Leader-Follower Stochastic Differential Game with Asymmetric Information and Applications

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1. A motivation

**Example: Continuous Time Newsvendor Problems**

$D(\cdot)$ is the demand rate

\[
\begin{cases}
    dD(t) = a(\mu - kR(t) - D(t))dt + \sigma dW(t) + \tilde{\sigma} d\tilde{W}(t), \\
    D(0) = d_0 \in \mathbb{R},
\end{cases}
\]

retail price $R(t)$. 
The retailer will obtain an expected profit

\[ J_1(q(\cdot), R(\cdot), w(\cdot)) \]

\[ = \mathbb{E} \int_0^T \left[ (R(t) - S) \min[D(t), q(t)] 
- (w(t) - S)q(t) \right] dt. \]

order rate \( q(t) \); wholesale price \( w(t) \); salvaged at unit price \( S \geq 0 \).
the manufacturer has a fixed production cost per unit $M \geq 0$, an expected profit

$$J_2(q(\cdot), R(\cdot), w(\cdot)) = \mathbb{E} \int_0^T (w(t) - M)q(t)dt.$$
The information $\mathcal{G}_t^1, \mathcal{G}_t^2$ available to the retailer and the manufacturer at time $t$, respectively. $\mathcal{G}_t^1 \neq \mathcal{G}_t^2$.

For any $w(\cdot)$, retailer selects a $\mathcal{G}_t^1$-adapted processes pair $(q^*(\cdot), R^*(\cdot))$ for the retailer such that

\[
J_1(q^*(\cdot), R^*(\cdot), w(\cdot))
\equiv J_1(q^*(\cdot; w(\cdot)), R^*(\cdot; w(\cdot)), w(\cdot))
= \max_{q(\cdot), R(\cdot)} J_1(q(\cdot), R(\cdot), w(\cdot)),
\]
Manufacturer selects a $G_t^2$-adapted process $w^*(\cdot)$ for the manufacturer such that

$$J_2(q^*(\cdot), R^*(\cdot), w^*(\cdot))$$

$$\equiv J_2(q^*(\cdot; w^*(\cdot)), R^*(\cdot; w^*(\cdot)), w^*(\cdot))$$

$$= \max_{w(\cdot)} J_2(q^*(\cdot; w(\cdot)), R^*(\cdot; w(\cdot)), w(\cdot)),$$
Example: Cooperative Advertising and Pricing Problems

\[
\begin{aligned}
\frac{dx(t)}{dt} &= \left[ \rho u(t) \sqrt{1 - x(t)} - \delta x(t) \right] dt + \sigma(x(t)) dW(t) \\
&\quad + \tilde{\sigma}(x(t)) d\tilde{W}(t), \\
x(0) &= x_0 \in [0, 1],
\end{aligned}
\]

\(u(t)\) advertisement effort rate, \(x(t)\) awareness share, \(\rho\) response constant, and \(\delta\) the rate at which potential consumers are lost.

Manufacturer decide:
wholesale price \(w(t)\), cooperative participation rate \(\theta(t)\).
Retailer decides: effort \(u(t)\), retail price \(p(t)\).
Given $w(t)$ and $\theta(t)$, retailer solves an optimization problem to maximize his expected profit

$$J_1(w(\cdot), \theta(\cdot), u(\cdot), p(\cdot)) = \mathbb{E} \int_0^T e^{-rt} \left[ (p(t) - w(t)) D(p(t)) x(t) - (1 - \theta(t)) u^2(t) \right] dt,$$

$D$ is demand function.
The manufacturer’s optimization problem is to maximize his expected profit

\[ J_2(w(\cdot), \theta(\cdot), u(\cdot), p(\cdot)) \]

\[ = \mathbb{E} \int_{0}^{T} e^{-rt} \left[ (w(t) - c)D(p(t))x(t) - \theta(t)u^2(t) \right] dt, \]

where \( c \geq 0 \) is the constant unit production cost.
The information $\mathcal{G}_t^1, \mathcal{G}_t^2$ available to the retailer and the manufacturer at time $t$, respectively. \( \mathcal{G}_t^1 \neq \mathcal{G}_t^2 \).

Given $w$ and $\theta$, retailer chooses $\mathcal{G}_{1,t}$-adapted pair $(u^*(\cdot), p^*(\cdot))$,

\[
J_1(w(\cdot), \theta(\cdot), u^*(\cdot), p^*(\cdot)) = \max_{u(\cdot), p(\cdot) \geq 0} J_1(w(\cdot), \theta(\cdot), u(\cdot), p(\cdot)).
\]
Select a $G_{2,t}$-adapted pair $(w^*(\cdot), \theta^*(\cdot))$ for the manufacturer

$$J_2(w^*(\cdot), \theta^*(\cdot), u^*(\cdot), p^*(\cdot))$$

$$\equiv J_2(w^*(\cdot), \theta^*(\cdot), u^*(\cdot; (w^*(\cdot), \theta^*(\cdot))),$$

$$p^*(\cdot; (w^*(\cdot), \theta^*(\cdot)))$$

$$= \max_{w(\cdot), 0 \leq \theta(\cdot) \leq 1} J_2(w(\cdot), \theta(\cdot), u^*(\cdot; (w(\cdot), \theta(\cdot))),$$

$$p^*(\cdot; (w(\cdot), \theta(\cdot)))$$,
2. Problem formulation

State equation:

\[
\begin{aligned}
    dx^{u_1,u_2}(t) &= b(t, x^{u_1,u_2}(t), u_1(t), u_2(t)) \, dt \\
    &\quad + \sigma(t, x^{u_1,u_2}(t), u_1(t), u_2(t)) \, dW(t) \\
    &\quad + \tilde{\sigma}(t, x^{u_1,u_2}(t), u_1(t), u_2(t)) \, d\tilde{W}(t), \\
    x^{u_1,u_2}(0) &= x_0,
\end{aligned}
\]  

(2.1)
$G^1_t \neq G^2_t \subseteq \mathcal{F}_t$.

