INTERIOR GRADIENT ESTIMATES FOR WEAK SOLUTIONS OF QUASI-LINEAR p-LAPLACIAN TYPE EQUATIONS

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Abstract. We study the interior weighted Sobolev regularity for weak solutions of the quasilinear equations of the form $\operatorname{div}\mathbf{A}(x,u,\nabla u) = \operatorname{div}\mathbf{F}$. The vector field $\mathbf{A}$ is allowed to be discontinuous in x, Hölder continuous in u and its growth in the gradient variable is like the p-Laplace operator with $1 < p < \infty$. We establish interior weighted $W^{1,q}$-regularity estimates for weak solutions to the equations for every $q > p$ assuming that the weak solutions are in the local John-Nirenberg BMO space. This paper therefore improves available results because it replaces the boundedness or continuity assumption on weak solutions by the borderline BMO one. Our regularity estimates also recover known results in which $\mathbf{A}$ is independent of the variable u. Our regularity theory complements the classical $C^{1,\alpha}$- regularity theory developed by many mathematicians including DiBenedetto and Tolksdorf for this general class of quasi-linear elliptic equations.

Keywords: Quasilinear elliptic equations, Quasilinear p-Laplacian type equations, Calderón-Zygmund regularity estimates, weighted Sobolev spaces.
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1. Introduction

This paper establishes interior regularity estimates in weighted Sobolev spaces for weak solutions to the following general quasi-linear p-Laplacian type equations

$$\operatorname{div}[\mathbf{A}(x,u,\nabla u)] = \operatorname{div}[|\mathbf{F}|^{p-2}\mathbf{F}] \quad \text{in} \quad B_{2R}, \tag{1.1}$$

where B_{2R} is the ball in $\mathbb{R}^n$ centered at the origin and has radius $2R$ for some $R > 0$, $\mathbf{F}$ is a given measurable vector field function, u is an unknown solution, and

$$\mathbf{A} = \mathbf{A}(x,z,\xi) : B_{2R} \times \mathbb{K} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$$

is a given vector field. We assume that $\mathbf{A}(\cdot,z,\xi)$ is measurable in B_{2R} for every $(z,\xi) \in \mathbb{K} \times (\mathbb{R}^n \setminus \{0\})$; $\mathbf{A}(x,\cdot,\xi)$ Hölder continuous in $\mathbb{K}$ for a.e. $x \in B_{2R}$ and for all $\xi \in \mathbb{R}^n \setminus \{0\}$; and $\mathbf{A}(x,z,\cdot)$ differentiable in $\mathbb{R}^n \setminus \{0\}$ for each $z \in \mathbb{K}$ and for a.e. $x \in B_{2R}$. Here, $\mathbb{K}$ is an open interval in $\mathbb{R}$, which could be the same as $\mathbb{R}$. We assume in addition that there exist constants $\Lambda > 0, \alpha \in (0,1]$, and $1 < p < \infty$ such that $\mathbf{A}$ satisfies the following natural growth conditions

$$\langle \partial_\xi \mathbf{A}(x,z,\xi)\eta,\eta\rangle \geq \Lambda^{-1}|\xi|^{p-2}|\eta|^2, \qquad \text{for a.e. } x \in B_{2R}, \quad \forall\, z \in \mathbb{K}, \quad \forall\, \xi,\eta \in \mathbb{R}^n \setminus \{0\}, \tag{1.2}$$

$$|\mathbf{A}(x,z,\xi)| + |\xi||\partial_\xi \mathbf{A}(x,z,\xi)| \leq \Lambda|\xi|^{p-1}, \qquad \text{for a.e. } x \in B_{2R}, \quad \forall\, z \in \mathbb{K}, \quad \forall\, \xi \in \mathbb{R}^n \setminus \{0\}, \tag{1.3}$$

$$|\mathbf{A}(x,z_1,\xi) - \mathbf{A}(x,z_2,\xi)| \leq \Lambda|\xi|^{p-1}|z_1 - z_2|^\alpha \qquad \forall\, z_1,z_2 \in \mathbb{K}, \quad \text{for a.e. } x \in B_{2R}, \quad \forall \xi \in \mathbb{R}^n \setminus \{0\}. \tag{1.4}$$

Observe that under the conditions (1.2)–(1.4), the class of equations of the form (1.1) contains the well-known p-Laplace equations.

The focus of this paper is to investigate the regularity in weighted Sobolev spaces for weak solutions u of (1.1) when the nonlinearity of $\mathbf{A}$ depends on u as its variable. In this perspective, we would like to point out that, on one hand, the $C^{1,\alpha}$-regularity theory for *bounded, weak solutions* of this class of equations has been investigated extensively, see [7, 14, 15, 25–28, 37–39], assuming some regularity of $\mathbf{A}$ in both x and z variables. On the other hand, when $\mathbf{A}$ is discontinuous in x or $\mathbf{F}$ is not sufficiently regular, one does not expect those mentioned Schauder's type estimates for weak solutions of (1.1) to hold, and it is natural to

