Part I

1. Let $p$ be a prime, and let $S_p$ denote the symmetric group of degree $p$. Prove that if $P$ is a subgroup of $S_p$ of order $p$, then the normalizer of $P$ in $S_p$ has order $p(p-1)$.

2. Classify, up to isomorphism, the groups of order 63.

Part II

1. A local ring is a commutative ring with $1 \neq 0$ that has a unique maximal ideal. Prove that if $R$ is a local ring with maximal ideal $M$, then every element of $R \setminus M$ is a unit. Also prove that if $R$ is a commutative ring with $1 \neq 0$, in which the set of nonunits forms an ideal $M$, then $R$ is a local ring with maximal ideal $M$.

2. Let $p \in \mathbb{Z}_+$ be prime, and let $\mathbb{Z}[i]$ denote the usual ring of Gaussian integers $\{ a + bi \mid a, b \in \mathbb{Z} \}$. For which $p$ is the quotient ring $\mathbb{Z}[i]/(p)$ (i) a field, (ii) a product of fields? Justify your answer. (You may use the following facts: (i) $\mathbb{Z}[i]$ is a Euclidean Domain with respect to the field norm, hence is also a Unique Factorization Domain, and (ii) a prime $p \in \mathbb{Z}_+$ with $p \equiv 1 \pmod{4}$ can be written as the sum of two integer squares.)

Hint: Use the Chinese Remainder Theorem where appropriate. Also note that a product of fields cannot contain a nonzero nilpotent element.

Part III

1. Let $V$ be a finite dimensional vector space over a field $F$, and let $v_1, v_2$ be nonzero elements of $V$. Prove that $v_1 \otimes v_2 = v_2 \otimes v_1$ in $V \otimes_F V$ if and only if $v_1 = \lambda v_2$ for some $\lambda \in F$.

2. Let $R$ be a ring with $1 \neq 0$, let $P, M, N$ be $R$–modules, and let there be an exact sequence of $R$–module homomorphisms $M \to N \to 0$.

(a) Prove that if $P$ is a direct summand of a free $R$–module, then the induced sequence of Abelian group homomorphisms

$$\text{Hom}_R(P, M) \to \text{Hom}_R(P, N) \to 0$$

is exact. (Here $\phi'$ is the homomorphism $\psi \mapsto \phi \circ \psi$.)

(b) Show by means of an example that in general the induced sequence $\text{Hom}_R(P, M) \to \text{Hom}_R(P, N) \to 0$ need not be exact.

Note: For this question do not assume any result concerning projective modules.
Part IV  

In this part, $x$ denotes an indeterminate.

1. This question concerns the splitting field over $\mathbb{Q}$ of the polynomial $x^4 - 2x^2 - 2 \in \mathbb{Q}[x]$.

   (a) Prove that $x^4 - 2x^2 - 2$ is irreducible over $\mathbb{Q}$, and that its roots in $\mathbb{C}$ are $\pm \alpha, \pm \beta$, where $\alpha = \sqrt{1 + \sqrt{3}}, \beta = \sqrt{1 - \sqrt{3}}$.

   (b) Prove that $\mathbb{Q}(\alpha) \neq \mathbb{Q}(\beta)$, and that $[\mathbb{Q}(\alpha, \beta) : \mathbb{Q}(\alpha)] = 2$.

   (c) Prove that the splitting field of $x^4 - 2x^2 - 2$ has degree 8 over $\mathbb{Q}$, and that the Galois group of this polynomial over $\mathbb{Q}$ is dihedral of order 8.

   *Hint for (c)*: The Galois group acts faithfully on the set of roots of the polynomial.

2. Let $\mathbb{F}_p$ denote the field of order $p$, let $f \in \mathbb{F}_p[x]$ be irreducible over $\mathbb{F}_p$, and let $K$ be a splitting field for $f$ over $\mathbb{F}_p$.

   Let $L$ be an intermediate field, i.e. $\mathbb{F}_p \subseteq L \subseteq K$. Prove that the irreducible factors of the polynomial $f$ in $L[x]$ all have the same degree.

   *Hint*: Here is one approach. Let $g \in L[x]$ be a factor of $f$ that is irreducible in $L[x]$, and let $\alpha$ be a root of $g$ in $K$. Consider the relationship between $[L(\alpha) : L]$ and $[K : L]$. 
Algebra Preliminary Examination
August 2020

Attempt all questions, and justify each answer.

Part I

1. Let $P$ be a Sylow $p$–subgroup of a finite group $G$. If $p$ is the smallest prime dividing $|G|$ and $P$ is cyclic, prove that $N_G(P) = C_G(P)$. (Recall that $N_G(P)$, $C_G(P)$ denote the normalizer and centralizer of $P$ in $G$, respectively.)

   (Hint: Consider the order of the automorphism group of $P$ and the action of $N_G(P)$ on $P$ by conjugation.)

2. (a) Prove that a group of order 105 contains a cyclic normal subgroup of order 35.
   (b) Prove that, up to isomorphism, there is just one non-Abelian group of order 105.

   In parts II, III and IV, $X$ denotes an indeterminate.

Part II

1. Let $R$ be a commutative ring with $1 \neq 0$. Recall that $R$ is Artinian if it satisfies the descending chain condition on ideals, i.e. if $I_1 \supseteq I_2 \supseteq \ldots$ is a descending chain of ideals of $R$, then there exists $k \in \mathbb{Z}_+$ such that $I_m = I_k$ for all $m > k$.

   Let $S$ be an arbitrary commutative ring with $1 \neq 0$, and let $J$ denote the Jacobson radical of $S[X]$. Prove that $S[X]/J$ is not Artinian.

2. Let $R$ be the subring of $\mathbb{Q}[X]$ consisting of all polynomials whose constant term is an integer.
   (a) Prove that $R$ is an integral domain in which every irreducible element is prime.
   (b) Prove that $R$ is not a Unique Factorization Domain.
      (Hint: Consider factorizations of the element $X$.)