The admissible control

$$\mathcal{U}_1 := \left\{ u_1 \mid u_1 : \Omega \times [0, T] \to U_1 \text{ is } G^1_t\text{-adapted} \right\},$$

and $\sup_{0 \leq t \leq T} \mathbb{E}|u_1(t)|^i < \infty, \ i = 1, 2, \cdots$, \hspace{1cm} (2.2)

$$\mathcal{U}_2 := \left\{ u_2 \mid u_2 : \Omega \times [0, T] \to U_2 \text{ is } G^2_t\text{-adapted} \right\},$$

and $\sup_{0 \leq t \leq T} \mathbb{E}|u_2(t)|^i < \infty, \ i = 1, 2, \cdots$. \hspace{1cm} (2.3)
Problem of the follower. For any chosen $u_2(\cdot) \in \mathcal{U}_2$ by the leader, choose a $\mathcal{G}_t^1$-adapted control $u_1^*(\cdot) = u_1^*(\cdot; u_2(\cdot)) \in \mathcal{U}_1$, such that

$$J_1(u_1^*(\cdot), u_2(\cdot)) \equiv J_1(u_1^*(\cdot; u_2(\cdot)), u_2(\cdot)) = \inf_{u_1 \in \mathcal{U}_1} J_1(u_1(\cdot), u_2(\cdot)), \tag{2.4}$$

$$u_1^*(\cdot) = u_1^*(\cdot; u_2(\cdot)), \text{ (partial information) optimal control.}$$
The leader would like to choose a $\mathcal{G}_t^2$-adapted control $u_2^*(\cdot)$ to minimize

$$J_2(u_1^*(\cdot), u_2(\cdot)) = \mathbb{E} \left[ \int_0^T g_2(t, x_1^{u_1^*, u_2}(t), u_1^*(t; u_2(t)), u_2(t)) \, dt \right] + G_2(x_1^{u_1^*, u_2}(T)) = (2.5)$$
Problem of the leader. Find a $\mathcal{G}_t^2$-adapted control $u_2^*(\cdot) \in \mathcal{U}_2$, such that

\[
J_2(u_1^*(\cdot), u_2^*(\cdot)) = J_2(u_1^*(\cdot; u_2^*(\cdot)), u_2^*(\cdot)) \\
= \inf_{u_2 \in \mathcal{U}_2} J_2(u_1^*(\cdot; u_2(\cdot)), u_2(\cdot)), \tag{2.6}
\]
Issacs (1954-5) differential games
Basar and Olsder (1982) Monograph to summarize
Stackelberg (1934) Leader-follower game introduced
Yong (2002) LQ, complete info.
3. LQ Leader-Follower Stochastic Differential Game with Asymmetric Information; Stochastic maximum principle under partial information.

The state equation

\[
\begin{align*}
    dx_{u_1,u_2}(t) &= \left[ A(t)x_{u_1,u_2}(t) + B_1(t)u_1(t) + B_2(t)u_2(t) \right] dt \\
    &+ \left[ C(t)x_{u_1,u_2}(t) + D_1(t)u_1(t) + D_2(t)u_2(t) \right] dW(t) \\
    &+ \left[ \tilde{C}(t)x_{u_1,u_2}(t) + \tilde{D}_1(t)u_1(t) + \tilde{D}_2(t)u_2(t) \right] d\tilde{W}(t), \\
    x_{u_1,u_2}(0) &= x_0,
\end{align*}
\]
Given $u_2 \in \mathcal{U}_2$, follower minimizes his cost functional

$$J_1(u_1(\cdot), u_2(\cdot))$$

$$= \frac{1}{2} \mathbb{E} \left[ \int_0^T \left( \langle Q_1(t)x_{u_1,u_2}(t), x_{u_1,u_2}(t) \rangle ight. ight.$$  

$$+ \left. \langle N_1(t)u_1(t), u_1(t) \rangle \right) dt$$

$$+ \langle G_1 x_{u_1,u_2}(T), x_{u_1,u_2}(T) \rangle \right],$$  

(3.8)

Info: $\mathcal{G}_t^1 = \sigma\{\widetilde{W}(s); 0 \leq s \leq t\}$
Stochastic maximum principle under partial information.

Probability space \((\Omega, \mathcal{F}, \hat{P})\).