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search for L^q- estimates for the gradients instead, see [15,22,25,28,29] for example. In this line of research, we note that in case $\mathbf{A} = \mathbf{A}_0$ for some $\mathbf{A}_0$ which is independent on the variable $z \in \mathbb{K}$, the equation (1.1) is reduced to

$$\operatorname{div}[\mathbf{A}_0(x, \nabla u)] = \operatorname{div}[|\mathbf{F}|^{p-2}\mathbf{F}] \quad \text{in} \quad B_{2R}, \tag{1.5}$$

and the $W^{1,q}$-regularity estimates of Calderón-Zygmund type for weak solutions to the class of equations (1.5) has been studied by many authors, for example see [2,3,5,8–10,12,13,18,19,22,23,29,30]. However, if $\mathbf{A}$ depends on the z-variable as in (1.1) and even with $\mathbf{F} = 0$, the $W^{1,q}$-regularity estimates become much more challenging, and not very well-understood. This is due the fact that the Calderón-Zygmund theory relies heavily on the scaling and dilation invariances of the considered class of equations, see [40] for the geometric intuition of this fact. Since the class of equations (1.5) is invariant under the scalings:

$$u \mapsto u/\lambda, \quad \text{and} \quad u(x) \mapsto \frac{u(rx)}{r}, \quad \text{for all positive numbers} \quad r, \lambda, \tag{1.6}$$

the $W^{1,q}$-regularity of Calderón-Zygmund for weak solutions of (1.5) is therefore naturally expected. Meanwhile, the invariant homogeneity with respect to (1.6) is no longer available for (1.1). This fact presents a serious obstacle in obtaining $W^{1,q}$-estimates for the weak solutions of (1.1) as they do not generate enough estimates to carry out the proof by using existing methods.

In the recent work [17, 34], the $W^{1,q}$-regularity estimates for weak solutions of (1.1) are addressed, and the $W^{1,q}$-regularity estimates are established assuming that the weak solutions are bounded. To overcome the loss of the homogeneity that we mentioned, we introduced in [17, 34] some "double-scaling parameter" technique. Essentially, we study an enlarged class of "double parameter" equations of the type (1.1). Then, by some compactness argument, we successfully applied the perturbation method in [5] to tackle the problem. Carefull analysis is required to ensure that all intermediate steps in the perturbation process are uniformly with respect to the scaling parameters. See also [4, 35] for further implementation of this idea, and the work [11] for some other related results in this line of research. In the papers [4, 17, 34], the a priori boundedness assumption on the weak solutions is essential to start the investigation of $W^{1,q}$-theory. This is because the approach uses the maximum principle for the unperturbed equations to implement the perturbation technique of [5]. We also would like to refer [1] for which the same $W^{1,p}$-theory for parabolic equations of type (1.1) is also achieved for continuous weak solutions.

A natural question arises from the mentioned work: *Is it necessary to assume that solutions are bounded, both for Sobolev regularity theory and Schauder's regularity one*? In this paper, we will give an answer to this question in the Sobolev regularity setting. We particularly establish the $W^{1,q}$-regularity estimates for weak solutions of (1.1) by assuming that the solutions are in the BMO John-Nirenberg space, i.e. the borderline case. This is achieved in Theorem 1.1 below. Our paper therefore generalizes all results in [1,4,17,34]. More than that, this paper also simplifies many technical issues in [17,34], and gives a generic approach to unify and treat both classes of equations (1.1) and (1.5) at the same time. Unlike [4,17,34] we only use "one parameter" in the class of our equations. Precisely, we investigate the following equation

$$\operatorname{div}\Big[\mathbf{A}(x, \lambda u, \nabla u)\Big] = \operatorname{div}[|\mathbf{F}|^{p-2}|\mathbf{F}], \quad \text{in } B_{2R}, \tag{1.7}$$

with the parameter $\lambda \geq 0$. The class of equations (1.7) is indeed the smallest one that is invariant with respect to the scalings and dilation (1.6) and that includes (1.1). When $\lambda = 0$, the equation (1.7) clearly becomes the equation (1.5). This paper therefore recovers known results such as [2, 3, 5, 8, 9, 12, 13, 18, 19, 29, 30] regarding the interior regularity of weak solutions of (1.5).

From now, the notation A_q with $q \geq 1$ stands for the class of Muckenhoupt weights whose definition will be recalled in Section 2.2. Also, $B_R(y)$ is the ball in $\mathbb{R}^n$ with radius $R > 0$ and centered at $y \in \mathbb{R}^n$. For simplicity, we also write $B_R = B_R(0)$. Moreover, for some locally integrable function $f : U \to \mathbb{R}$ with some measurable set $U \subset \mathbb{R}^n$ and with $\rho_0 > 0$, the BMO semi-norm of bounded mean oscillation of f is defined

by

$$[[f]]_{\mathrm{BMO}(U,\rho_0)} = \sup_{y\in U, 0<\rho<\rho_0} \frac{1}{|B_\rho(y)|}\int_{B_\rho(y)\cap U} |f(x) - \bar{f}_{B_\rho(y)\cap U}|dx, \quad \text{where}$$
$$\bar{f}_{B_\rho(y)\cap U} = \frac{1}{|B_\rho(y)|}\int_{B_\rho(y)\cap U} f(x)dx.$$

The main result of this paper is the following interior regularity estimates for weak solutions of (1.7) in weighted Lebesgue spaces.

Theorem 1.1. *Let $\Lambda > 0, M > 0, p, q > 1$, $\gamma \geq 1$, and $\alpha \in (0,1]$. Then, there exists a sufficiently small constant $\delta = \delta(p,q,n,\Lambda,M,\gamma,\alpha) > 0$ such that the following statement holds true. Assume that $\mathbf{A} : B_{2R}\times\mathbb{K}\times\mathbb{R}^n \longrightarrow \mathbb{R}^n$ is a Carathéodory map satisfying* (1.2)-(1.4) *and*

$$[[\mathbf{A}]]_{\mathrm{BMO}(B_R,R)} := \sup_{0<\rho\leq R}\sup_{y\in B_R}\frac{1}{|B_\rho(y)|}\int_{B_\rho(y)}\Big[\sup_{\substack{z\in\mathbb{K}\\ \xi\in\mathbb{R}^n\setminus\{0\}}}\frac{|\mathbf{A}(x,z,\xi)-\bar{\mathbf{A}}_{B_\rho(y)}(z,\xi)|}{|\xi|^{p-1}}\Big]dx \leq \delta, \tag{1.8}$$