Part III

1. Let $k$ be a field, and let $R = M_2(k)$ be the ring of $2 \times 2$ matrices over $k$. Let $P$ be the set of $2 \times 1$ matrices over $k$; then $P$ is an Abelian group under matrix addition, and left matrix multiplication of elements of $P$ by elements of $R$ accords $P$ the structure of a left $R$–module.

   Prove that the $R$–module $P$ is projective, but not free.

2. Let $R = \mathbb{Z}[X]$, let $I \subset R$ be the ideal generated by $2, X$, and let $M = I \otimes_R I$.

   Prove that the element $2 \otimes 2 + X \otimes X \in M$ cannot be written as a simple tensor $a \otimes b$ ($a, b \in I$).
   (Hint: Use a suitable $R$–module homomorphism defined on $M$.)
Part IV

1. Prove that $\mathbb{Q}(\sqrt{5 + 2\sqrt{5}})$ is a Galois extension of $\mathbb{Q}$, and determine its Galois group.

2. Let $F$ be a field (possibly infinite) of finite characteristic $p$, and let $a \in F$ be an element not of form $b^p - b$ for any $b \in F$. Let $f = X^p - X - a \in F[X]$.

   (a) Prove that the polynomial $f$ is separable and irreducible over $F$.

   (b) Prove that if $\alpha$ is a root of $f$ in an extension field of $F$, then $F(\alpha)$ is a splitting field for $f$ over $F$.

   (Hint: Consider the effect of substituting $X + 1$ for $X$ in the polynomial $f$.)
ALGEBRA PRELIMINARY EXAM

JANUARY 2020

Instructions: Attempt all problems in all four parts. Justify each answer.

General Assumptions: Unless explicitly stated otherwise, all rings have $1 \neq 0$ [and their subrings contain 1] and all modules are unitary.

Part I

1. Let $G$ be a finite group and $\phi : G \rightarrow H$ a surjective homomorphism. Prove that if $y \in H$ is such that $|y| = p^r$, for some prime $p$ and $r \in \mathbb{Z}_{>0}$, then there is $x \in G$ such that $\phi(x) = y$ and $|x| = p^s$, for some $s \in \mathbb{Z}_{>0}$.
   
   [Hint: Let $g \in G$ such that $\phi(g) = y$, and write $|g| = n \cdot p^k$, where $p \nmid n$.]

2. Let $G$ be a group of order 60 and assume that 4 divides $|Z(G)|$ [where $Z(G)$ denotes the center of $G$]. Prove that $G$ must be Abelian.

Part II

1. Let $I$ be the ideal of $\mathbb{Z}[x]$ generated by 7 and $x^2 + 1$. Prove that $I$ is a maximal ideal.

2. Let $R$ be an integral domain such that for any descending chain of ideals

   \[ I_1 \supseteq I_2 \supseteq I_3 \supseteq \cdots \]

   there is a positive integer $N$ such that $I_i = I_N$ for all $i \geq N$. Prove that $R$ is a field.

Part III

1. Let $R$ be a subring of $S$. Prove that $S \otimes_R S \neq 0$.

2. Let $R$ be a ring containing $\mathbb{Z}$ such that $R$ is a free $\mathbb{Z}$-module of finite rank $n > 0$ and every non-zero ideal of $R$ has a non-zero element of $\mathbb{Z}$. Prove that for every non-zero ideal $I$ we have that $R/I$ is finite.

Part IV

1. Given a prime $p$ and a positive integer $n$, show that there is an Abelian extension [i.e., Galois with Abelian Galois group] $K$ of $\mathbb{Q}$ with $[K : \mathbb{Q}] = p^n$.

2. Let $F$ be a field of characteristic $p$ with exactly $p^n$ elements. If $K$ is a finite extension of $F$ with $K = F[\alpha]$, for some $\alpha \in K$, and $f$ is the minimal polynomial of $\alpha$ over $F$, then show that if $\beta$ is another root of $f$, then $\beta \in K$ and $\beta = \alpha^{pk}$ for some $k \in \mathbb{Z}$.
ALGEBRA PRELIMINARY EXAM

AUGUST 2019

Instructions: Attempt all problems in all four parts. Justify each answer.

General Assumptions: Unless explicitly stated otherwise, all rings have $1 \neq 0$ [and their subrings contain 1] and all modules are unitary.

Part I

1. Let $G_1, G_2$ be groups, $N \leq G_1$, and $\phi : G_1 \to G_2$ be an onto homomorphism such that $N \cap \ker(\phi) = \{1\}$. Prove that for $x \in N$ we have that $C_{G_2}(\phi(x)) = \phi(C_{G_1}(x))$. [Remember: $C_G(x) \overset{\text{def}}{=} \{g \in G : gx = xg\}$ is the centralizer of $x$ in $G$.]

2. Let $G$ be a group of order $992 = 2^5 \cdot 31$. Prove that either $G$ has a normal subgroup of order $32 = 2^5$ or it has a normal subgroup of order 62.

Part II

1. Let $R$ be a UFD with exactly two non-associate prime elements $p$ and $q$ [i.e., $p$ and $q$ are non-associate primes and every prime is an associate of either $p$ or $q$]. Prove that $R$ is a PID.

2. Let $R$ be a PID and $P$ a prime ideal of $R[x]$ such that $P \cap R \neq \{0\}$. Prove that there is $p \in R$ prime [in $R$] such that either $P = (p)$ or $P = (p, f)$ for some $f$ prime in $R[x]$.

Part III

1. Let $R$ be a commutative ring and $M$ an $R$-module. Prove that $R \otimes_R \text{Hom}_R(R \oplus R, M)$ is projective if and only if $M$ is projective.

2. Let $R$ be a commutative ring, $M$ and $N$ be $R$-modules and $M'$ and $N'$ be submodules of $M$ and $N$ respectively. Define $L$ as the submodule of $M \otimes_R N$ generated by the set

$$\{x \otimes y \in M \otimes_R N : x \in M' \text{ or } y \in N'\}.$$

Show that $M/M' \otimes_R N/N' \cong (M \otimes_R N)/L$.