State equation: FBSDE

\[
\begin{cases}
\quad dx = b(t, x, v)dt + \sigma(t, x, v)dW + \tilde{\sigma}(t, x, v)d\tilde{W}, \\
-\quad dy = g(t, x, y, z, \tilde{z}, v)dt - zdW - \tilde{z}dY, \\
\quad x(0) = x_0, \quad y(1) = f(x(1)).
\end{cases}
\tag{3.9}
\]

Observation

\[
\begin{cases}
\quad dY = h(t, x)dt + d\tilde{W}, \\
\quad Y(0) = 0.
\end{cases}
\tag{3.10}
\]
Admissible controls $v \in \mathcal{A}$

$v$ is $\mathcal{G}_t$-adapted, where $\mathcal{G}_t = \mathcal{F}_t^Y$.

Control problem: Find $u \in \mathcal{A}$ s.t.

$$J(u) = \min_{v \in \mathcal{A}} J(v),$$

where

$$J(v) = \hat{E} \left( \int_0^1 \ell(t, x^v, y^v, z^v, \tilde{z}^v, v) dt + \phi(x^v(1)) + \gamma(y^v(0)) \right).$$
Hypothesis (H1):
- Coeff. conti. diff. w/ bounded 1st order partial derivatives.
- $\tilde{\sigma}$, $h$ bounded.

Hypothesis (H2):
- $\ell$, $\phi$, $\gamma$ conti. diff.

$$\hat{E}\left(\int_0^1 |\ell(t, x^v, y^v, z^v, \tilde{z}^v, v)| dt + |\phi(x^v(1))| + |\gamma(y^v(0))|\right) < \infty.$$
when $b = \sigma = \tilde{\sigma} = 0$, studied by Huang-Wang-Xiong (SICON 2009)
General form by Wang-Wu-Xiong (SICON 2013).
Detail on LQ case by Wang-Wu-Xiong (IEEE TAC 2015).
Replace \( b, \ W \) in state equation by \( \tilde{b}, \ Y \), resp, where

\[
\tilde{b} = b - \tilde{\sigma} h.
\]

State equation:

\[
\begin{aligned}
\begin{cases}
  dx &= \tilde{b}(t, x, v)dt + \sigma(t, x, v)dW + \tilde{\sigma}(t, x, v)dY, \\
  -dy &= g(t, x, y, z, \tilde{z}, v)dt - zdW - \tilde{z}dY,
\end{cases}
\end{aligned}
\]

\( x(0) = x_0, \ y(1) = f(x(1)) \).

(3.11)
Let
\[
Z^v(t) = \exp \left\{ \int_0^t h(s, x^v) dY - \frac{1}{2} \int_0^t h^2(s, x^v) ds \right\}.
\]

So
\[
dZ^v = Z^v h(t, x^v) dY. \tag{3.12}
\]

Let \( \mathbb{P} \sim \hat{\mathbb{P}} \) be s.t.
\[
d\hat{\mathbb{P}} = Z^v(1) d\mathbb{P}.
\]

Then, under \( \mathbb{P} \), \((W, Y)\) is a B.M.

\[
J(v) = \mathbb{E} \left( \int_0^1 Z^v \ell(t, x^v, y^v, z^v, \tilde{z}^v, v) dt + Z^v(1) \phi(x^v(1)) + \gamma(y^v(0)) \right).
\]
Hypothesis (H3):

- For $\xi$ bdd, $G_t$-measurable,

$$v(s) \equiv \xi1_{[t,t+\tau)}(s),$$

then $v \in \mathcal{A}$.

- For any $u \in \mathcal{A}$ and $v$ being $G_s$-adapted and bdd, $\exists \delta > 0$, if $|\epsilon| < \delta$, then $u + \epsilon v \in \mathcal{A}$. 
Adjoint equations:

\[
\begin{align*}
\frac{dp}{dt} &= (g_y(t, x, y, z, \tilde{z}, u)p - \ell_y(t, x, y, z, \tilde{z}, u)) \, dt \\
&\quad + (g_z(t, x, y, z, \tilde{z}, u)p - \ell_z(t, x, y, z, \tilde{z}, u)) \, dW \\
&\quad + (g_{\tilde{z}}(t, x, y, z, \tilde{z}, u)p - h(t, x)) \, d\tilde{W}, \\
-dq &= \left\{ (b_x(t, x, u) - \tilde{\sigma}(t, x, u)h_x(t, x))q + \sigma_x(t, x, u)k \\
&\quad + \tilde{\sigma}_x(t, x, u)k - g_x(t, x, y, z, \tilde{z}, u)p \\
&\quad - \ell_x(t, x, y, z, \tilde{z}, u) \right\} dt - kdW - \tilde{k}d\tilde{W}, \\
p(0) &= -\gamma_y(y(0)) \\
q(1) &= -f_x(x(1))p(1) + \phi_x(x(1)),
\end{align*}
\]
(3.13)
and

\[
\begin{align*}
-dP &= \ell(t, x, y, z, \tilde{z}, u)dt - QdW - \tilde{Q}d\tilde{W} \\
P(1) &= \phi(x(1)).
\end{align*}
\]  

(3.14)

Hamitonian

\[
H(t, x, y, z, \tilde{z}, v; p, q, k, \tilde{k}, \tilde{Q}) = bq + \sigma k + \tilde{\sigma} \tilde{k} + h\tilde{Q} + \ell - (g - h\tilde{z})p.
\]
Theorem