for some $R > 0$ and for some open interval $\mathbb{K}\subset\mathbb{R}$. Then, for every $\mathbf{F}\in L^p(B_{2R},\mathbb{R}^n)$, if u is a weak solution of

$$\mathrm{div}[\mathbf{A}(x,\lambda u,\nabla u)] = \mathrm{div}[|\mathbf{F}|^{p-2}\mathbf{F}] \quad in \quad B_{2R}$$

with $[[\lambda u]]_{\mathrm{BMO}(B_R,R)} \leq M$ for some $\lambda \geq 0$, the following weighted regularity estimate holds

$$\int_{B_R}|\nabla u|^{pq}\omega(x)dx \leq C\left[\int_{B_{2R}}|\mathbf{F}|^{pq}\omega(x)dx + \omega(B_{2R})\left(\frac{1}{|B_{2R}|}\int_{B_{2R}}|\nabla u|^p dx\right)^q\right],$$

as long as its right hand side is finite, where $\omega\in A_q$ with $[\omega]_{A_q}\leq\gamma$, $\bar{\mathbf{A}}_{B_\rho(y)}(z,\xi) := \fint_{B_\rho(y)}\mathbf{A}(x,z,\xi)\,dx$, and C is a constant depending only on q, p, n, Λ, α, M, $\mathbb{K}$, R, and γ.

We emphasize that the significant contribution in Theorem 1.1 is that it relaxes and do not requires the considered weak solutions to be bounded as in [1,4,17,34]. This is completely new even for the case $\omega = 1$, in comparison to the known work that we already mentioned for both the Schauder's regularity theory and the Sobolev one regarding weak solutions of (1.1). Certainly, removing the boundedness assumption on solutions and replacing it by the condition that weak solutions are in BMO is valuable in the critical cases in which the L^∞-bound for solutions are not available, see [8] for example. When $p = n$, our weak solutions are in $W^{1,n}$, hence they are in BMO by the Sobolev imbedding theorem. Therefore, in this case, our theorem is applicable directly while results [1, 4, 17, 34] may be not. Note that M is not required to be small, our $[[\lambda u]]_{\mathrm{BMO}(B_R,R)}$ is not necessary small. When $\lambda = 0$, the condition $[[\lambda u]]_{\mathrm{BMO}(B_R,R)} \leq M$ is certainly held for every function u. Therefore, Theorem 1.1 recovers results in [2, 3, 5, 9, 12, 13, 18, 19, 30] in which the case that $\mathbf{A}$ is independent on $z\in\mathbb{K}$ is studied. This paper therefore unifies $W^{1,q}$-regularity estimates for both (1.1) and (1.5). We also would like to note that the fact that $\mathbf{A}$ is defined in $z\in\mathbb{K}$ only is important in many applications. A simple example is $\mathbb{K} = (0,\infty)$, meaning that (1.2)-(1.4) only hold for positive solution u. In the study of cross-diffusion equations in [17], $K = (0, M_0)$ for some $M_0 > 0$.

We remark that the smallness condition (1.8) on the mean oscillation of $\mathbf{A}$ with respect to x-variable is necessary as there is a counterexample provided in [31] for linear equations. In this regard, we also would like to point out that in the work [10], regularity estimates for weak solutions of equations with measurable coefficients that are small in partial BMO-semi norm are established.

This paper follows the perturbation approach [5] and makes use of Hardy-Littlewood maximal function, see also [2,4,17,34,35,40]. One can also find in the work [10,11,22,23] for a similar perturbation approach which uses Fefferman-Stein sharp function. To overcome the loss of boundedness of solutions due to our assumption, instead of applying maximum principle during the perturbation process as in the known work, we directly derive and delicately use Hölder's regularity estimates for solutions of the corresponding homogeneous equations, see the estimates (3.4) and (3.15) for example. The well-known John-Nirenberg's theorem and reverse Hölder's inequality also play very important role in our approach.

We now conclude this section by outlining the organization of this paper. Section 2 reviews some definitions, and some known results needed in the paper. Intermediate steps in the approximation estimates required in the proof of Theorem 1.1 are established and proved in Section 3. The last section, Section 4 gives the proof of Theorem 1.1.

2. Definitions and preliminaries

2.1. **Scaling invariances, and definitions of weak solutions.** Let $\lambda' \geq 0$, and let us consider a function $u \in W^{1,p}_{loc}(U)$ satisfying

$$\operatorname{div}[\mathbf{A}(x, \lambda' u, \nabla u)] = \operatorname{div}[|\mathbf{F}|^{p-2}\mathbf{F}] \quad \text{in} \quad U,$$

in the sense of distribution, for some open bounded set $U \subset \mathbb{R}^n$. Then for some fixed $\lambda > 0$, the rescaled function

$$v(x) = \frac{u(x)}{\lambda} \quad \text{for} \quad x \in U \tag{2.1}$$

solves the equation

$$\operatorname{div}[\hat{\mathbf{A}}(x, \hat{\lambda} v, \nabla v)] = \operatorname{div}[|\hat{\mathbf{F}}|^{p-2}\hat{\mathbf{F}}] \quad \text{in} \quad U$$

in the distributional sense, where $\hat{\lambda} = \lambda\lambda' \geq 0$, and $\hat{\mathbf{A}} : U \times \mathbb{K} \times \mathbb{R}^n \to \mathbb{R}^n$ is defined by

$$\hat{\mathbf{A}}(x, z, \xi) = \frac{\mathbf{A}(x, z, \lambda\xi)}{\lambda^{p-1}} \quad \text{and} \quad \hat{\mathbf{F}}(x) = \frac{\mathbf{F}(x)}{\lambda^{p-1}}. \tag{2.2}$$

