Analysis Diagnostic Exam August 13, 2018

NAME: ______________________________

#1.) ____/15 #2.) ____/15 #3.) ____/15 #4.) ____/20 #5.) ____/15 #6.) ____/20

Total: ______/

Instructions: There are 100 points possible on this exam. If you have any question about the notation or meaning of any question, please ask the exam proctor. You must show all necessary steps to get full credit. Partial credit will only be given for progress toward a correct solution.

1.) (15 points) Let A and B be sets. Prove that $(A \setminus B) \cup (B \setminus A) = (A \cup B) \setminus (A \cap B)$.

2.) (15 points) Prove directly from the definition of the limit of a sequence (that is, without using any limit theorems) that

$$\lim_{n\to\infty} \frac{4n+2}{3n-5} = \frac{4}{3}.$$

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3.) (15 points) Using only the definition of open sets (i.e., $O \subseteq \mathbb{R}$ is open if, for each $x \in O$, there exists $\epsilon > 0$, where ϵ may depend on x, such that $(x - \epsilon, x + \epsilon) \subseteq O$), prove that the interval $(-1, 1)$ is open.

4.) (20 points) Suppose $f : (0, 1) \to \mathbb{R}$ is uniformly continuous. Prove that there exists a function $g : [0, 1] \to \mathbb{R}$ such that g is continuous and $g(x) = f(x)$ for all $x \in (0, 1)$ (i.e., g is an extension of f). You can assume the Bolzano-Weierstrass theorem and the fact that f is bounded. It is enough to give the argument at one endpoint and say that the argument for the other endpoint is similar.

5.) (15 points) Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = \begin{cases} x^4 \sin\left(\frac{1}{x^3}\right) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$. Determine whether f is differentiable at 0. If f is differentiable at $x = 0$, determine whether the derivative function is continuous at 0. You can assume the usual calculus facts about the functions $\sin x$ and $\cos x$.

6.) (20 points) Suppose $a, b \in \mathbb{R}$ with $a < b$. Suppose $f : [a, b] \to \mathbb{R}$ is increasing: if $x, y \in [a, b]$ and $x < y$ then $f(x) \leq f(y)$. Prove that f is Riemann integrable on $[a, b]$.