If $u$ is a local minimum for $J(\cdot)$, then

$$
\mathbb{E} \left( H_u(t, x, y, z, \tilde{z}, v; p, q, k, \tilde{k}, \tilde{Q}) \bigg| \mathcal{G}_t \right) = 0.
$$

Idea of proof Set

$$
\frac{d}{d\epsilon} J(u + \epsilon v) \bigg|_{\epsilon=0} = 0.
$$
Remark

- (3.14) is for $Z^v(t)$. If $h = 0$, this eq. is not needed.
- Suppose $h = b = \sigma = \tilde{\sigma}$, it is proved in Huang-Wang-Xiong (2008, SICON)
4. Follower’s optimization problem

Step 1. (Optimal control)

Hamiltonian function:

\[
H_1(t, x, u_1, u_2; q, k, \tilde{k}) = \langle q, A(t)x + B_1(t)u_1 + B_2(t)u_2 \rangle \\
+ \langle k, C(t)x + D_1(t)u_1 + D_2(t)u_2 \rangle \\
+ \langle \tilde{k}, \tilde{C}(t)x + \tilde{D}_1(t)u_1 + \tilde{D}_2(t)u_2 \rangle \\
- \frac{1}{2} \langle Q_1(t)x, x \rangle - \frac{1}{2} \langle N_1(t)u_1, u_1 \rangle.
\]
Then,
\[
0 = N_1(t)u_1^*(t) - B_1^\top(t)\mathbb{E}[q(t)|\mathcal{G}_t^1] - D_1^\top(t)\mathbb{E}[k(t)|\mathcal{G}_t^1] - \tilde{D}_1^\top(t)\mathbb{E}[\tilde{k}(t)|\mathcal{G}_t^1], \ a.e. t \in [0,T],
\]
where $\mathcal{F}_t$-adapted $(q(\cdot), k(\cdot), \tilde{k}(\cdot))$ satisfies the BSDE
\[
\begin{aligned}
-dq(t) &= \left[ A^\top(t)q(t) + C^\top(t)k(t) + \tilde{C}^\top(t)\tilde{k}(t) - Q_1(t)x^{u_1^*, u_2}(t) \right] dt \\
&\quad - k(t)dW(t) - \tilde{k}(t)d\tilde{W}(t), \\
q(T) &= - G_1 x^{u_1^*, u_2}(T).
\end{aligned}
\]
Step 2. (Optimal filtering)

Let

$$\hat{f}(t) := \mathbb{E}[f(t)|G^1_t], \quad f = q, k, \tilde{k}.$$ 

How to derive filtering equation? Direct from (4.17) is impossible.

Solve (4.17) first!

Guess:

$$q(t) = -P_1(t)x^{u_1^*, u_2}(t) - \varphi(t), \quad t \in [0, T],$$ 

(4.18)

Set

$$\left\{ \begin{array}{l}
    d\varphi(t) = \alpha(t)dt + \beta(t)d\tilde{W}(t), \\
    \varphi(T) = 0.
\end{array} \right.$$ 

(4.19)
Applying $dq(t)$ and comparing, we get

$$k(t) = -P_1(t) \left[ C(t)x_{u_1, u_2}(t) + D_1(t)u_1^*(t) + D_2(t)u_2(t) \right], \quad (4.20)$$

$$\tilde{k}(t) = - P_1(t) \left[ \tilde{C}(t)x_{u_1, u_2}(t) + \tilde{D}_1(t)u_1^*(t) + \tilde{D}_2(t)u_2(t) \right] - \beta(t), \quad (4.21)$$

$$\alpha(t) = \left[ - \dot{P}_1(t) - P_1(t)A(t) - A^\top(t)P_1(t) - Q_1(t) \right] x_{u_1, u_2}(t) - P_1(t)B(t)u_1^*(t) - P_1(t)B_2(t)u_2(t) - A^\top(t)\varphi(t) + C^\top(t)k(t) + \tilde{C}^\top(t)\tilde{k}(t). \quad (4.22)$$
Hence,

\[ \hat{q}(t) = -P_1(t) \hat{x}^{u_1^*, \hat{u}_2(t)} - \varphi(t), \]  
\[ \hat{k}(t) = -P_1(t) \left[ C(t) \hat{x}^{u_1^*, \hat{u}_2(t)} + D_1(t) u_1^*(t) + D_2(t) \hat{u}_2(t) \right], \]  
\[ \hat{k}(t) = -P_1(t) \left[ \tilde{C}(t) \hat{x}^{u_1^*, \hat{u}_2(t)} + \tilde{D}_1(t) u_1^*(t) \right. \]
\[ \left. + \tilde{D}_2(t) \hat{u}_2(t) \right] - \beta(t) \]  

with \( \hat{u}_2(t) := \mathbb{E}[u_2(t)|G_{1,t}] \).
Using filtering technique (Lemma 5.4 in Xiong (2008)), we get

\[
\begin{cases}
\hat{d}\hat{x}^{\hat{u}_1,\hat{u}_2}(t) \\
= \left[ A(t)\hat{x}^{\hat{u}_1,\hat{u}_2}(t) + B_1(t)\hat{u}_1(t) + B_2(t)\hat{u}_2(t) \right] dt \\
+ \left[ \tilde{C}(t)\hat{x}^{\hat{u}_1,\hat{u}_2}(t) + \tilde{D}_1(t)\hat{u}_1(t) + \tilde{D}_2(t)\hat{u}_2(t) \right] d\tilde{W}(t),
\end{cases}
\tag{4.26}
\]