Remark 2.1. *If $\mathbf{A} : U \times \mathbb{K} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ satisfies the conditions* (1.2)–(1.4) *on $U \times \mathbb{K} \times \mathbb{R}^n$, then the rescaled vector field $\hat{\mathbf{A}} : U \times \mathbb{K} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ defined in* (2.2) *also satisfies the structural conditions* (1.2)–(1.4) *with the same constants Λ, p, and α. Moreover, $[[\mathbf{A}]]_{\mathrm{BMO}(U,\rho_0)} = [[\hat{\mathbf{A}}]]_{\mathrm{BMO}(U,\rho_0)}$ for any $\rho_0 > 0$.*

In this paper, $C^\infty_0(U)$ is the set of all smooth compactly supported functions in U, $L^p(U, \mathbb{R}^n)$ with $1 \leq p < \infty$ is the Lebesgue space consists all measurable functions $f : U \to \mathbb{R}^n$ such that $|f|^p$ is integrable on U, and $W^{1,p}(U)$ is the standard Sobolev space on U. Moreover, $\langle \cdot, \cdot \rangle$ is the Euclidean inner product in $\mathbb{R}^n$. Let us now recall the definitions of weak solutions that we use throughout the paper.

Definition 2.2. *Let $\mathbb{K} \subset \mathbb{R}$ be an interval, let $\Lambda > 0, p > 1, \alpha \in (0, 1]$. Also, let $U \subset \mathbb{R}^n$ be an open bounded set in $\mathbb{R}^n$ with sufficiently smooth boundary ∂U, and let $\mathbf{A} : U \times \mathbb{K} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ satisfy conditions* (1.2)–(1.4) *on $U \times \mathbb{K} \times \mathbb{R}^n$.*

(i) *For every $\mathbf{F} \in L^p(U; \mathbb{R}^n)$ and $\lambda \geq 0$, a function $u \in W^{1,p}_{loc}(U)$ is called a weak solution of*

$$\operatorname{div}\Big[\mathbf{A}(x, \lambda u, \nabla u)\Big] = \operatorname{div}[|\mathbf{F}|^{p-2}\mathbf{F}], \quad \textit{in} \quad U$$

if $\lambda u(x) \in \mathbb{K}$ for a.e. $x \in U$, and

$$\int_U \Big\langle \mathbf{A}(x, \lambda u, \nabla u), \nabla\varphi \Big\rangle \, dx = \int_U \langle |\mathbf{F}|^{p-2}\mathbf{F}, \nabla\varphi \rangle \, dx \qquad \forall\, \varphi \in C^\infty_0(U).$$

(ii) *For every $\mathbf{F} \in L^p(U; \mathbb{R}^n)$, $g \in W^{1,p}(U)$, and $\lambda \geq 0$, a function $u \in W^{1,p}(U)$ is a weak solution of*

$$\begin{cases} \operatorname{div}[\mathbf{A}(x, \lambda u, \nabla u)] &= \operatorname{div}[|\mathbf{F}|^{p-2}\mathbf{F}], & \textit{in} \quad U, \\ u &= g, & \textit{on} \quad \partial U, \end{cases}$$

if $\lambda u(x) \in \mathbb{K}$ for a.e. $x \in U$, $u - g \in W^{1,p}_0(U)$, and

$$\int_U \Big\langle \mathbf{A}(x, \lambda u, \nabla u), \nabla\varphi \Big\rangle \, dx = \int_U \langle \mathbf{F}, \nabla\varphi \rangle \, dx \qquad \forall\, \varphi \in C^\infty_0(U).$$

2.2. **Muckenhoupt weights, weighted inequalities, and crawling ink-spots lemma.** This section recalls several analysis results and definitions that are needed in the paper. Firstly, we recall the definition of A_p-Muckenhoupt class of weights introduced in [33].

Definition 2.3. *Let $1 \le p < \infty$, a non-negative, locally integrable function $\omega : \mathbb{R}^n \to [0, \infty)$ is said to be in the class A_p of Muckenhoupt weights if*

$$[\omega]_{A_p} := \sup_{\text{balls } B\subset\mathbb{R}^n} \left(\fint_B \omega(x)dx\right)\left(\fint_B \omega(x)^{\frac{1}{1-p}}dx\right)^{p-1} < \infty, \quad \text{if} \quad p > 1,$$

$$[\omega]_{A_1} := \sup_{\text{balls } B\subset\mathbb{R}^n} \left(\fint_B \omega(x)dx\right)\left\|\omega^{-1}\right\|_{L^\infty(B)} < \infty \quad \text{if} \quad p = 1.$$

It turns out that the class of A_p-Muckenhout weights satisfies the reverse Hölder's inequality and the doubling properties. In particular, a measure of any A_p-weight is comparable with the Lebesgue measure in some sense. This is in fact a well-known result due to R. Coifman and C. Fefferman [6], and it is an important ingredient in the paper.

Lemma 2.4 ([6]). *For $1 < p < \infty$, the following statements hold true*

(i) *If $\mu \in A_p$, then for every ball $B \subset \mathbb{R}^n$ and every measurable set $E \subset B$,*

$$\mu(B) \le [\mu]_{A_p}\left(\frac{|B|}{|E|}\right)^p \mu(E).$$

(ii) *If $\mu \in A_p$ with $[\mu]_{A_p} \le \gamma$ for some given $\gamma \ge 1$, then there is $C = C(\gamma, n)$ and $\beta = \beta(\gamma, n) > 0$ such that*

$$\mu(E) \le C\left(\frac{|E|}{|B|}\right)^\beta \mu(B),$$

for every ball $B \subset \mathbb{R}^n$ and every measurable set $E \subset B$.

