Algebra Diagnostic Exam: Sample Questions

All vector spaces are assumed to be finite dimensional.

Rationale. The aim is to produce questions on topics listed at *http://www.math.utk.edu/diagnostic*. The level of sophistication is mandated to be higher than that of 200-courses, and the questions are to test the ability of students to construct a short proof. The envelope has been pushed slightly in both directions: one or two of the questions might be on the easy side, and one or two might be on the verge of being a little hard.

The accompanying document contains solutions to the questions.

1. Let W_1, W_2 be subspaces of a vector space V, such that neither of W_1, W_2 is contained in the other. Show that $W_1 \cup W_2$ is not a subspace of V.

2. Let W_1, W_2 be finite dimensional subspaces of a vector space V, such that $W_1 \cap W_2 = \{0\}$. Let $W_1 + W_2$ denote the subspace $\{w_1 + w_2\} \mid w_1 \in W_1, w_2 \in W_2\}$. Show that $\dim(W_1 + W_2) = \dim(W_1) + \dim(W_2)$.

3. Suppose that $\{v_1, \ldots, v_m\}$ is a linearly independent subset of V, and that w is a vector in V. Show that if $\{v_1 + w, \ldots, v_m + w\}$ is linearly dependent, then w is a linear combination of $v_1, \ldots, v_m$.

4. Prove or give a counterexample: if $\{v_1, v_2, v_3, v_4\}$ is a basis of V and U is a subspace of V such that $v_1, v_2 \in U$ and $v_3, v_4 \notin U$, then $\{v_1, v_2\}$ is a basis of U.

5. Let V be a finite dimensional vector space of dimension at least 2. Formulating your answer without using matrices, show that there exist linear maps S, T from V to V with $ST \neq TS$.

6. Let V_1, V_2 be vector spaces of dimensions $4, 5$ respectively. Give an example of a linear map $T : V_1 \to V_1$ whose range (image) is equal to its nullspace (kernel), and show that there does not exist a linear map $T : V_2 \to V_2$ whose range is equal to its nullspace.

7. Let $T : V \to W$ be a linear map of vector spaces, and let $v_1, \ldots, v_m$ be elements of V such that the subset $\{T(v_1), \ldots, T(v_m)\}$ of W is linearly independent. Show that $\{v_1, \ldots, v_m\}$ is linearly independent.

8. Let V be a vector space, and let $S, T : V \to V$ be linear maps satisfying $\text{range}(S) \subseteq \text{nullspace}(T)$. Show that $(ST)^2 = 0$, and give an example of such linear maps for which $ST \neq 0$.

9. Let V be a real vector space, and let φ_1, φ_2 be linear maps from V to $\mathbb{R}$ that have the same nullspace. Show that there exists $c \in \mathbb{R}$ such that $\varphi_1 = c\,\varphi_2$.

10. Let $T : V \to V$ be a linear map, and let $v_1 , v_2 , \ldots , v_m$ be eigenvectors of T whose corresponding eigenvalues $\lambda_1 , \lambda_2 , \ldots , \lambda_m$ are all distinct. Show that the set $\{v_1 , v_2 , \ldots , v_m\}$ is linearly independent. (*Hint:* Assume linear dependence; let k be the first index for which v_k is a linear combination of the previous v_i , and obtain a contradiction.)

11. Let $S, T : V \to V$ be linear maps that commute, *i.e.* $ST = TS$. Show that both the nullspace (kernel) of S and the range (image) of S are invariant under T .

12. Let $S, T : V \to V$ be linear maps. Show that ST , TS have the same eigenvalues.

13. Let $v_1 , \ldots , v_m$ be linearly independent vectors in a finite dimensional vector space V . Show that there exists a linear map $T : V \to V$ for which $v_1 , \ldots , v_m$ are eigenvectors of T with distinct eigenvalues.

14. Let $T : V \to V$ be a linear map satisfying $T^n = I$ for some positive integer n . Show that V has a basis with respect to which the matrix for T is diagonal. (*Hint:* Consider the Jordan Canonical Form for T .)

15. Let P be a linear map from the finite dimensional vector space V to itself such that $P^2 = P$. Prove (i) $\text{nullspace}(P) \cap \text{image}(P) = \{0\}$ and (ii) V has a basis with respect to which the matrix for P is diagonal, with entries all 0 or 1 .

16. Let A be an $n \times n$ matrix over the reals such that the diagonal entries are all positive, the off-diagonal entries are all negative and the row sums are all positive. Show that $\det A \neq 0$.
(*Hint:* Consider the homogeneous linear system $AX = 0$.)

17. A linear map $T : \mathbb{C}^3 \to \mathbb{C}^3$ is defined by $T(z_1, z_2, z_3) = (z_2, z_3, 0)$. Show that T does not have a square root, *i.e.* show that there does not exist a linear map $S : \mathbb{C}^3 \to \mathbb{C}^3$ such that $S^2 = T$.

18. Let $\mathcal{P}_m(\mathbb{C})$ denote the $\mathbb{C}$ –vector space consisting of all polynomials in x of degree at most m , with coefficients in $\mathbb{C}$. Suppose that $p_0, p_1, \ldots, p_m$ are elements of $\mathcal{P}_m(\mathbb{C})$ such that $p_j(2) = 0$ $(0 \leq j \leq m)$. Show that $\{p_0, p_1, \ldots, p_m\}$ is not linearly independent in $\mathcal{P}_m(\mathbb{C})$.

19. Suppose that $\{v_1, \ldots, v_m\}$ is a linearly independent subset of the vector space V , and that $w \in V$. Show that the span of $\{v_1 + w , \ldots , v_m + w\}$ has dimension at least $m - 1$.

20. Suppose that $T : \mathbb{R}^3 \to \mathbb{R}^3$ is a linear map and that -4 , 5 , $\sqrt{7}$ are eigenvalues of T . Show that there exists $x \in \mathbb{R}^3$ such that $Tx - 9x = (-4 , 5 , \sqrt{7})$.

21. Suppose that $T : V \to V$ is a linear map such that $T^2 = I$ and suppose that -1 is not an eigenvalue of T. Show that $T = I$.

22. Let $T : \mathbb{C}^3 \to \mathbb{C}^3$ be a linear map such that 6 and 7 are eigenvalues of T. Suppose also that T does not have a diagonal matrix with respect to any basis of $\mathbb{C}^3$. Show that T is invertible.

23. Let V be a vector space of dimension n over the complex numbers, and let $T : V \to V$ be a linear map with eigenvalues 5 and 6, and with no other eigenvalues. Show that
$(T - 5I)^{n-1}(T - 6I)^{n-1} = 0$.

24. The following matrix M is given:

$$M = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix} , \quad \lambda \neq 0 .$$

Find the Jordan canonical form of M^2, explaining your result.

25. The following matrix M is given:

$$M = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} .$$

Find the Jordan canonical form of M^2, explaining your result.

26. Suppose that V is a complex vector space and that $T : V \to V$ is an invertible linear map. Show that there exists a polynomial p with complex coefficients such that $T^{-1} = p(T)$.

27. Let V be a complex vector space of dimension $n > 0$ and let $I : V \to V$ be the identity map. Either find linear maps $A, B : V \to V$ such that $AB - BA = -I$, or show that no such A, B exist.

28. Let $T : V \to V$ be non-singular, and let $W \subseteq V$ be an eigenspace of T. Show that W is an eigenspace of T^{-1}.