\[
\hat{x}^{\hat{u}_1,\hat{u}_2}(0) = x_0,
\]

and

\[
\begin{cases}
-d\hat{q}(t) = \left\{ A^\top(t)\hat{q}(t) + C^\top(t)\hat{k}(t) + \tilde{C}^\top(t)\tilde{k}(t) \\
- Q_1(t)\hat{x}^{\hat{u}_1,\hat{u}_2}(t) \right\} dt - \hat{k}(t)d\tilde{W}(t),
\end{cases}
\tag{4.27}
\]

\[
\hat{q}(T) = - G_1\hat{x}^{\hat{u}_1,\hat{u}_2}(T),
\]
Step 3. (Optimal state feedback control)

By (4.16) we arrive at

\[ u_1^*(t) = -\tilde{N}_1(t)^{-1} \left[ \left( B_1^T(t)P_1(t) + D_1^T(t)P_1(t)C(t) \right. \right. \]
\[ \left. + \tilde{D}_1^T(t)P_1(t)\tilde{C}(t) \right) \hat{x}_{u_1,\hat{u}_2}(t) \]
\[ \left. + \left( D_1^T(t)P_1(t)D_2(t) + \tilde{D}_1^T(t)P_1(t)\tilde{D}_2(t) \right) \hat{u}_2(t) \right. \]
\[ \left. + B_1^T(t)\varphi(t) + \tilde{D}_1^T(t)\beta(t) \right] , \]

where

\[ \tilde{N}_1(t) := N_1(t) + D_1^T(t)P_1(t)D_1(t) + \tilde{D}_1^T(t)P_1(t)\tilde{D}_1(t) \tag{4.29} \]
Substituting (4.28) into (4.22), we obtain Riccati equation

\[
\begin{align*}
\dot{P}_1(t) + P_1(t)A(t) + A^\top(t)P_1(t) + C^\top(t)P_1(t)C(t) \\
+ \tilde{C}^\top(t)P_1(t)\tilde{C}(t) + Q_1(t) \\
- \left( P_1(t)B_1(t) + C^\top(t)P_1(t)D_1(t) \\
+ \tilde{C}^\top(t)P_1(t)\tilde{D}_1(t) \right) \tilde{N}_1^{-1}(t) \left( B_1^\top(t)P_1(t) \\
+ D_1^\top(t)P_1(t)C(t) + \tilde{D}_1^\top(t)P_1(t)\tilde{C}(t) \right) = 0,
\end{align*}
\]

\( P_1(T) = G_1. \)

Suppose it admits a unique differentiable solution \( P_1(\cdot) \)
\[ \alpha(t) = \left[ A^\top(t) - \tilde{S}_1(t)\tilde{N}_1(t)^{-1}B_1^\top(t) \right] \varphi(t) \]
\[ \quad + \left[ \tilde{C}^\top(t) - \tilde{S}_1(t)\tilde{N}_1(t)^{-1}\tilde{D}_1^\top(t) \right] \beta(t) \]
\[ \quad + \left[ -\tilde{S}_1(t)\tilde{N}_1(t)^{-1}\tilde{S}(t) + \tilde{S}_2(t) \right] \hat{u}_2(t), \]

which is \( G_t^1 \)-adapted, where

\[ \tilde{S}(t) := D_1^\top(t)P_1(t)D_2(t) + \tilde{D}_1^\top(t)P_1(t)\tilde{D}_2(t), \]
\[ \tilde{S}_1(t) := P_1(t)B_1(t) + \tilde{C}^\top(t)P_1(t)D_1(t) \]
\[ \quad + \tilde{C}^\top(t)P_1(t)\tilde{D}_1(t), \]
\[ \tilde{S}_2(t) := P_1(t)B_2(t) + \tilde{C}^\top(t)P_1(t)D_2(t) \]
\[ \quad + \tilde{C}^\top(t)P_1(t)\tilde{D}_2(t), t \in [0, T]. \]
Then,

\[
-d\varphi(t) = \begin{cases} 
    \left[ A^\top(t) - \tilde{S}_1(t)\tilde{N}_1(t)^{-1}B_1^\top(t) \right] \varphi(t) \\
    + \left[ \tilde{C}^\top(t) - \tilde{S}_1(t)\tilde{N}_1(t)^{-1}\tilde{D}_1^\top(t) \right] \beta(t) \\
    + \left[ -\tilde{S}_1(t)\tilde{N}_1(t)^{-1}\tilde{S}(t) + \tilde{S}_2(t) \right] \hat{u}_2(t) \\
    - \beta(t)d\tilde{W}(t), \\
    \varphi(T) = 0,
\end{cases}
\]