Part IV

1. Let $F = \mathbb{Q}(\sqrt[3]{2}: \zeta)$, where $\zeta = -1/2 + \sqrt{3}i/2$ [a primitive third root of unity]. Prove that $-1$ is not a sum of squares in $F$, i.e., there is no positive integer $n$ and $\alpha_1, \ldots, \alpha_n \in F$ such that $-1 = \alpha_1^2 + \cdots + \alpha_n^2$.

2. Let $F$ be a field of characteristic 0 and $K/F$ be a field extension of degree $n$ such that there is a root of unity $\zeta$ in the algebraic closure of $K$ such that $K \subseteq F[\zeta]$. Prove that if $d \mid n$, there is $\alpha \in K$ such that the minimal polynomial of $\alpha$ over $F$ has degree $d$. 
ALGEBRA PRELIMINARY EXAM

AUGUST 2018

Instructions: Attempt all problems in all four parts. Justify your answers.

General assumptions: All rings have $1 \neq 0$, their subrings contain 1, and all modules are unitary.

Part I

1. Let $G$ be a (possibly infinite) group, and suppose that $G$ contains a subgroup $H \neq G$ whose index $[G : H]$ is finite. Prove that $G$ contains a normal subgroup $N \neq G$ of finite index.
2. Prove that every group of order 70 contains a cyclic, normal subgroup of order 35.

Part II

1. Let $R$ be a commutative ring in which every element is either a unit or nilpotent. Prove that $R$ has exactly one prime ideal.
2. If $R$ is an integral domain, prove that there are infinitely many ideals in $R[x]$ that are both prime and principal.

Part III

1. Let $R$ be a ring, possibly non-commutative, and suppose that $0 \to M' \to M \to M'' \to 0$ is a short exact sequence of left $R$-modules, with $M'$ and $M''$ finitely generated. Prove that $M$ is finitely generated.
2. Let $M$ be a finitely-generated $\mathbb{Z}$-module, and let $T \subset M$ be its torsion submodule. Show that the torsion submodule of $M \otimes_{\mathbb{Z}} M$ has at least $|T|$ elements.

Part IV

1. Let $p$ be a prime and suppose that $f \in \mathbb{F}_p[x]$ is an irreducible polynomial. Let $K$ be a degree 2 extension of $\mathbb{F}_p$ and suppose that there exist non-constant polynomials $g, h \in K[x]$ such that $f = gh$. If $g$ is an irreducible polynomial of degree 5, what is the degree of $f$?
2. Suppose that $f \in \mathbb{Q}[x]$ is an irreducible degree 4 polynomial, and $K/\mathbb{Q}$ is an extension such that $f$ has exactly one root in $K$. Let $G$ be the Galois group of $f$, and show that $|G|$ is divisible by 12.
ALGEBRA PRELIMINARY EXAM

AUGUST 2017

Instructions: Attempt all problems in all four parts. Justify your answer.

General Assumptions: Unless explicitly stated otherwise, all rings have \( 1 \neq 0 \) [and their subrings contain 1] and all modules are unitary.

Part I

1. Suppose that \( H \) is a subgroup of a finite group \( G \) of index \( p \), where \( p \) is the smallest prime dividing the order of \( G \). Prove that \( H \) is normal in \( G \).

2. Show that every group of order 222 is solvable.
   
   Fun fact: The University of Tennessee was established 222 years ago.

Part II

1. Let \( I \) and \( J \) be ideals of a ring \( R \) and assume that \( P \) is a prime ideal of \( R \) that contains \( I \cap J \). Prove that either \( I \) or \( J \) is contained in \( P \).

2. Let \( R \) be an integral domain and suppose that every prime ideal in \( R \) is principal. Prove that \( R \) is a PID.

Part III

1. Let \( V \) be a Noetherian right \( R \)-module, and \( \theta : V \to V \) a homomorphism.
   (a) Show that \( \ker(\theta^{n+1}) = \ker(\theta^n) \) for some \( n \geq 1 \).
   (b) If \( \theta \) is onto, show that it is one-to-one.

2. An \( R \)-projection is defined to be an \( R \)-module homomorphism \( \varphi : R^n \to R^n \) such that \( \varphi^2 = \varphi \). Prove that a finitely generated \( R \)-module \( M \) is projective if and only if it is isomorphic to the image of some \( R \)-projection.

Part IV

1. Let \( F \subseteq E \) be fields and suppose \( 0 \neq \alpha \in E \) with \( E = F(\alpha) \). Assume that some power of \( \alpha \) lies in \( F \) and let \( n \) be the smallest positive integer such that \( \alpha^n \in F \).
   (a) If \( \alpha^m \in F \) with \( m > 0 \), show that \( m \) is a multiple of \( n \).
   (b) If \( E \) is a separable extension of \( F \), prove that the characteristic of \( F \) does not divide \( n \).
   (c) If every root of unity of \( E \) lies in \( F \), show that \( [E : F] = n \).

2. Let \( F \) be a field of characteristic 0 and let \( E \) be a finite Galois extension of \( F \).
   (a) If \( 0 \neq \alpha \in E \) with \( E = F(\alpha) \), show that \( F(\alpha^2) \neq E \) if and only if there exists \( \sigma \in \text{Gal}(E/F) \) with \( \sigma(\alpha) = -\alpha \).
   (b) Prove that there exists an element \( \alpha \in E \) with \( E = F(\alpha^2) \).
• Attempt all four parts. Justify your answers.

Part I.

1. Show that a group of order 255 is not a simple group.

2. A group $G$ has a cyclic normal subgroup of order 2016. If $G$ also has a subgroup of order 2017, then show that $G$ has a cyclic subgroup of order $(2016) \times (2017)$.

Part II.

Note: Rings are assumed to be commutative and with $1 \neq 0$.

1. Let $A$ and $B$ be rings. Show that each ideal of $A \times B$ is of the form $I \times J$, where $I$ is an ideal of $A$ and $J$ is an ideal of $B$.

2. Let $R$ be a ring, let $X$ be an indeterminate and let $S := \{X^n \mid 0 \leq n \in \mathbb{Z}\}$. If $S^{-1}R[[X]]$ is a field, then show that $R$ is a field.

