Analysis Diagnostic Exam, August 10, 2020

1. (12) Prove directly from the definition of limit of a sequence (that is without use of limit theorems) that $\lim_{n\to\infty} \frac{8n^2+10}{n^2-2020} = 8$.

2. (12) Let A, B, X, Y be sets with $A, B \subseteq X$, and let $f : X \to Y$ be a function.

Prove: If f is 1-1, then $f(A \setminus B) = f(A) \setminus f(B)$.

3. (12) Prove: If $A \subseteq \mathbb{R}$, then the boundary of A is closed.

4. (12) Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ such that f is differentiable at 0 and f is not continuous at any point $x \neq 0$.

5. (12) Let $A \subseteq \mathbb{R}$ be a compact set, and let $f : A \to \mathbb{R}$ be continuous. Show that $f(A)$ is compact.

6. (a) (4) Give an example of an unbounded continuous function $f : (0,1) \to \mathbb{R}$. You don't need to prove that your function is continuous or unbounded, it just needs to be correct.

(b) (12) Prove: If $f : (0,1) \to \mathbb{R}$ is uniformly continuous, then f must be a bounded function.

7. (12) Suppose that $f_n, f : \mathbb{R} \to \mathbb{R}$ are functions such that $f_n \to f$ uniformly on $\mathbb{R}$.

Show that if each f_n is continuous, then f is continuous.

8. (12) Recall that a bounded function $f : [a,b] \to \mathbb{R}$ is Riemann integrable, if and only if for every $\varepsilon > 0$ there is a partition P of $[a,b]$ such that $U(f,P) - L(f,P) < \varepsilon$. Here $U(f,P)$ and $L(f,P)$ are the upper and lower sums of f over P.

Let

$$f(x) = \begin{cases} \sin\frac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$$

Prove: f is Riemann integrable on $[0,1]$. You may use all elementary calculus facts about sine.