n = g(x)u \) on \( \partial \Omega \times (0, T], g \in C^0(\partial \Omega) \).

Problem 7. Let \( G(x, y) \) be the Green's function for \(-\Delta\) on \( \Omega \) with Dirichlet boundary conditions. Define \( g(x) = \int_\Omega G(x, y)dy, x \in \overline{\Omega} \). Suppose \( u \in C^2(\Omega) \cap C^0(\overline{\Omega}) \) is a solution of

\[
-\Delta u = e^{-u} \quad \text{on} \quad \Omega, \\
\]

\( u = 0 \) on \( \partial \Omega \).

(a) Find \(-\Delta g\).

(b) Prove there exists a constant \( m > 0 \) such that \( mg \leq u \leq g \) on \( \Omega \). Express \( m \) in some explicit form involving \( g \).
In the following $\Omega \subset \mathbb{R}^n$ is an open, bounded set with $C^\infty$-smooth boundary $\partial \Omega$. Denote $\Omega_T = \Omega \times (0, T]$, $\Gamma_T = \text{parabolic boundary of } \Omega_T = \overline{\Omega} \setminus \Omega_T$.

**Problem 1.** Find all positive solutions $u$ defined on all of $\mathbb{R}^2$ to the equation
\[ xu_x + yu_y = \frac{(x^2 + y^2)}{u}. \]

**Problem 2.** Suppose $f \in C^0(\partial \Omega)$, $f \geq 0$ on $\partial \Omega$. Show that if a solution $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ to the boundary-value problem
\[ -\Delta u = \frac{1}{1 + u^2} \quad \text{on } \Omega, \]
\[ u = f \quad \text{on } \partial \Omega, \]
exists, then it is unique.

**Problem 3.** Suppose $u \in C^2(\mathbb{R}^3 \times [0, \infty))$ is a solution of
\[ u_{tt} - \Delta u = 0 \quad \text{on } \mathbb{R}^3 \times [0, \infty), \]
\[ u(x, 0) = 0, \quad x \in \mathbb{R}^3, \]
\[ u_t(x, 0) = g(x), \quad x \in \mathbb{R}^3, \]
where $g \in C^2(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$. Prove that there exists $C > 0$ such that
\[ \sup_{x \in \mathbb{R}^3} \int_0^\infty u(x, t)^2 \, dt \leq C \|g\|_{L^2(\mathbb{R}^3)}^2. \]

**Problem 4.** Let $T > 0$ and suppose $f \in C^1(\mathbb{R})$, $f(0) = 0$. Consider the problem
\[ u_t = \Delta u + f(u) \quad \text{on } \Omega_T, \]
\[ u = 0 \quad \text{on } \Gamma_T. \]
Prove this has a solution $u \in C^{2,1}(\Omega_T) \cap C^0(\overline{\Omega_T})$ and that the solution is unique.
Problem 5. Let $\Omega = (0, \pi), Q = \Omega \times (0, \infty), f \in C^0([0, \pi]), f(0) = f(\pi) = 0$. 
Prove the problem
\[
\begin{align*}
  u_t &= u_{xx} + u^2 & \text{on } Q, \\
  u &= 0 & \text{on } \partial\Omega \times (0, \infty), \\
  u &= f & \text{on } \Omega \times \{t = 0\},
\end{align*}
\]
has no solution $u \in C^{2,1}(Q) \cap C^0(\overline{Q})$ if $I = \int_0^\pi f(x) \sin x \, dx$ is sufficiently large and positive. 
Hint: Derive a differential inequality for $F(t) = \int_0^\pi u(x, t) \sin x \, dx$ and obtain a contradiction.

Problem 6. Suppose $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ is a solution of
\[
\Delta u = u^3 - u \quad \text{on } \Omega,
\]
\[
  u = 0 \quad \text{on } \partial\Omega.
\]
Prove 
(a) $-1 \leq u \leq 1$ on $\Omega$, 
(b) $|u(x)| \neq 1$ for all $x \in \Omega$.