Part III.

Note: Rings are assumed to be commutative with $1 \neq 0$ and modules are assumed to be unitary.

1. Let $A$ be a ring and let $M$, $N$ be finitely generated projective (left) $A$-modules. Show that $\text{Hom}_A(M, N)$ is a finitely generated projective $A$-module.

2. Let $R$ be a PID and let $I$, $J$ be ideals of $R$. If $I \neq R \neq J$, then show that $(R/I) \oplus (R/J)$ and $(R/I) \otimes_R (R/J)$ are not isomorphic as (left) $R$-modules.

Part IV.

Note: In what follows, $X$ is an indeterminate.

1. Let $K$ be an extension-field of $\mathbb{Q}$ such that $K/\mathbb{Q}$ is Galois with Galois group $\mathbb{Z}_{30}$. Suppose each of $f, g \in \mathbb{Q}[X]$ is an irreducible polynomial of degree 6 and $f$ has a root $a \in K$. If $g$ has a root in $K$, then show that $g$ has all its roots in $\mathbb{Q}[a]$.

2. Let $F \subset K$ be finite fields of characteristic 5 and suppose $g \in F[x]$ is irreducible in $F[x]$. If $g$ has degree 11, then show that either $g$ is irreducible in $K[x]$ or all its roots are in $K$. 
• Attempt all four parts. Justify your answers.

Part I.

1. Let $p$ be a prime number and $G$ be a non-Abelian group of order $p^3$. Show that $G$ has at least 3 (distinct) subgroups of index $p$.

2. Let $G$ be a group of order $p^3q$, where $p$, $q$ are distinct prime numbers. If no Sylow $p$-subgroup of $G$ is normal and also no Sylow $q$-subgroup of $G$ is normal, then show that $G$ has order 24.

Part II.

Note: Rings are tacitly assumed to be commutative and with $1 \neq 0$.

1. Let $R$ be a ring, $X$ an indeterminate and $h : R[X] \rightarrow R[[X]]$ a ring-homomorphism such that $h(a) = a$ for all $a \in R$. Show that $h$ is not surjective.

2. Let $R$ be an integral domain with at least 3 (distinct) maximal ideals. Given maximal ideals $M$ and $N$ of $R$, show that $R_M \cap R_N \neq R$. (Here localization of $R$ at a prime ideal is naturally identified as a ring in between $R$ and the quotient-field of $R$.)

Part III.

Note: Rings are assumed to be commutative and with $1 \neq 0$ and modules are assumed to be unitary.

1. Let $R$ be a ring and let $a \in R$ be a nonzero element of $R$ such that $a^3 = a$. Show that the ideal $Ra$ is a projective $R$-module.

2. Let $R$ be a PID and let $M$ be a finitely generated $R$-module. For a maximal ideal $Q$ of $R$, let $\delta(Q, M)$ denote the dimension of $M \otimes_R R/Q$ as a vector-space over the field $R/Q$. Let $\delta(M)$ denote the sup\{\delta(Q, M)\}, where the supremum is taken over all maximal ideals $Q$ of $R$. Show that as an $R$-module, $M$ has a generating set of cardinality $\delta(M)$ and any generating set of $M$ has cardinality at least $\delta(M)$.

Part IV.

Note: In what follows, $X$ is an indeterminate.

1. Let $f(X)$ be a monic polynomial with rational coefficients. Assume $f(X)$ is irreducible in $\mathbb{Q}[X]$ and the Galois-group of $f(X)$ over $\mathbb{Q}$ is a group of order 99. What is the degree of $f(X)$?

2. Compute the Galois group of $X^6 - 9$ over $\mathbb{Q}$. 
ALGEBRA PRELIMINARY EXAM

JANUARY 2016

Instructions: Attempt all problems in all four parts. Justify each answer.

General Assumptions: Unless explicitly stated otherwise, all rings have $1 \neq 0$ [and their subrings contain 1] and all modules are unitary.

Part I

1. Let $G$ be a finite group and $H$ be a subgroup of $G$. Prove that $n_p(H) \leq n_p(G)$, where $n_p(X)$ denotes the number of Sylow $p$-subgroups of $X$.

2. Let $G$ be a group of order $p^n$ for some prime $p$ and positive integer $n$. Prove that if $1 \neq H \leq G$, then $Z(G) \cap H \neq 1$. [Here $Z(G)$ denotes the center of $G$.]

Part II

1. Let $R$ be a Boolean ring, i.e., a ring [with 1] for which $a^2 = a$ for all $a \in R$. [You can use without proof the well known fact that if $R$ is Boolean, then it is commutative of characteristic 2.]
   (a) Prove that if $R$ is finite, then its order is a power of 2.
   (b) Prove that every prime ideal of $R$ is maximal.

2. Show that $R \overset{\text{def}}{=} \mathbb{Z}[x_1, x_2, x_3, \ldots] / \langle x_1 x_2, x_3 x_4, x_5 x_6, \ldots \rangle$ has infinitely many distinct minimal prime ideals. [$P$ is a minimal prime ideal if it is prime and whenever $Q \subseteq P$, with $Q$ also prime, we have $Q = P$.]

Part III

1. Let $F$ be a field and $M$ be a torsion $F[x]$-module. Prove that if there is $m_0 \in M$, with $m_0 \neq 0$, and an irreducible $f \in F[x]$ such that $f \cdot m_0 = 0$, then $\text{Ann}(M) \subseteq \langle f \rangle$.

2. Let $R$ be an integral domain and $I$ a principal ideal of $R$. Prove that $I \otimes_R I$ has no non-zero torsion element [i.e., if $m \in I \otimes_R I$, with $m \neq 0$, and $r \in R$ with $rm = 0$, then $r = 0$].