Observe that in the above statement and in this paper, the following notation is used

$$|U| = \int_U dx, \quad \mu(U) = \int_U \mu(x)dx,$$

for every measurable set $U \subset \mathbb{R}^n$.

Secondly, we state a standard result in measure theory.

Lemma 2.5. *Assume that $g \ge 0$ is a measurable function in a bounded subset $U \subset \mathbb{R}^n$. Let $\theta > 0$ and $N > 1$ be given constants. If μ is a weight function in $\mathbb{R}^n$, then for any $1 \le p < \infty$*

$$g \in L^p(U, \mu) \Leftrightarrow S := \sum_{j\ge1} N^{pj}\mu(\{x \in U : g(x) > \theta N^j\}) < \infty.$$

Moreover, there exists a constant $C > 0$ depending only on θ, N and p. such that

$$C^{-1}S \le \|g\|^p_{L^p(U,\mu)} \le C(\mu(U) + S),$$

where $L^p(U, \mu)$ is the weighted Lesbesgue space with norm

$$\|g\|_{L^p(U,\mu)} = \left(\int_U |g(x)|^p\mu(x)dx\right)^{1/p}.$$

Thirdly, we discuss the Hardy-Littlewood maximal operator and its boundedness in weighted spaces. For a given locally integrable function $f : \mathbb{R}^n \to \mathbb{R}$, the Hardy-Littlewood maximal function is defined as

$$\mathcal{M}f(x) = \sup_{\rho>0} \fint_{B_\rho(x)} |f(y)|dy, \quad x \in \mathbb{R}^n. \tag{2.3}$$

For a function f that is defined on a bounded domain U, we write

$$\mathcal{M}_U f(x) = \mathcal{M}(f\chi_U)(x),$$

where χ_U is the characteristic function of the set U. The following boundedness of Hardy-Littlewood maximal operator $\mathcal{M} : L^q(\mathbb{R}^n, \omega) \to L^q(\mathbb{R}^n, \omega)$ is classical.

Lemma 2.6. *Let $\gamma \geq 1$, and $\omega \in A_q$ with $[\omega]_{A_q} \leq \gamma$. The followings hold.*

(i) *Strong (q, q): Let $1 < q < \infty$, then there exists a constant $C = C(\gamma, q, n)$ such that*

$$\|\mathcal{M}\|_{L^q(\mathbb{R}^n,\omega)\to L^q(\mathbb{R}^n,\omega)} \leq C.$$

(ii) *Weak $(1, 1)$: There exists a constant $C = C(n)$ such that for any $\lambda > 0$, we have*

$$\left|\left\{x \in \mathbb{R}^n : \mathcal{M}(f) > \lambda\right\}\right| \leq \frac{C}{\lambda}\int_{\mathbb{R}^n} |f|dx.$$

Finally, we recall the following important lemma that is needed in this paper. This lemma is usually referred to "crawling ink-spots" lemma, which is originally due to N. V. Krylov and M. V. Safonov, see [24, 36].

Lemma 2.7 (crawling ink-spots)**.** *Suppose $\omega \in A_q$ with $[\omega]_{A_q} \leq \gamma$ for some $1 < q < \infty$ and some $\gamma \geq 1$. Suppose also that $R > 0$, and suppose that C, D are measurable sets satisfying $C \subset D \subset B_R$. Assume that there are $\rho_0 \in (0, R/2)$, and $0 < \epsilon < 1$ such that the followings hold*

(i) *$\omega(C) < \epsilon\omega(B_{\rho_0}(y))$ for almost every $y \in B_R$, and*
(ii) *for all $x \in B_R$ and $\rho \in (0, \rho_0)$, if $\omega(C \cap B_\rho(x)) \geq \epsilon\omega(B_\rho(x))$, then $B_\rho(x) \cap B_R \subset D$.*

Then

$$\omega(C) \leq \epsilon_1\omega(D), \quad \textit{for} \quad \epsilon_1 = \epsilon 20^{nq}\gamma^2.$$

2.3. **Hölder regularity, and self-improving regularity.** We recall some classical regularity results that are needed in the paper. The first result is about the interior Hölder's regularity for weak solutions of homogeneous p-Laplacian type equations (1.5). This result is indeed a consequence of the well-known De Giorgi-Nash-Moser theory, see [16, Theorem 7.6] and [25, Theorem 1.1, p. 251].

Lemma 2.8. *Let $\Lambda > 0, p > 1$, and let $\mathbb{A}_0 : B_r \times \mathbb{R}^n \to \mathbb{R}^n$ be a Carathéodory map and satisfy* (1.2)-(1.3) *on $B_r \times \mathbb{R}^n$ with some $r > 0$. If $v \in W^{1,p}(B_r)$ is a weak solution of the equation*

$$\operatorname{div}[\mathbb{A}_0(x, \nabla v)] = 0, \quad \textit{in} \quad B_r.$$

Then, there is $C_0 > 0$ depending only on Λ, n, p such that

$$\|v\|_{L^\infty(B_{5r/6})} \leq C_0\left(\fint_{B_r} |v|^p dx\right)^{1/p}.$$

Moreover, there exists a constant $\beta \in (0, 1)$ depending only on Λ, n, p and $\|v\|_{L^\infty(B_{5r/6})}$ such that

$$|v(x) - v(y)| \leq C_0\|v\|_{L^\infty(B_{5r/6})}\left(\frac{|x-y|}{r}\right)^\beta, \quad \forall\, x, y \in \overline{B}_{2r/3}.$$

We now recall a classical result on self-improving regularity estimates for weak solutions of p-Laplacian type equations. The following result is due to N. Meyers and A. Elcrat in [32, Theorem 1] (see also [8]). For the parabolic version of this result, see [20].

