Analysis Diagnostic Exam, May 6, 2020

The exam is closed book/closed notes. Sign the honor statement and submit it with your exam solutions:

As a student of the University of Tennessee, I pledge that I will neither knowingly give nor receive any inappropriate assistance in academic work, thus affirming my own personal commitment to honor and integrity.

1. (15) From the $\varepsilon - \delta$ definition of limit, show that $\lim_{x\to 4} \frac{4x-10}{x-1} = 2$.

2. (15) Let A, X, Y be sets with $A \subseteq X$, and let $f : X \to Y$ be a function.

Prove: If f is 1-1, then $A = f^{-1}(f(A))$.

3. (10) Let S be a set and let $\{A_\lambda\}_{\lambda\in S}$ be a collection of subsets of the real line $\mathbb{R}$. Prove that if for each $\lambda \in S$ the set A_λ is open in $\mathbb{R}$, then $\bigcup_{\lambda\in S} A_\lambda$ is open in $\mathbb{R}$.

4. (15) Let

$$f(x) = \begin{cases} x^2 \sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Prove that f is differentiable at 0, but that f' is not continuous at 0. You may use all elementary calculus facts about sine and cosine.

5. (10) Let $f : [2,5] \to \mathbb{R}$ be continuous and assume there is a sequence of points $x_n \in [2,5]$ such that $f(x_n) = 4 - \frac{1}{n}$ for each $n \in \mathbb{N}$. Show that there is $x \in [2,5]$ such that $f(x) = 4$.

6. (10) Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of real numbers and suppose that $\lim_{n\to\infty} x_n = L \in \mathbb{R}$. Use the open cover definition of compactness (not the Bolzano-Weierstrass or Heine Borel theorems) to show that the set

$$K = \{x_n : n \in \mathbb{N}\} \cup \{L\}$$

is compact.

7. (15) Suppose that $A \subseteq \mathbb{R}$ and $f_n, g_n, f, g : A \to \mathbb{R}$ are functions such that $f_n \to f$ and $g_n \to g$ uniformly on A.

(a) Show that if f and g are bounded, then $f_n g_n \to fg$ uniformly on A.

(b) Is (a) still true without the hypothesis that f and g are bounded? Prove or disprove.

8. (10) Recall that a bounded function $f : [a, b] \to \mathbb{R}$ is Riemann integrable, if and only if for every $\varepsilon > 0$ there is a partition P of $[a, b]$ such that $U(f, P) - L(f, P) < \varepsilon$. Here $U(f, P)$ and $L(f, P)$ are the upper and lower sums of f over P.

Prove: If $f : [a, b] \to \mathbb{R}$ is continuous, then f is Riemann integrable.