Part IV

1. Let $K/F$ be an algebraic field extension and $\text{Emb}(K/F)$ denote the set of field homomorphisms $\sigma : K \to \bar{K}$ such that $\sigma(a) = a$ for all $a \in F$. [Here $\bar{K}$ is a fixed algebraic closure of $K$.]
   (a) Prove that if $\alpha$ is a root of a [not necessarily irreducible] non-zero polynomial $f \in F[x]$ with $\text{deg}(f) = n$, then $\text{Emb}(F[\alpha]/F)$ has at most $n$ elements.
   (b) Give an example of an algebraic extension $K/F$ of degree greater than one for which $\text{Emb}(K/F)$ has a single element.

2. Let $F = \mathbb{Q}[\sqrt{2}]$ and $K = \mathbb{Q}[\sqrt{2}, i]$.
   (a) Prove that $K/F$ is Galois with $[K : F] = 8$.
   (b) Prove that $\text{Gal}(K/F)$ has a non-normal subgroup. [This implies that it is the dihedral group of order 8, as it is the only group of order 8 with this property.]
ALGEBRA PRELIMINARY EXAM

AUGUST 2015

Instructions: Attempt all problems in all four parts. Justify each answer.

General Assumptions: Unless explicitly stated otherwise, all rings have \( 1 \neq 0 \) [and their subrings contain 1] and all modules are unitary.

Part I

1. Let \( G \) be a non-Abelian group of order \( p^3 \), \( [G,G] = \langle xyx^{-1}y^{-1} : x, y \in G \rangle \) be its commutator subgroup and \( Z(G) \) be its center. Show that \( |Z(G)| = p \) and that \( Z(G) = [G,G] \).

2. Let \( G_1 \) and \( G_2 \) be groups of order 81 acting faithfully [i.e., only 1 acts as the identity function] on sets \( X_1 \) and \( X_2 \), respectively, with 9 elements each. Show that there is an isomorphism \( \psi : G_1 \to G_2 \).

Part II

1. Let \( D \) be a finite division ring. Prove that \( D \) has a prime power number of elements. [Hint: Consider the center \( Z(D) = \{ a \in D : ax = xa \text{ for all } x \in D \} \].

2. Let \( p \in \mathbb{Z} \) prime and

\[
f = a_{2n+1}x^{2n+1} + \cdots + a_1x + a_0 \in \mathbb{Z}[x].
\]

Prove that if \( p^3 \nmid a_0, p^2 \mid a_0, a_1, \ldots, a_n, p \mid a_{n+1}, a_{n+2}, \ldots, a_{2n} \) and \( p \nmid a_{2n+1} \), then \( f \) is irreducible in \( \mathbb{Q}[x] \).

Part III

1. Let \( R \) be a commutative ring. An \( R \)-module is Artinian if it satisfies the descending chain condition for submodules. [i.e., if \( S_1 \supseteq S_2 \supseteq S_3 \supseteq \cdots \) is a chain of submodules, then there is a \( i_0 \) such that for all \( i \geq i_0 \), we have \( S_i = S_{i_0} \).] Show that if \( L \) and \( N \) are Artinian \( R \)-modules and we have a short exact sequence

\[
0 \longrightarrow L \overset{\psi}{\longrightarrow} M \overset{\phi}{\longrightarrow} N \longrightarrow 0,
\]

then \( M \) is also Artinian.

2. Let \( R \) be a commutative ring such that every \( R \)-module is free. Prove that \( R \) is a field.

Part IV

1. Let \( \mathbb{F}_p \) be the field with \( p \) elements, and \( t \) be an indeterminate. Let \( f(t), g(t) \in \mathbb{F}_p[t] \setminus \{0\} \), with \( \max\{\deg f, \deg g\} < p \) and \( f(t), g(t) \not\in \mathbb{F}_p \). Show that the extension \( \mathbb{F}_p(t)/\mathbb{F}_p(f(t)/g(t)) \) is separable.

2. Suppose that \( f = \prod_{i=1}^{N} (x - \alpha_i) \in \mathbb{Q}[x] \) [with \( \alpha_i \in \mathbb{C} \)] is irreducible in \( \mathbb{Q}[x] \) and let \( f_n \overset{\text{def}}{=} \prod_{i=1}^{N} (x - \alpha_i^n) \). Prove that for each \( n \), there is \( g_n \in \mathbb{Q}[x] \) irreducible and a positive integer \( k_n \) such that \( f_n = g_n^{k_n} \).
ALGEBRA PRELIMINARY EXAMINATION
Fall 2014

Attempt all four parts. Justify your answers.

Part I.

1. Show that $S_4$ (the group of permutations of $\{1, 2, 3, 4\}$) does not have a subgroup isomorphic to $Q_8$ (the quaternion-group of order 8).

2. Let $G$ be a group of order 2014. Show that $G$ is cyclic if and only if $G$ has a normal subgroup of order 2.

Part II.

Note: Rings are assumed to be commutative and with $1 \neq 0$, subrings are assumed to contain 1 and ring-homomorphisms are assumed to map 1 to 1.

1. Let $R$ be an integral domain with only finitely many units. Show that the intersection of all maximal ideals of $R$ is 0.

2. Let $R$ be a ring such that each non-unit of $R$ is nilpotent. Let $X$ be an indeterminate and let $f \in R[[X]]$. Show that $f^n = f$ for some integer $n \geq 2$ if and only if either $f = 0$ or $f^{n-1} = 1$.

Part III.

Note: Rings are assumed to be commutative and with $1 \neq 0$ and modules are assumed to be unitary.

1. Let $L$ be a module over a ring $R$ and let $M, N$ be $R$-submodules of $L$. Show that if $(M + N)/(M \cap N)$ is a projective $R$-module then $M/(M \cap N)$ is also a projective $R$-module.

2. Let $R$ be a PID with infinitely many prime ideals and let $M$ be a finitely generated $R$-module. Show that $M$ is a torsion $R$-module if and only if $M \otimes_R R/P = 0$ for all but finitely many prime ideals $P$ of $R$.

Part IV.

Note: In what follows, $X$ is an indeterminate.

1. Let $f(X) := X^5 + 3X^3 + X^2 + 3 \in \mathbb{Q}[X]$. Let $K$ be the splitting field of $f(X)$ over $\mathbb{Q}$. Compute $[K : \mathbb{Q}]$.