(4.32)

which admits a unique $\mathcal{G}_t^1$-adapted solution $(\varphi(\cdot), \beta(\cdot))$. 
Finally,

\[ d\hat{x}^{u_1^*, \hat{u}_2}(t) = \left[ \tilde{A}(t)\hat{x}^{u_1^*, \hat{u}_2}(t) + \tilde{F}_1(t)\varphi(t) \right. \]

\[ + \tilde{B}_1(t)\beta(t) + \tilde{B}_2(t)\hat{u}_2(t) \] \[ dt \]

\[ + \left[ \tilde{C}(t)\hat{x}^{u_1^*, \hat{u}_2}(t) + \tilde{B}_1^\top(t)\varphi(t) \right. \]

\[ + \tilde{F}_3(t)\beta(t) + \tilde{D}_2(t)\hat{u}_2(t) \] \[ d\tilde{W}(t), \] 

\[ \hat{x}^{u_1^*, \hat{u}_2}(0) = x_0, \]

solves \( \hat{x}^{u_1^*, \hat{u}_2} \), where
\[
\begin{align*}
\tilde{A}(t) &:= A(t) - B_1(t)\tilde{N}_1(t)^{-1}\tilde{S}_1^\top(t), \\
\tilde{B}_2(t) &:= B_2(t) - B_1(t)\tilde{N}_1(t)^{-1}\tilde{S}(t), \\
\tilde{F}_1(t) &:= -B_1(t)\tilde{N}_1(t)^{-1}B_1^\top(t), \\
\tilde{B}_1(t) &:= -B_1(t)\tilde{N}_1(t)^{-1}\tilde{D}_1^\top(t), \\
\tilde{C}(t) &:= \tilde{C}(t) - \tilde{D}_1(t)\tilde{N}_1(t)^{-1}\tilde{S}_1^\top(t), \\
\tilde{F}_3(t) &:= -\tilde{D}_1(t)\tilde{N}_1(t)^{-1}\tilde{D}_1^\top(t), \\
\tilde{D}_2(t) &:= \tilde{D}_2(t) - \tilde{D}_1(t)\tilde{N}_1(t)^{-1}\tilde{S}(t) \\
\tilde{F}_4(t) &:= -\tilde{S}_1(t)\tilde{N}_1(t)^{-1}\tilde{S}(t) + \tilde{S}_2(t).
\end{align*}
\]
Theorem 3.1  Let Riccati equation admit a differentiable solution $P_1(\cdot)$ and let matrix be convertible. For chosen $u_2(\cdot)$ of the leader, Problem of the follower admits an optimal control $u_1^*(\cdot)$ of the state feedback, where processes triple $(\hat{x}^u_1, \hat{u}_2(\cdot), \varphi(\cdot), \beta(\cdot))$ is the unique $\mathcal{G}^1_t$-adapted solution to the FBSDE (4.32)-(4.33).
5. Complete Information LQ Stochastic Optimal Control Problem of the Leader

The leader knows the complete information $\mathcal{F}_t$ at any time $t$. The state equations faced by the leader consist of BSDE (4.32) and the SDE (3.7). By (4.28), it can be written as
\[
\begin{align*}
\dot{x}^{u_2}(t) &= \left[ A(t)x^{u_2}(t) + (\tilde{A}(t) - A(t))\dot{x}^{\hat{u}_2}(t) + \tilde{F}_1(t)\varphi(t) \\
&\quad + \tilde{B}_1(t)\beta(t) + B_2(t)u_2(t) + (\tilde{B}_2(t) - B_2(t))\dot{u}_2(t) \right] dt \\
&\quad + \left[ C(t)x^{u_2}(t) + \tilde{F}_5(t)\dot{x}^{\hat{u}_2}(t) + \tilde{B}_1(t)^\top\varphi(t) + \tilde{D}_1(t)\beta(t) \\
&\quad + D_2(t)u_2(t) + \tilde{F}_2(t)\dot{u}_2(t) \right] dW(t) \\
&\quad + \left[ \tilde{C}(t)x^{u_2}(t) + (\tilde{C}(t) - \tilde{C}(t))\dot{x}^{\hat{u}_2}(t) + \tilde{B}_1(t)^\top\varphi(t) \\
&\quad + \tilde{F}_3(t)\beta(t) + \tilde{D}_2(t)u_2(t) + (\tilde{D}(t) - \tilde{D}(t))\dot{u}_2(t) \right] d\tilde{W}(t),
\end{align*}
\]
\[-d\varphi(t) = \left[ \tilde{A}^\top(t)\varphi(t) + \tilde{C}^\top(t)\beta(t) \\
&\quad + \tilde{F}_4(t)\dot{u}_2(t) \right] dt - \beta(t)d\tilde{W}(t),
\]
\[x^{u_2}(0) = x_0, \quad \varphi(T) = 0. \tag{5.34}\]
The leader’s cost functional

$$\tilde{J}_2(u_2(\cdot)) := J_2(u_1^*(\cdot), u_2(\cdot))$$

$$= \frac{1}{2} \mathbb{E} \left[ \int_0^T \left( \langle Q_2(t)x^{u_1^*, u_2}(t), x^{u_1^*, u_2}(t) \rangle + \langle N_2(t)u_2(t), u_2(t) \rangle \right) dt + \langle G_2x^{u_1^*, u_2}(T), x^{u_1^*, u_2}(T) \rangle \right]$$

(5.35)