Problem 7. Let $T > 0, 1 < p \leq m$. Suppose $\phi, \psi \in C^\infty(\overline{\Omega})$ and $u \in C^2(\Omega_T) \cap C^0(\overline{\Omega}_T)$ is a solution of
\[
\begin{align*}
  u_{tt} - \Delta u + u_t |u_t|^{m-1} &= u|u|^{p-1} & \text{on } \Omega_T, \\
  u &= 0 & \text{on } \partial\Omega \times (0, T], \\
  u &= \phi & \text{on } \Omega \times \{t = 0\}, \\
  u_t &= \psi & \text{on } \Omega \times \{t = 0\}.
\end{align*}
\]
Denote $H(t) = \frac{1}{2} \|u_t(\cdot, t)\|_{L^2(\Omega)}^2 + \frac{1}{2} \|\nabla u(\cdot, t)\|_{L^2(\Omega)}^2 + \frac{1}{p+1} \|u(\cdot, t)\|_{L^{p+1}(\Omega)}^{p+1}$, 
t $\in [0, T]$ ($H$ is not the energy for the p.d.e.). Prove that for some constant $c > 0, H(t) \leq H(0)e^{ct}$ for all $t \in [0, T]$.
Hint: Calculate $\dot{H}(t)$. 

2
Problem 1:
Prove that every positive harmonic function in all of $\mathbb{R}^n$ is a constant. Conclude that every semi-bounded harmonic function in all of $\mathbb{R}^n$ is a constant.

Problem 2:
Show that the damped Burger's equation $u_t + uu_x = -u$, for $x \in \mathbb{R}$, $t \geq 0$, with initial data $u(x, 0) = \phi(x)$ (for a positive $C^1$ function $\phi$) has a global solution for $t \geq 0$, provided $\phi'(x) > -1$.

Problem 3:
Let $Q = \mathbb{R}^n \times (0, \infty)$, $f \in L^1(\mathbb{R}^n)$, and let $u \in C^{2,1}(Q) \cap C^0(\bar{Q})$ be the solution of the problem

$$
\begin{align*}
     u_t - \Delta u + u &= 0 & \text{for } t > 0, x \in \mathbb{R}^n \\
     u(x, 0) &= f(x) & \text{for } x \in \mathbb{R}^n.
\end{align*}
$$

subject to the growth condition $|u(x, t)| \leq A e^{\alpha x^2}$ for $x \in \mathbb{R}^n$ and $t \geq 0$, with certain positive constants $A, \alpha$. Show that

$$
\|u(\cdot, t)\|_{L^\infty(\mathbb{R}^n)} \leq C t^{-n/2} e^{-t} \|f\|_{L^1(\mathbb{R}^n)}
$$

for all $t > 0$.

Problem 4:
Let $Q = \mathbb{R}^n \times (0, \infty)$, $f \in L^1(\mathbb{R}^n)$, and $g \in C^0[0, \infty) \cap L^1(0, \infty)$. Assume that $\lim_{t \to \infty} g(t)$ exists. Suppose $u \in C^{2,1}(Q) \cap C^0(\bar{Q})$ satisfies

$$
\begin{align*}
     u_t - \Delta u &= g(t) & \text{on } Q \\
     u &= f & \text{on } \mathbb{R}^n \times \{t = 0\}
\end{align*}
$$

and that the usual growth condition that implies uniqueness is satisfied. Show

$$
\lim_{t \to \infty} u(x, t) = \int_0^\infty g(t) \, dt \text{ and } \lim_{t \to \infty} u_t(x, t) = 0
$$

for each $x \in \mathbb{R}^n$. 

1
Problem 5:
Assume in a bounded domain $\Omega \subset \mathbb{R}^n$, we have a solution $u \in C^0(\overline{\Omega}) \cap C^2(\Omega)$ to $\Delta u = u^3 - 1$ and a solution $v$ to $\Delta v = v - 1$, each vanishing at the boundary. Show that $0 < v \leq u \leq 1$ in $\Omega$.