2. Let $f(X) := X^3 + X + 1 \in \mathbb{Q}[X]$. Let $F$ be a finite Galois extension of $\mathbb{Q}$ such that the Galois group of $F$ over $\mathbb{Q}$ is an Abelian group. Show that $f$ is irreducible in $F[X]$. 
Algebra Preliminary Exam        January 2014

Attempt all problems and justify all your answers. All rings have a 1 ≠ 0, all ring homomorphisms send 1 to 1, and all R-modules are unitary.

Part I. Groups

1. Show that every group of order 1,225 is abelian.
2. Let n ≥ 2. Show that there is a nontrivial homomorphism
   \( f : S_n \to \mathbb{Z}/n\mathbb{Z} \) (i.e., \( \ker f \neq S_n \)) if and only if n is even.

Part II. Rings

1. Let R be a commutative ring. Show that \( J(R[X]) = \text{nil}(R[X]) \).
   \( J(A) \) and \( \text{nil}(A) \) are the Jacobson and nil radicals of A.

2. Let R be a PID.
   (a) Show that R satisfies ACC on ideals.
   (b) Show that every nonzero prime ideal of R is maximal.

Part III. Modules

1. Let R be a ring and M a nonzero R-module. Show that \( M = A \oplus B \) for proper submodules A and B of M if and only if there is a nonzero, nonidentity homomorphism \( f : M \to M \) with \( f^2 = f \).

2. Let R be a commutative ring, I a proper ideal of R, and M an R-module. Show that \( (R/I) \otimes_R M \) and \( M/IM \) are isomorphic as R-modules.

Part IV. Fields

1. Let K a subfield of a field F. Show that there is a subring of F containing K that is a PID, but not a field, if and only if the extension F/K is not algebraic.

2. Determine the Galois group of \( f(X) = X^{10} + X^8 + X^6 + X^2 \) over \( \mathbb{Z}/2\mathbb{Z} \).
Algebra Preliminary Exam

August 2013

Attempt all problems and justify all your answers. All rings have an identity 1 ≠ 0, all ring homomorphisms send 1 to 1, and all R-modules are unitary.

Part I.

1. (a) Let p and q be (not necessarily distinct) prime numbers. Show that a group G with |G| = pq is either abelian or Z(G) = {e}.
   (b) Give an example of a nonabelian group G whose order is the product of three (not necessarily distinct) primes and Z(G) ≠ {e}.

2. (a) Let G be a group with |G| = 100. Show that G is abelian if and only if its Sylow 2-subgroup is normal.
   (b) Give an example of a nonabelian group of order 100.

Part II.

1. Let R and S be a commutative rings with 1 ≠ 0. Show that every ideal of R×S has the form I×J for I an ideal of R and J an ideal of S.

2. Let R be a commutative ring with 1 ≠ 0. Show that f(X) = a_0 + a_1X + ⋯ + a_nX^n is a unit in R[X] if and only if a_0 is a unit in R and a_1, ..., a_n are nilpotent.
Part III

1. Let $P$ and $Q$ be finitely generated projective $R$-modules over a commutative ring $R$ with $1 \neq 0$. Show that $\text{Hom}_R(P,Q)$ is a finitely generated projective $R$-module.

2. Let $R$ be a commutative ring with $1 \neq 0$, $S$ a nonempty multiplicatively closed subset of $R$, and $M$ an $R$-module. Show that $(S^{-1}R) \otimes_R M$ and $S^{-1}M$ are isomorphic as $S^{-1}R$-modules.

Part IV.

1. Let $p$ and $q$ be distinct prime numbers, $F$ a subfield of a field $K$, and $f(X), g(X) \in F[X]$ be irreducible with $\deg(f(X)) = p$ and $\deg(g(X)) = q$. Let $a, b \in K$ be roots of $f(x)$ and $g(X)$, respectively. Show that $[F(a,b):F] = pq$.

2. (a) Let $F$ be a splitting field for $f(X) \in \mathbb{Q}[X]$ over $\mathbb{Q}$ with abelian Galois group $G$. Show that every subfield $L$ of $F$ is a splitting field over $\mathbb{Q}$ for some polynomial $g(X) \in \mathbb{Q}[X]$.

(b) Give an example to show that if $G$ is not abelian in part (a), then some $L$ need not be a splitting field.
ALGEBRA PRELIMINARY EXAM

JANUARY 2013

Instructions: Attempt all problems in all four parts. Justify each answer.

General Assumptions: Unless explicitly stated otherwise, all rings have 1 ≠ 0 [and their subrings contain 1] and all modules are unitary.

Part I

1. Let p and q be prime numbers such that q < p and q does not divide $p^2 - 1$. Prove that every group of order $p^2q$ is Abelian.

2. Let $G$ be a finite simple group. Show that if $p$ is the largest prime dividing $|G|$, then there is no subgroup $H \leq G$ such that $1 < |G : H| < p$.

Part II

1. Let $R$ be a ring not necessarily having 1 [or commutative], with at least two elements and such that for all non-zero $a \in R$ there is a unique $b \in R$ such that $aba = a$.
   (a) Show that $R$ has no [non-zero] zero divisors.
   (b) Show that for $a$ and $b$ as above, we also have $bab = b$.
   (c) Show that $R$ has 1.

2. Let $R$ be a commutative ring and $a \in R$ such that $a^n \neq 0$ for all positive integers $n$. Let $I$ be an ideal maximal with respect to the property that $a^n \not\in I$ for any positive integer $n$. Show that $I$ is prime.

Part III

1. Let $V = \mathbb{R}^2$ and \{e_1, e_2\} be a basis of $V$. Show that $e_1 \otimes e_2 + e_2 \otimes e_1 \in V \otimes_R V$ cannot be written as a single tensor.

2. Let $R$ be a PID.
   (a) Prove that a finitely generated $R$-module $M$ is free if and only if it is torsion free.
   (b) Prove that if a finitely generated $R$-module $M$ is projective, then it is free.