This complete info conditional mean-field LQ problem with state (5.34) and cost (5.34) can be solved explicitly using a general result.
General conditional mean-field SMP. State

\[
\begin{align*}
    \frac{dx_{u2}(t)}{dt} &= b(t)dt + \sigma(t)dW(t) + \tilde{\sigma}(t)d\tilde{W}(t), \\
    -dq(t) &= \left\{ b_x(t)q(t) + \sum_{j=1}^{d_1} \sigma^j_x(t)k^j(t) \\
    &\quad + \sum_{j=1}^{d_2} \tilde{\sigma}^j_x(t)\tilde{k}^j(t) - g_{1x}(t) \right\} dt \\
    &= k(t)dW(t) - \tilde{k}(t)d\tilde{W}(t), \\
    x_{u2}(0) &= x_0, \quad q(T) = -G_{1x}(x_{u2}(T)),
\end{align*}
\]

where \(b\) is a function of \((t, x, \hat{x}, u_2, \hat{u}_2, q, k, \tilde{k}, \hat{k}, \tilde{k})\).
Cost:

\[ J_2(u_2(\cdot)) = \mathbb{E}\left[ \int_0^T g_2(t) dt + G_2(x^{u_2}(T)) \right] \]

**Problem of the leader.** Find a \( G^2_t \)-adapted control \( u_2^*(\cdot) \in \mathcal{U}_2 \), such that

\[ J_2(u_2^*(\cdot)) = \inf_{u_2 \in \mathcal{U}_2} J_2(u_2(\cdot)), \tag{5.37} \]

subject to (5.36).
Define Hamiltonian

\[ H_2(t, x, u_2, \hat{x}, \hat{u}_2, q, k, \tilde{k}; y, z, \tilde{z}, p) \]
\[ = \langle y, b() \rangle + \text{tr}\{z^\top \sigma(t)\} + \text{tr}\{\tilde{z}^\top \tilde{\sigma}(t)\} + g_2(t) \]
\[ - \langle p, b_x(t)q + \sum_{j=1}^{d_1} \sigma^j_x(t)k^j + \sum_{j=1}^{d_2} \tilde{\sigma}^j_x(t)\tilde{k}^j - g_{1x}(t) \rangle. \]
Let \((y(\cdot), z(\cdot), \tilde{z}(\cdot), p(\cdot))\) be the unique \(\mathcal{F}_t\)-adapted solution to the adjoint conditional mean-field FBSDE of the leader

\[
dp(t) = \left\{ b_x^*(t)p(t) + \mathbb{E} \left[ b_x^*(t)p(t) \middle| \mathcal{G}_t^1 \right] \right\} dt \\
+ \sum_{j=1}^{d_1} \left\{ \sigma_{x}^{*j}(t)p(t) + \mathbb{E} \left[ \sigma_{x}^{*j}(t)p(t) \middle| \mathcal{G}_t^1 \right] \right\} dW^j(t) \\
+ \sum_{j=1}^{d_2} \left\{ \tilde{\sigma}_{x}^{*j}(t)p(t) + \mathbb{E} \left[ \tilde{\sigma}_{x}^{*j}(t)p(t) \middle| \mathcal{G}_t^1 \right] \right\} d\tilde{W}^j(t),
\] (5.39)
\[-dy = \left\{ b_{xy} + \mathbb{E}[b_{\hat{x}y} | \mathcal{G}_t^1] + \sum_{j=1}^{d_1} \left[ \sigma^j_x z^j + \mathbb{E}[\sigma^j_{\hat{x}} z^j | \mathcal{G}_t^1] \right] \right. \\
+ \sum_{j=1}^{d_2} \left[ \tilde{\sigma}^j_x(t) z^j(t) + \mathbb{E}[\tilde{\sigma}^j_{\hat{x}}(t) z^j(t) | \mathcal{G}_t^1] \right] \\
- \sum_{i=1}^{n} \left\{ \frac{\partial b_x}{\partial x_i}(t) q(t)p_i(t) + \mathbb{E}\left[ \frac{\partial b_x}{\partial \hat{x}_i}(t) q(t)p_i(t) | \mathcal{G}_t^1 \right] \right\} \\
- \sum_{i=1}^{n} \left\{ \frac{\partial}{\partial x_i} \left( \sum_{j=1}^{d_1} \sigma^j_x k^j \right) p_i + \mathbb{E}\left[ \frac{\partial}{\partial \hat{x}_i} \left( \sum_{j=1}^{d_1} \sigma^j_{\hat{x}} k^j \right) p_i | \mathcal{G}_t^1 \right] \right\} \\
- \sum_{i=1}^{n} \left\{ \frac{\partial}{\partial x_i} \left( \sum_{j=1}^{d_1} \tilde{\sigma}^j_x \tilde{k}^j \right) p_i + \mathbb{E}\left[ \frac{\partial}{\partial \hat{x}_i} \left( \sum_{j=1}^{d_1} \tilde{\sigma}^j_{\hat{x}} \tilde{k}^j \right) p_i | \mathcal{G}_t^1 \right] \right\} \\
+ g_{1xx} p + \mathbb{E}\left[ g_{1\hat{x}x} p | \mathcal{G}_t^1 \right] + g_{2x} + \mathbb{E}\left[ g_{2\hat{x}} | \mathcal{G}_t^1 \right] \right\} dt - zdW - \tilde{z}d\tilde{W} ,\]
with boundary

\[ p(0) = 0, \]
\[ y(T) = G_{1xx}(x^*(T))p(T) + G_{2x}(x^*(T)), \]

(5.41)

where we have used \( \phi \equiv \phi(t, x^*(t), \hat{x}^*(t), u_2^*(t), \hat{u}_2^*(t)) \) for \( \phi = b, \sigma, \tilde{\sigma}, g_1, g_2 \) and all their derivatives.