Lemma 2.9. *Let $\Lambda > 0, p > 1$. Then, there exists $p_0 = p_0(\Lambda, n, p) > p$ such that the following statement holds true. Suppose that $\mathbb{A}_0 : B_{2r} \times \mathbb{R}^n \to \mathbb{R}^n$ is a Carathéodory map satisfying* (1.2)-(1.3) *on $B_{2r} \times \mathbb{R}^n$ with some $r > 0$. If $v \in W^{1,p}(B_{2r})$ is a weak solution of the equation*

$$\operatorname{div}[\mathbb{A}_0(x, \nabla v)] = 0, \quad \textit{in} \quad B_{2r}$$

then, for every $p_1 \in [p, p_0]$, there exists a constant $C = C(\Lambda, p_1, p, n) > 0$ such that

$$\left(\frac{1}{|B_r|}\int_{B_r}|\nabla v|^{p_1}dx\right)^{1/p_1} \le C\left(\frac{1}{|B_{2r}|}\int_{B_{2r}}|\nabla v|^{p}dx\right)^{1/p}.$$

2.4. **Some simple energy estimates.** In this section we derive some elementary estimates which will be used frequently in the paper.

Lemma 2.10. *Let $\Lambda > 0, p > 1$, and let $U \subset \mathbb{R}^n$ be a bounded open set, and let $\mathbb{K}$ be an interval in $\mathbb{R}$. Assume that $\mathbf{A} : U \times \mathbb{K} \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ satisfies (1.2) - (1.3) on $U \times \mathbb{K} \times \mathbb{R}^n$. Then for any functions $u, v \in W^{1,p}(U)$ and any nonnegative function $\phi \in C(\overline{U})$, it holds that*

(i) *If $1 < p < 2$, then for every $\tau > 0$,*

$$\int_U |\nabla u - \nabla v|^p \phi\, dx \le \tau \int_U |\nabla u|^p \phi\, dx + C(\Lambda, p)\tau^{\frac{p-2}{p}} \int_U \langle \mathbf{A}(x, u, \nabla u) - \mathbf{A}(x, u, \nabla v), \nabla u - \nabla v\rangle \phi\, dx.$$

(ii) *If $p \ge 2$, then*

$$\int_U |\nabla u - \nabla v|^p \phi\, dx \le C(\Lambda, p) \int_U \langle \mathbf{A}(x, u, \nabla u) - \mathbf{A}(x, u, \nabla v), \nabla u - \nabla v\rangle \phi\, dx.$$

Proof. This lemma is well-known, see [37, Lemma 1] and [34, Lemma 3.1]. However, because it is important and also for completeness, we provide the proof. We first claim that from (1.2), the following monotonicity property of $\mathbf{A}$ holds true

$$\langle \mathbf{A}(x, z, \xi) - \mathbf{A}(x, z, \eta), \xi - \eta\rangle \ge \begin{cases} \gamma_0|\xi - \eta|^p, & \text{if} \quad p \ge 2, \\ \gamma_0(|\xi| + |\xi - \eta|)^{p-2}|\xi - \eta|^2 & \text{if} \quad 1 < p < 2, \end{cases} \tag{2.4}$$

for all $(x, z) \in U \times \mathbb{K}$ and for all $\xi, \eta \in \mathbb{R}^n \setminus \{0\}$, where $\gamma_0 = \gamma_0(\Lambda, p) > 0$ is a constant. To prove the claim, observe that for each $(x, z) \in U \times \mathbb{K}$ and each $\xi, \eta \in \mathbb{R}^n \setminus \{0\}$, we can write

$$\langle \mathbf{A}(x, z, \xi) - \mathbf{A}(x, z, \eta), \xi - \eta\rangle = \int_0^1 \langle \mathbf{A}_\xi(x, z, \xi + t(\eta - \xi))(\xi - \eta), \xi - \eta\rangle dt, \tag{2.5}$$

where $\mathbf{A}_\xi(x, z, \cdot)$ is the matrix of partial derivatives of $\mathbf{A}$ with respect to the third componental variable in $\mathbb{R}^n \setminus \{0\}$ of $\mathbf{A}$. It follows from (1.2) that

$$\langle \mathbf{A}_\xi(x, z, \xi + t(\eta - \xi))(\xi - \eta), \xi - \eta\rangle \ge \Lambda^{-1}|\xi + t(\eta - \xi)|^{p-2}|\xi - \eta|^2. \tag{2.6}$$

Then, if $p \in (1, 2)$, we see that $|\xi + t(\eta - \xi)| \le |\xi| + |\xi - \eta|$ and therefore,

$$\langle \mathbf{A}(x, z, \xi) - \mathbf{A}(x, z, \eta), \xi - \eta\rangle \ge \Lambda^{-1}(|\xi| + |\xi - \eta|)^{p-2}|\xi - \eta|^2.$$

Hence, the second estimate in (2.4) is proved. On the other hand, when $p \ge 2$, by (2.5)-(2.6), we see that

$$\langle \mathbf{A}(x, z, \xi) - \mathbf{A}(x, z, \eta), \xi - \eta\rangle \ge \Lambda^{-1}|\xi - \eta|^2 \int_0^{1/4} |\xi + t(\eta - \xi)|^{p-2} dt.$$

We may now assume without loss of generality that $|\xi - \eta| \ne 0$ and $|\eta| \le |\xi|$. Let us define $t_0 = \frac{|\xi|}{|\xi - \eta|}$. Note that if $|\xi - \eta| \le 2|\xi|$, then $t_0 \ge \frac{1}{2}$ and

$$|\xi + t(\eta - \xi)| \ge ||\xi| - t|\xi - \eta|| = |t - t_0||\xi - \eta| \ge \frac{1}{4}|\xi - \eta|, \quad \forall\, t \in (0, 1/4).$$