Part IV

1. Let $\mathbb{F}_p$ be the field with $p$ elements, $\bar{\mathbb{F}}_p$ be a fixed algebraic closure of $\mathbb{F}_p$ and let
   $$L = \{ \alpha \in \bar{\mathbb{F}}_p : p \nmid [\mathbb{F}_p[\alpha] : \mathbb{F}_p] \}.$$  
   Show that $L$ is a field.

2. Let $p$ be a prime, $F$ be a field of characteristic different from $p$ and $f = x^p - a \in F[x]$ [not necessarily irreducible]. Let $K$ be the splitting field of $x^p - 1$ over $F$ and assume that all roots of $f$ lie in $K$.
   (a) Show that if $f(\alpha) = 0$ with $\alpha \notin F$, then $F[\alpha] = K$.
   (b) Prove that $f$ has a root in $F$. 
ALGEBRA PRELIMINARY EXAM

AUGUST 2012

Instructions: Attempt all problems in all four parts. Justify each answer.

General Assumptions: All rings have $1 \neq 0$ [and their subrings contain 1] and all modules are unitary.

Part I

1. Let $G$ and $H$ be finite Abelian groups. Prove that if $G \times H \times H \cong G \times G \times H$, then $G \cong H$.

2. Let $p$ be a prime and $G$ be a group of order $p^n$. For $k \in \{1, 2, 3, \ldots, (n - 1)\}$, let $s_k$ and $n_k$ denote the number of subgroups and normal subgroups of $G$ of order $p^k$ respectively. Show that $s_k - n_k$ is divisible by $p$.

Part II

1. Let $R$ be a commutative ring for which every proper ideal is prime. Show that $R$ is a field.

2. Let $F$ be a field and consider the subring $R$ of $F[t]$ given by polynomials with the coefficient of $t$ equal to zero, i.e., $R = F + t^2F[t]$.
   (a) Show that $R$ has an irreducible element which is not prime. [Hence, $R$ is not PID.]
   (b) Show that $R$ is Noetherian. [Hint: Consider a connection between $R$ and $F[x, y]$.

Part III

1. Let $R$ be a commutative ring, $S$ be a subring of $R$, $A$ be an $R$-module and

   $\mathcal{H} \overset{\text{def}}{=} \text{Hom}_R(R \otimes_S (S \oplus S), A)$.

   Show that for every surjective homomorphism of $R$-modules $\phi : M \to N$ and $R$-module homomorphism $f : \mathcal{H} \to N$ there is an $R$-module homomorphism $F : \mathcal{H} \to M$ such that $\phi \circ F = f$ if and only if the same if true if we replace $\mathcal{H}$ by $A$.

2. Let $R$ be a commutative ring, $D$, $M$ and $N$ be $R$-modules, $\phi : M \to N$ be an $R$-module homomorphism and $1 \otimes \phi : D \otimes_R M \to D \otimes_R N$ be the homomorphism for which

   $(1 \otimes \phi)(d \otimes m) = d \otimes \phi(m)$.

   (a) Assume that $\phi$ is injective. Show that if $D$ is free and of finite rank, then $1 \otimes \phi$ is also injective. [The finite rank is not necessary, but we assume it here for simplicity.]
   (b) Show that the above statement is not true for an arbitrary $D$.

Part IV

1. Let $F$ be a field and $K/F$ be an algebraic extension. Show that if $R$ is a subring of $K$ with $F \subseteq R \subseteq K$, then $R$ is a field.

2. Let $F$ be a field, $K/F$ be a Galois extension and $f \in F[x]$ be monic, separable and irreducible. Show that if $f = f_1 \cdots f_k$ is the factorization of $f$ in $K[x]$, with $f_i$ irreducible and monic, then the $f_i$'s are distinct, of the same degree and $G \overset{\text{def}}{=} \text{Gal}(K/F)$ acts transitively on $\{f_1, \ldots, f_k\}$. [i.e., given $\sigma \in G$, the map $f_i \mapsto f_i^\sigma$ is a permutation of the $f_i$'s and given $i, j \in \{1, \ldots, k\}$, there is a $\tau \in G$ such that $f_i^\tau = f_j$.]
Attempt all four parts. Justify your answers.

Part I.

1. Show that a group of order 455 is necessarily cyclic.

2. Let $G$ be a group of order 56. Show that $G$ is solvable.

Part II.

1. Let $f : \mathbb{Q} \to \mathbb{Z}$ be a function such that $f(ab) = f(a)f(b)$ for all $a, b \in \mathbb{Q}$. Show that the image of $f$ has at most three elements and there exist an infinite number of such functions whose image has three elements.

2. Let $R$ be a PID and let $J$ denote the intersection of all maximal ideals of $R$. If $a^2 - a$ is in $J$ for all $a \in R$, then show that $R$ has only finitely many maximal ideals.

Part III.

Note: Rings are assumed to be commutative and with $1 \neq 0$ and modules are assumed to be unitary.

1. Let $R$ be an integral domain and let $M, N$ be projective $R$-modules. Show that $M \otimes_R N$ is a projective $R$-module.

2. Suppose $R$ is a principal ideal domain that is not a field. Suppose $M$ is a finitely generated $R$-module such that for every maximal ideal $P$ of $R$, $M/PM$ is a cyclic $R/P$-module. Show that $M$ itself is cyclic.

Part IV.

1. Let $f(X)$ be a monic polynomial of degree 9 having rational coefficients. Assume that $f(X)$ is irreducible in $\mathbb{Q}[X]$. Let $K$ denote the splitting field of $f$ over $\mathbb{Q}$ and let $u \in K$ be a root of $f$. If $[K : \mathbb{Q}] = 27$, then show that $\mathbb{Q}[u]$ has a subfield $L$ with $[L : \mathbb{Q}] = 3$.

2. Let $F, K$ be fields such that $K$ is a finite Galois extension of $F$ with Galois group $G$. Suppose $a \in K$ is such that $\sigma(a) - a \in F$ for all $\sigma \in G$. If the characteristic of $F$ does not divide the order of $G$, then show that $a \in F$. Assuming $F$ to be the field of two elements, construct a quadratic field extension $K := F[a]$ of $F$ such that $\sigma(a) - a \in F$ for all $\sigma \in G$. 
Algebra Preliminary Exam

January 2011

Attempt all problems and justify all your answers. All rings have an identity 1 ≠ 0, all ring homomorphisms send 1 to 1, and all R-modules are unitary.