Related work: Anderson and Djehiche (2011), Li (2012), Yong (2.13).
Proposition (Maximum principle of conditional mean-field FBSDE with partial information)

Let $u^*_2(\cdot) \in \mathcal{U}_2$ be the optimal control for Problem of the leader and $(x^*(\cdot), q^*(\cdot), k^*(\cdot), \tilde{k}^*(\cdot))$ be the corresponding optimal state which is the unique $\mathcal{F}_t$-adapted solution to (5.36). Let $(y(\cdot), z(\cdot), \tilde{z}(\cdot), p(\cdot))$ be the unique $\mathcal{F}_t$-adapted solution to the adjoint equation. Then

$$
\mathbb{E} \left\{ \frac{\partial H_2}{\partial u_2} + \mathbb{E} \left[ \frac{\partial H_2}{\partial \hat{u}_2} \bigg| \mathcal{G}^1_t \right] \bigg| u_2 = u^*_2(t) \right\} \bigg| \mathcal{G}^2_t \right\} = 0, \ a.e. t \in [0, T], \ (5.42)
$$
Now, we come back to LQ problem. Define Hamiltonian

\[
H_2(t, x^{u_2}, u_2, \varphi, \beta; y, z, \tilde{z}, p)
\]

\[
= \langle y, A(t)x^{u_2} + (\tilde{A}(t) - A(t))\hat{x}^{\hat{u}_2} + \tilde{F}_1(t)\varphi \\
+ \tilde{B}_1(t)\beta + B_2(t)u_2 + (\tilde{B}_2(t) - B_2(t))\hat{u}_2 \rangle \\
+ \langle z, C(t)x^{u_2} + \tilde{F}_5(t)\hat{x}^{\hat{u}_2} + \tilde{B}_1(t)\varphi \\
+ \tilde{D}_1(t)\beta + D_2(t)u_2 + \tilde{F}_2(t)u_2 \rangle \\
+ \langle \tilde{z}, \tilde{C}(t)x^{u_2} + (\tilde{C}(t) - \tilde{C}(t))\hat{x}^{\hat{u}_2} + \tilde{B}_1(t)\varphi \\
+ \tilde{F}_3(t)\beta + \tilde{D}_2(t)u_2 + (\tilde{D}_2(t) - D_2(t))\hat{u}_2 \rangle \\
+ \langle p, \tilde{A}^\top\varphi + \tilde{C}^\top\beta + \tilde{F}_4\hat{u}_2 \rangle + \frac{1}{2} [\langle Q_2x^{u_2}, x^{u_2} \rangle + \langle N_2u_2, u_2 \rangle].
\]  

(5.43)
Applying the general thm, if there exists an $\mathcal{F}_t$-adapted optimal control $u_2^*(\cdot) \in \mathcal{U}_2$ for the leader, then

$$0 = N_2(t)u_2^*(t) + \tilde{F}_4^\top(t)\hat{p}(t) + B_2^\top(t)y(t)$$

$$+ (\tilde{B}_2(t) - B_2(t))^\top\hat{y}(t) + D_2^\top(t)\tilde{z}(t)$$

$$+ \tilde{F}_2^\top(t)\hat{z}(t) + \tilde{D}_2^\top(t)\tilde{\tilde{z}}(t)$$

$$+ (\tilde{D}_2(t) - \tilde{\tilde{D}}_2(t))^\top\tilde{\tilde{z}}(t), \text{ a.e. } t \in [0, T],$$

(5.44)

where $\mathcal{F}_t$-adapted processes quadruple $(p(\cdot), y(\cdot), z(\cdot), \tilde{\tilde{z}}(\cdot))$ satisfies the conditional mean-field FBSDE
\[
\begin{aligned}
dp(t) &= \left[ \tilde{A}(t)p(t) + \tilde{F}_1^\top(t)y(t) + \tilde{B}_1(t)z(t) + \tilde{B}_1(t)\tilde{z}(t) \right] dt \\
&\quad + \left[ \tilde{C}(t)p(t) + \tilde{B}_1^\top(t)y(t) \\
&\quad \quad + \tilde{D}_1(t)z(t) + \tilde{F}_3^\top(t)\tilde{z}(t) \right] d\tilde{W}(t), \\
-dy(t) &= \left[ A^\top(t)y(t) + (\tilde{A}(t) - A(t))^\top \hat{y}(t) \\
&\quad + C^\top(t)z(t) + \tilde{F}_5^\top(t)\hat{z}(t) + \tilde{C}^\top(t)\tilde{z}(t) \\
&\quad + (\tilde{C}(t) - \tilde{C}(t))^\top \hat{z}(t) + Q_2(t)x^*(t) \right] dt \\
&\quad - z(t)dW(t) - \tilde{z}(t)d\tilde{W}(t),
\end{aligned}
\]

\[ p(0) = 0, \quad y(T) = G_2x^*(T). \]

(5.45)
Thanks!