Problem 6:
Let $g \in C^2(\mathbb{R}^3)$ satisfy the conditions
\[ |g(x)| < C \quad \text{and} \quad \int_{\mathbb{R}^3} |\nabla g(x)| \, dx < 4\pi C \quad \text{and} \quad \lim_{|x| \to \infty} g(x) = 0 \]
and consider a classical solution $u$ to the wave equation
\[
\begin{align*}
\partial_{tt} - \Delta u &= 0 & \text{in } \mathbb{R}^3 \times (0, \infty) \\
u(x, 0) &= C & \text{for } x \in \mathbb{R}^3 \\
u_t(x, 0) &= g(x) & \text{for } x \in \mathbb{R}^3.
\end{align*}
\]
where $C$ is a given positive constant. Prove that $u(x, t) > 0$ for all $(x, t) \in \mathbb{R}^3 \times [0, \infty)$.

Problem 7:
Suppose $\phi \in C^\infty(\mathbb{R}^n)$ and $\psi \in C^\infty(\mathbb{R}^n)$ have support contained in the ball $B(0, r)$, and that $u \in C^2(\mathbb{R}^n \times [0, \infty))$ is a solution to
\[
\begin{align*}
\partial_{tt} - \Delta u + \frac{1}{1 + |x|}|u_t| &= 0 & \text{on } \mathbb{R}^n \times (0, \infty) \\
u(x, 0) &= \phi(x) & \text{for } x \in \mathbb{R}^n \\
u_t(x, 0) &= \psi(x) & \text{for } x \in \mathbb{R}^n
\end{align*}
\]
Define $E(t) := \frac{1}{2} \int_{\mathbb{R}^n} (u_t^2 + |\nabla u|^2) \, dx$ and $I(t) := \int_0^t \int_{\mathbb{R}^n} \frac{1}{1 + |x|} (u_t^2 + |\nabla u|^2) \, dx \, ds$.
(a) Prove that $\int_t^{\infty} \int_{\mathbb{R}^n} \frac{1}{1 + |x|} |u_s^2| \, dx \, ds \leq E(t)$.
(b) For your information: it can be proved that $I(t) \leq CE(t)$. You do not need to do this; only be assured of the corollary that $I(t)$ is finite.

(b) Prove that there exists a positive constant $C$ such that $I(t) \geq CE(2t)$ for all $t \geq r$ (with the $r$ from the support of the data). Hints: $I(t) \geq \int_t^{2t} \ldots$. You may assume that the support of $u$ has the same properties as solutions to the wave equation whose initial data have support in $B(0, r)$. And you may assume that $E(t)$ is non-increasing in $t$. 

2
In the following $\Omega \subset \mathbb{R}^n$ is an open, bounded set with $C^\infty$- smooth boundary $\partial \Omega$. Denote $\Omega_T = \Omega \times (0, T]$.

**Problem 1.** Prove the pde $u_x + 2xu_y = (y^2 - x^2)u^2 + 1$ cannot have a solution $u \in C^1(\mathbb{R}^2)$ in the entire plane $\mathbb{R}^2$.

**Problem 2.** Let $a \in \mathbb{R}$. Show the problem

$$
\Delta u = u^5 + a \quad \text{on} \quad \Omega,
$$

$$
u = 0 \quad \text{on} \quad \partial \Omega,
$$

has at most one solution $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$.

**Problem 3.** Let $Q = \mathbb{R}^n \times (0, \infty)$ and suppose $u \in C^{2,1}(Q) \cap C^0(\overline{Q})$ is a solution of

$$
u_t - \Delta u = 0 \quad \text{on} \quad Q,
$$

$$
u = g(x) \quad \text{on} \quad \mathbb{R}^n \times \{t = 0\},
$$

satisfying the growth condition

$$
|u(x, t)| \leq Ae^{\alpha|x|^2}, \quad (x, t) \in Q,
$$

where $A, \alpha$ are positive constants.

(a) Assume that $g \in C^0(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n)$ does not depend on a variable $x_j$ for some fixed $j$. Prove that the same is true for $u$.

(b) Prove that if $g \in C^\infty(\mathbb{R}^n)$ is a harmonic function on $\mathbb{R}^n$, the solution $u$ is time independent.

