LINEAR ALGEBRA DIAGNOSTIC TEST–May 6, 2020 (Online)

All vector spaces assumed to be finite-dimensional, over the complex numbers unless stated otherwise.

1 . Let $T \in \mathcal{L}(V, W)$. Show there exists a subspace $U \subset V$ such that:

$$U \cap null(T) = \{0\} \text{ and } ran(T) = \{Tu; u \in U\}.$$

2. Let $T \in \mathcal{L}(\mathbb{C}^4)$ be a linear map such that $2, 4$, and 8 are eigenvalues of T . Suppose also that T does not have a diagonal matrix with respect to any basis of $\mathbb{C}^4$. Show that T is invertible.

3. Let A be an $n \times n$ matrix with complex entries, such that the entries in each column add up to 1. Prove that 1 is an eigenvalue of A.

4. Consider the quadratic form in $\mathbb{R}^3$:

$$q(v) = x^2 + 10xz + z^2, \quad v = (x, y, z) \in \mathbb{R}^3.$$

Find the maximum and minimum values of q over the unit sphere in $\mathbb{R}^3$.

5. Let $T \in \mathcal{L}(V)$ be self-adjoint (where V is a real or complex inner product space). Suppose there exists $v \in V$ non-zero so that:

$$||Tv - 5v|| < 10||v||.$$

Prove there exists an eigenvalue of T in the interval $(-5, 15) \subset \mathbb{R}$.

6. Let $\mathcal{P}_m(\mathbb{C})$ denote the $\mathbb{C}$–vector space consisting of all polynomials in x of degree at most m, with coefficients in $\mathbb{C}$. Suppose that $\{p_0, p_1, \ldots, p_m\}$ are elements of $\mathcal{P}_m(\mathbb{C})$ such that $p_j(9) = 0$ for $j = 0, 1, \ldots, m$. Show that $\{p_0, p_1, \ldots, p_m\}$ is not linearly independent in $\mathcal{P}_m(\mathbb{C})$.

7. Let $T \in \mathcal{L}(\mathbb{R}^3)$ be given by:

$$T(x) = (2x_2, 3x_3, 0).$$

(i) Show that T is nilpotent.

(ii) Let A be the matrix of T in the standard basis of $\mathbb{R}^3$. Compute the exponential matrix $e^{4\mathbb{I}+A}$.

8. Give an example of two 5×5 matrices A, B which have the same characteristic polynomial and same minimal polynomial but are not similar (that is, there is no invertible matrix P such that $B = P^{-1}AP$). State the minimal and characteristic polynomials for your examples, and explain why the matrices are not similar.