Otherwise, we have $|\eta| \le |\xi| \le \frac{1}{2}|\xi - \eta|$ and then

$$|\xi + t(\eta - \xi)| = |(1 - t)(\xi - \eta) + \eta| \ge (1 - t)|\xi - \eta| - |\eta| \ge \frac{3}{4}|\xi - \eta| - \frac{1}{2}|\xi - \eta| = \frac{1}{4}|\xi - \eta|, \quad \forall t \in (0, 1/4).$$

Hence, in conclusion, we have $|\xi + t(\eta - \xi)| \geq |\xi - \eta|/4$ for all $t \in (0, 1/4)$ and therefore,

$$\langle \mathbf{A}(x, z, \xi) - \mathbf{A}(x, z, \eta), \xi - \eta \rangle \geq \frac{1}{4^{p-1}\Lambda}|\xi - \eta|^p.$$

This proves the first estimate in (2.4) when $p \geq 2$, and also completes the proof of (2.4).

Finally, observe that from (2.4), (ii) becomes trivial. Therefore, it remains to prove (i) with $1 < p < 2$. In this case, for each $\xi, \eta \in \mathbb{R}^n \setminus \{0\}$, and for each $\tau \in (0, 1)$, we can use Young's inequality to obtain

$$\begin{aligned} |\xi - \eta|^p &= (|\xi| + |\xi - \eta|)^{\frac{p(2-p)}{2}}(|\xi| + |\xi - \eta|)^{\frac{p(p-2)}{2}}|\xi - \eta|^p \\ &\leq \frac{\tau}{3^{-p}}(|\xi| + |\xi - \eta|)^p + C_p\tau^{\frac{p-2}{p}}(|\xi| + |\xi - \eta|)^{p-2}|\xi - \eta|^2. \end{aligned}$$

From this and (2.4), we infer that

$$\begin{aligned} |\xi - \eta|^p &\leq \tau|\xi|^p + C_p\tau^{\frac{p-2}{p}}(|\xi| + |\xi - \eta|)^{p-2}|\xi - \eta|^2 \\ &\leq \tau|\xi|^p + C(\Lambda, p)\tau^{\frac{p-2}{p}}\langle \mathbf{A}(x, z, \xi) - \mathbf{A}(x, z, \eta), \xi - \eta \rangle. \end{aligned}$$

Then (i) follows and the proof of Lemma 2.10 is complete. □

Lemma 2.11 (Caccioppoli's type estimates)**.** *Let $\Lambda > 0, p > 1$ be fixed. Then, for every $r > 0$, every $\mathbf{A}_0 : B_r \times \mathbb{R}^n$ satisfying* (1.2)-(1.3) *on $B_r \times \mathbb{R}^n$, if $v \in W^{1,p}(B_r)$ is a weak solution of*

$$\operatorname{div}[\mathbf{A}_0(x, \nabla v)] = 0, \quad \text{in} \quad B_r,$$

then, it holds that

$$\int_{B_r} |\nabla v|^p \phi(x)^p dx \leq C(\Lambda, p) \int_{B_r} |v - k|^p |\nabla \phi(x)|^p dx, \quad \forall\, \phi \in C_0^1(B_r), \quad \phi \geq 0,$$

and for all $k \in \mathbb{R}$.

Proof. Since $(v - k)\phi \in W_0^{1,p}(B_r)$, we can use it as a test function. This together with Hölder's inequality, and Young's inequality, we can infer that

$$\begin{aligned} \int_{B_r(x_0)} \langle \mathbf{A}_0(x, \nabla v) - \mathbf{A}_0(x, 0), \nabla v \rangle \phi^p dx &= -p \int_{B_r(x_0)} \langle \mathbf{A}_0(x, \nabla v), \nabla \phi \rangle (v - k)\phi^{p-1} dx \\ &\leq C(\Lambda, p) \int_{B_r(x_0)} |\nabla v|^{p-1}\phi^{p-1}|\nabla \phi||v - k| dx \\ &\leq \frac{1}{4}\int_{B_r(x_0)} |\nabla v|^p \phi^p(x) dx + C(\Lambda, p)\int_{B_r(x_0)} |v - k|^p |\nabla \phi|^p dx. \end{aligned}$$

Now, by Lemma 2.10, it follows that

$$\begin{aligned} \int_{B_r(x_0)} |\nabla v|^p \phi^p dx &\leq \frac{1}{4}\int_{B_r(x_0)} |\nabla v|^p \phi^p dx + C(\Lambda, p)\int_{B_r(x_0)} \langle \mathbf{A}_0(x, \nabla v) - \mathbf{A}_0(x, 0), \nabla v \phi^p \rangle dx \\ &\leq \frac{1}{2}\int_{B_r(x_0)} |\nabla v|^p \phi^p dx + C(\Lambda, p)\int_{B_r(x_0)} |v - k|^p |\nabla \phi|^p dx. \end{aligned}$$

Therefore,

$$\int_{B_r(x_0)} |\nabla v|^p \phi(x)^p dx \leq C(\Lambda, p)\int_{B_r(x_0)} |v - k|^p |\nabla \phi(x)|^p dx,$$

as desired. □

2.5. **A known approximation estimate.** We recall a known approximation estimate established in [2, 3] and many other papers for the solutions of equations of the type (1.5) in which the vector field $\mathbf{A}_0$ is independent on the variable $z \in \mathbb{K}$. This approximation estimate will be used in some intermediate step for the proof of Theorem 1.1.

