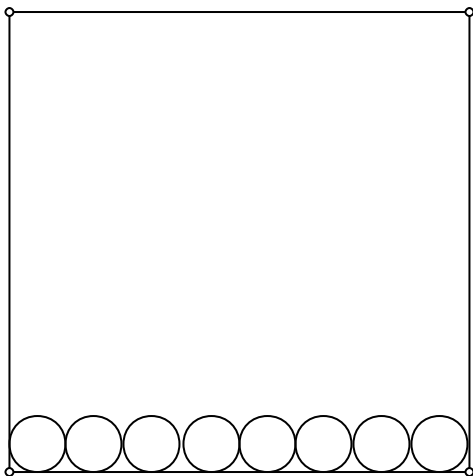


2001 Fermat II Exam

- 1) Find all integers (if any) a and b such that $\sqrt{19-6\sqrt{2}} = a+b\sqrt{2}$.
- 2) A cyclist follows a racetrack course that is divided into 4 parts of equal length. On the first part, which is flat and into the wind, she travels at 10 km/hr. On the second part, which is uphill, she travels at 5 km/hr. On the third part, which is downhill, she travels at 30 km/hr. On the fourth part, which is also flat but with the wind, she travels at 15 km/hr. What is the average speed of the cyclist over the entire course?
- 3) On a particular computer system, users must enter a password to log on. A password is a string of 5 uppercase letters from the usual 26 letter alphabet, for example: ABCDE, FERMA, or ZMTVZ. How many passwords contain exactly one T and exactly one N (simultaneously)?
- 4) Show that $\frac{1}{2001} < \left(\frac{1}{2}\right)\left(\frac{3}{4}\right)\left(\frac{5}{6}\right)\cdots\left(\frac{1999}{2000}\right) < \frac{1}{\sqrt{2001}}$.
- 5) Let P , Q , and R be three noncollinear points in the plane, each of whose coordinates is an integer. Show that the area of the triangle PQR is of the form $n/2$, where n is an integer.
- 6) A square dish rack with each side 16 inches will hold exactly 8 identical cylindrical glasses, each with a radius 1 inch, touching the bottom and each other (see diagram). Starting with the configuration shown, prove that you can put a total of 68 identical glasses in the rack (without breaking or stacking any). A drawing is not a proof.



- 7) Show that there are sequences $0=a_1 \leq a_2 \leq a_3 \leq \dots$ and $1=b_1 \geq b_2 \geq b_3 \geq \dots$ of rational numbers such that

$$(1) \quad a_n < \frac{1}{3} < b_n \text{ for all } n \geq 1,$$

$$(2) \quad b_{n+1} - a_{n+1} = \frac{b_n - a_n}{2} \text{ for all } n \geq 1, \text{ and}$$

$$(3) \quad (a_n - a_{n+1})(b_n - b_{n+1}) = 0 \text{ for all } n \geq 1.$$

- 8) Show that the polynomial $f(x) = x^8 + ax^6 + bx^4 - x^2 + 1$, where a and b are real numbers, cannot have all of its roots real numbers.