I. 1. Let G be a finite simple group. Show that if G has a subgroup H with \([G:H] = n \geq 2\), then \(|H|!(n - 1)!\).

2. List, up to isomorphism, all groups of order 153. Justify your answer.

II. 1. Let R be a commutative ring and I an ideal of R. Let \(I^* = \langle I, X \rangle\) be an ideal of the polynomial ring \(R[X]\). Determine, in terms of I, when \(I^*\) is a prime ideal of \(R[X]\) and when \(I^*\) is a maximal ideal of \(R[X]\). Justify your answers.

2. (a) Show that if a commutative ring R satisfies DCC on ideals (i.e., R is Artinian), then R has only a finite number of maximal ideals.

(b) Give an example to show that (a) may be false if DCC is replaced by ACC (i.e., if R is Noetherian).
III. 1. Let \( f: M \to M \) be an \( R \)-module homomorphism with \( f \cdot f = f \).

Show that the following statements are equivalent.

(a) \( f \) is injective.

(b) \( f \) is surjective.

(c) \( f = 1_M \).

2. (a) Let \( G \) and \( H \) be finitely generated abelian groups such that \( \mathbb{Z}_n \otimes G \cong \mathbb{Z}_n \otimes H \) for every integer \( n \geq 2 \). Show that \( G \cong H \).

(b) Give an example to show that (a) may be false if \( G \) and \( H \) are not both finitely generated.

IV. 1. Let \( F \) be a subfield of a field \( L \). Show that \( L/F \) is an algebraic extension if and only if every subring \( R \) of \( L \) containing \( F \) is a field.

2. Compute the Galois group of \( f(X) = X^4 + X + 1 \in \mathbb{Z}_2[X] \).
ALGEBRA PRELIMINARY EXAMINATION
Fall 2011

Attempt all four parts. Justify your answers.

Part I.

1. How many Sylow 2-subgroups does $S_5$ (the group of permutations of \{1, 2, 3, 4, 5\}) have?

2. Let $G$ be a group of order 231. Show that $G$ is Abelian if and only if $G$ has an Abelian subgroup of order 21.

Part II.

Note: Rings are assumed to be commutative and with $1 \neq 0$, subrings are assumed to contain 1 and ring-homomorphisms are assumed to map 1 to 1.

1. Let $R$ be a UFD such that each maximal ideal of $R$ is a principal ideal. Prove that $R$ is a PID.

2. Let $\mathbb{R}[[X]]$ denote the power-series ring in a single indeterminate $X$ over the field of real numbers $\mathbb{R}$. If $T$ is a multiplicative subset of $\mathbb{R}[[X]]$ containing 1 but not containing 0, then show that either $T^{-1}\mathbb{R}[[X]] = \mathbb{R}[[X]]$ or $T^{-1}\mathbb{R}[[X]]$ is a field.

Part III.

Note: Rings are assumed to be commutative and with $1 \neq 0$ and modules are assumed to be unitary.

1. Let $R$ be an integral domain and $I$ an ideal of $R$. Show that there exists a surjective $R$-module homomorphism $f : I \rightarrow R$ if and only if $I$ is a nonzero principal ideal.

2. Let $K$ be a field, $X$ an indeterminate over $K$ and $M$ a finitely generated $K[X]$-module. Show that $M$ is a projective $K[X]$-module if and only if $M$ is $K[X]$-module isomorphic to $K[X] \otimes_K V$ for some finite dimensional $K$-vector space $V$.

Part IV.

1. Let $K$ be a field and $F$ a subfield of $K$. The group of units of $K$ is denoted by $K^\times$. Suppose $f \in F[X]$ is a monic irreducible polynomial and $a, b \in K^\times$ are such that $f(a) = 0 = f(b)$. Show that the subgroup of $K^\times$ generated by $a$, is isomorphic to the subgroup of $K^\times$ generated by $b$.

2. Let $f \in \mathbb{Q}[X]$ be a polynomial of degree 4 such that the Galois group of $f$ (over $\mathbb{Q}$) is a group of order 6. Show that $f$ has a root in $\mathbb{Q}$.
Algebra Preliminary Exam
August 2010

Attempt all problems and justify all answers. All rings have an identity 1 ≠ 0, ring homomorphisms send 1 to 1, and all R-modules are unitary.

I. 1. Let f : G → H be a surjective homomorphism of finite groups and y ∈ H with |y| = n. Show that there is an x ∈ G with |x| = n.

2. Let p and q be primes, p ≥ q, n ≥ 1, and G a group with |G| = p^n q. Show that G has a normal subgroup H of order p^n. (Hint: do the p > q and p = q cases separately.)

II. 1. Let R be a commutative ring with distinct prime ideals P and Q with P ∩ Q = {0}. Show that R is isomorphic to a subring of the direct product of two fields.

2. Let p and q be positive primes. Show that the polynomial f(X) = X^3 + pX^2 + q ∈ ℤ[X] is irreducible in ℚ[X].

III. 1. Let A and B be finite abelian groups with |A| = m and |B| = n. Show that Homℤ(A, B) = 0 if and only if gcd(m, n) = 1.

2. Let A be a submodule of a projective R-module B. Show that A is projective if B/A is projective.
IV. 1. Let $K \subseteq F$ and $K \subseteq L$ be subfields of a field $M$ with $[F:K] = p$ and $[L:K] = q$ for distinct primes $p$ and $q$. Show that $F \cap L = K$, and that $F = K(\alpha)$ and $L = K(\beta)$ for any $\alpha \in F - K$ and $\beta \in L - K$.

2. Let $K$ be a field and $f(X) \in K[X]$ be irreducible and separable with $\deg(f(X)) = n$. Show that if the Galois group $G$ of $f(X)$ over $K$ is abelian, then $|G| = n$. 