**Problem 4.** Let $\alpha, T > 0, \gamma \in \mathbb{R}$. Suppose $\phi \in C^0(\overline{\Omega})$ and $c \in C^0(\overline{\Omega}_T)$ with $c \geq \gamma$ on $\overline{\Omega}_T$. Suppose $u \in C^{2,1}(\Omega_T) \cap C^1(\overline{\Omega}_T)$ is a solution of

$$
u_t - \Delta u + c(x, t)u = 0 \quad \text{on} \quad \Omega_T,
$$

$$
u = \phi \quad \text{on} \quad \Omega \times \{t = 0\},
$$

$$
\partial u / \partial n + \alpha u = 0 \quad \text{on} \quad \partial \Omega \times (0, T].
$$

Prove $|u| \leq \sup_{\overline{\Omega}} |\phi| e^{-\gamma t}$ on $\Omega_T$ and prove $u$ is unique.
Problem 5. Solve explicitly the initial-boundary value problem
\[ u_{tt} - 4u_{xx} = 0, \quad x > 0, \quad t > 0, \]
with initial data
\[ u(x, 0) = x, \quad x > 0, \]
\[ u_t(x, 0) = -2, \quad x > 0, \]
and boundary condition
\[ u_x(0, t) + tu(0, t) = 1, \quad t > 0. \]

Problem 6. Suppose \( \Omega \subset \mathbb{R}^2 \) and \( u \in C^2(\Omega) \cap C^0(\overline{\Omega}) \) is a solution of
\[
(1 + u_y^2)u_{xx} + (1 + u_x^2)u_{yy} - 2u_xu_yu_{xy} = 0 \quad \text{on} \quad \Omega.
\]
Show \( \inf_{\overline{\Omega}} u = \inf_{\partial\Omega} u \).

Problem 7. Let \( T > 0, a \in \mathbb{R} \). Suppose \( \phi, \psi \in C^\infty(\overline{\Omega}) \) and \( u \in C^2(\Omega_T) \cap C^1(\overline{\Omega_T}) \) is a solution of
\[
u_{tt} - \Delta u + au_t = 0 \quad \text{on} \quad \Omega_T,
\]
\[ u = \phi \quad \text{on} \quad \Omega \times \{t = 0\}, \]
\[ u_t = \psi \quad \text{on} \quad \Omega \times \{t = 0\}, \]
\[ \partial u / \partial n = 0 \quad \text{on} \quad \partial \Omega \times (0, T]. \]
Prove that for \( t \in [0, T] \) the following inequality holds \( E(t) \leq E(0)e^{a_0 t} \), where \( E(t) = \frac{1}{2} \int_{\Omega_T} (u_t^2 + |\nabla u|^2) \, dx \) and \( a_0 = \max\{0, -2a\} \).
In the following $\Omega \subset \mathbb{R}^n$ is an open, bounded set with $C^\infty$-smooth boundary $\partial \Omega$. Denote $\Omega_T = \Omega \times (0,T]$.

**Problem 1.** Suppose $u \in C^1(\mathbb{R}^2)$ is a solution of $yu_x - xu_y = u$ on the entire plane $\mathbb{R}^2$. Prove $u = 0$ on $\mathbb{R}^2$.

**Problem 2.** Suppose $f, g \in C^1(\mathbb{R})$ with $f(0) = g(0) = 0$, $f' > 0$ and $g' > 0$ on $\mathbb{R}\setminus\{0\}$. Suppose $u \in C^2(\Omega) \cap C^1(\overline{\Omega})$ is a solution of

$$
\Delta u = f(u) \text{ on } \Omega,
$$

$$
\frac{\partial u}{\partial n} + g(u) = 0 \text{ on } \partial \Omega.
$$

(a) Show $u = 0$ on $\Omega$ using the maximum principle.

(b) Show $u = 0$ on $\Omega$ using the energy method.

**Problem 3.** Let $T > 0, c \in C^0(\overline{\Omega}_T)$. Suppose $u \in C^{2,1}(\Omega_T) \cap C^0(\overline{\Omega}_T)$ satisfies

$$
u_t - \Delta u + c(x,t) u \leq 0 \text{ on } \Omega_T,
$$

$$
u \leq 0 \text{ on } \Gamma_T = \overline{\Omega}_T \setminus \Omega_T = \text{ parabolic boundary of } \Omega_T.
$$

Prove $u \leq 0$ on $\Omega_T$.

Hint: Consider $v = ue^{-Mt}$ for a suitable constant $M$.

**Problem 4.** Suppose $u \in C^2(\mathbb{R}^3 \times [0,\infty))$ is a solution of

$$
u_{tt} - \Delta u = 0 \text{ on } \mathbb{R}^3 \times [0,\infty),
$$

$$
u(x,0) = 0, \ x \in \mathbb{R}^3,
$$

$$
u_t(x,0) = g(x), \ x \in \mathbb{R}^3
$$

where $g \in C^2(\mathbb{R}^3)$ has compact support. Prove that there exists $C > 0$ such that

(a) $|u_t(x,t)| \leq C(1+t)^{-1}$ for all $(x,t) \in \mathbb{R}^3 \times [0,\infty)$, and
(b) \( (\int_{\mathbb{R}^n} |u|^6 dx)^{1/6} \leq C(1 + t)^{-2/3} \) for all \( t \geq 0 \).

**Problem 5.** Suppose \( u \in C^2(\mathbb{R}^n) \) satisfies \( \Delta u + u^2 + 2u \leq 0 \) on \( \mathbb{R}^n \). Show that the inequality \( u \geq 1 \) cannot hold on all of \( \mathbb{R}^n \).

Hint: Consider the auxiliary function \( v(x) = \frac{2}{2n}(R^2 - |x|^2) \) on \( B(0, R) \).

**Problem 6.** Suppose \( n \leq 3, \phi \in C^3(\mathbb{R}^n), \psi \in C^2(\mathbb{R}^n) \) and \( \phi, \psi \) have compact support. Suppose \( u \in C^2(\mathbb{R}^n \times [0, \infty)) \) is a solution of

\[
 u_{tt} - \Delta u = u^3 \quad \text{on} \quad \mathbb{R}^n \times (0, \infty), \\
u(x, 0) = \phi(x), \quad x \in \mathbb{R}^n, \\
u_t(x, 0) = \psi(x), \quad x \in \mathbb{R}^n,
\]

where \( \int_{\mathbb{R}^n} \phi(x)^2 dx > 0 \). Define the energy \( E(t) = \int_{\mathbb{R}^n} \left( \frac{1}{2} u_t^2 + \frac{1}{8} |\nabla u|^2 - \frac{1}{4} u^4 \right) dx \) and \( F(t) = \int_{\mathbb{R}^n} u^2 dx \) for \( t \geq 0 \). Assume \( E(0) < 0 \).

(a) Prove \( E(t) \) is constant in \( t \).

(b) Find a lower bound for \( \|u(\cdot, t)\|_{L^4(\mathbb{R}^n)} \) and prove \( F''(t) \geq 6\|u_t\|^2_{L^2(\mathbb{R}^n)} \) for each \( t \).

(c) Prove \( (F(t)^{-\frac{1}{2}})' \leq 0 \) for all \( t > 0 \) (note \( (F(t)^{-\frac{1}{2}})' = -\frac{1}{2}(FF'' - \frac{3}{2}F'^2)F^{-\frac{3}{2}} \)).

(d) Provided that \( F'(t) > 0 \) for some \( t > 0 \), show \( F(t) \to \infty \) as \( t \to t_0^- \) for some finite \( t_0 > 0 \).

**Problem 7.** Let \( Q = \mathbb{R}^n \times (0, \infty), n = 2, 3 \) and \( f \in C^0(\overline{Q}) \). Suppose \( u \in C^{2,1}(Q) \cap C^0(\overline{Q}) \) is a solution of

\[
 u_t - \Delta u = f(x, t) \quad \text{on} \quad Q, \\
u = 0 \quad \text{on} \quad \mathbb{R}^n \times \{0\}.
\]

Assume \( \int_{\mathbb{R}^n} f(x, t)^2 dx \leq k \) for all \( t \geq 0 \); and that for each \( \varepsilon > 0 \) there exists \( C_\varepsilon > 0 \) such that \( |f| \leq C_\varepsilon e^{\varepsilon |x|^2} \) on \( Q \). Assume \( |u| \leq A e^{\alpha |x|^2} \) holds on \( Q \) for some constants \( a, A > 0 \). Show, for some \( C, \alpha > 0 \), \( |u| \leq C t^\alpha \) holds on \( Q \).

Give \( \alpha \) explicitly and explain if your reasoning depends on \( n \). Explain the purpose of \( e^{\alpha |x|^2} \).