Lemma 2.12. *Let $\Lambda > 0, p > 1$ be fixed. Then, for every $\epsilon \in (0,1)$, there exists sufficiently small number $\delta_0 = \delta_0(\epsilon, \Lambda, n, p) \in (0, \epsilon)$ such that the following holds. Assume that $\mathbf{A}_0 : B_{2R} \times \mathbb{R}^n \to \mathbb{R}^n$ such that* (1.2)–(1.3) *hold, and*

$$\sup_{\substack{\xi\in\mathbb{R}^n\\ \xi\neq 0}} \sup_{\substack{x\in B_{2R}\\ 0<\rho<R}} \frac{1}{|B_\rho(x)|}\int_{B_\rho(x)} \frac{|\mathbf{A}_0(y,\xi) - \bar{\mathbf{A}}_{0,B_\rho(x)}|}{|\xi|^{p-1}} dy \le \delta_0.$$

Then, for every $x_0 \in B_R$ and $r \in (0, R/2)$ and for $\mathbf{G} \in L^p(B_{2R}, \mathbb{R}^n)$, if $w \in W^{1,p}(B_{2r}(x_0))$ is a weak solution of

$$\operatorname{div}[\mathbf{A}_0(x, \nabla w)] = \operatorname{div}[|\mathbf{G}|^{p-2}\mathbf{G}], \quad in \quad B_{2r}(x_0),$$

satisfying

$$\frac{1}{|B_{2r}(x_0)|}\int_{B_{2r}(x_0)} |\nabla w|^p dx \le 1,$$

and if

$$\frac{1}{|B_{2r}(x_0)|}\int_{B_{2r}(x_0)} |\mathbf{G}|^p dx \le \delta_0^p,$$

then there is $h \in W^{1,p}(B_{7r/4}(x_0))$ such that the following estimate holds

$$\frac{1}{|B_{7r/4}(x_0)|}\int_{B_{7r/4}(x_0)} |\nabla w - \nabla h|^p dx \le \epsilon^p, \quad \|\nabla h\|_{L^\infty(B_{3r/2}(x_0))} \le C(\Lambda, n, p).$$

3. Interior approximation estimates

In this section, let $\mathbf{A} : B_{2R} \times \mathbb{K} \times \mathbb{R}^n \to \mathbb{R}^n$ satisfy (1.2)–(1.4) on $B_{2R} \times \mathbb{K} \times \mathbb{R}^n$ for some $R > 0$ and some open interval $\mathbb{K} \subset \mathbb{R}$. We study the weak solutions $u \in W^{1,p}(B_{2R})$ of the scaling parameter equation

$$\operatorname{div}[\mathbf{A}(x, \lambda u, \nabla u] = \operatorname{div}[|\mathbf{F}|^{p-2}\mathbf{F}], \quad \text{in} \quad B_{2R}, \tag{3.1}$$

with the parameter $\lambda \ge 0$. Our goal in this section is to provide necessary estimates for proving Theorem 1.1. Our approach is based on the perturbation technique introduced in [5] together with the "scaling parameter" technique introduced in [17, 34]. The approach is also influenced by the recent developments, see [1–4, 35]. In our first step, we freeze u in $\mathbf{A}$, and then approximate the solution u of (3.1) by a solution of the corresponding homogeneous equations with the frozen u coefficient as in [1, 4].

Lemma 3.1. *Let $\Lambda, M > 0, p > 1$ be fixed and $\kappa \in (0, 1]$. Then, for every small $\epsilon \in (0, 1)$, there exists a sufficiently small number $\delta_1 = \delta_1(\epsilon, \Lambda, n, p, \kappa) \in (0, \epsilon)$ such that the following holds. Assume that $\mathbf{A} : B_{2R} \times \mathbb{K} \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies*(1.2)–(1.4) *with some $\mathbb{K} \subset \mathbb{R}$ and some $R > 0$, and assume that $\mathbf{F} \in L^p(B_{2R}, \mathbb{R}^n)$ satisfies*

$$\fint_{B_r(x_0)} |\mathbf{F}|^p dx \le \delta_1^p,$$

some $x_0 \in B_R$ and some $r \in (0, R)$. Suppose also that $u \in W^{1,p}(B_{2R})$ is a weak solution of (3.1) *satisfying*

$$\fint_{B_r(x_0)} |\nabla u|^p dx \le 1, \quad and \quad \lambda\left(\fint_{B_r(x_0)} |u - \bar{u}_{B_r(x_0)}|^p\right)^{1/p} \le M,$$

for some $\lambda \ge 0$. Then,

$$\fint_{B_r(x_0)} |\nabla u - \nabla v|^p dx \le \epsilon^p \kappa^n, \tag{3.2}$$

where $v \in W^{1,p}(B_r)$ is the weak solution of

$$\left\{\begin{array}{rcll} \operatorname{div}[\mathbf{A}(x,\lambda u,\nabla v)] & = & 0, & \text{in} \quad B_r(x_0), \\ v & = & u-\bar{u}_{B_r(x_0)}, & \text{on} \quad \partial B_r(x_0). \end{array}\right. \tag{3.3}$$

Moreover, it also holds that

$$\lambda\left(\fint_{B_r(x_0)}|v|^p dx\right)^{1/p} \le C(n,p)[M+\lambda r\epsilon\kappa^{\frac{n}{p}}]. \tag{3.4}$$

Proof. Note that for $\tilde{\mathbf{A}}_0(x,\xi) := \mathbf{A}(x,\lambda u(x),\xi)$. We see that $\tilde{\mathbf{A}}_0$ is independent on the variable $z \in \mathbb{K}$, and it satisfies the assumptions (1.2)–(1.3). The equation (3.3) is written as

$$\left\{\begin{array}{rcll} \operatorname{div}[\tilde{\mathbf{A}}_0(x,\nabla v)] & = & 0, & \text{in} \quad B_r(x_0), \\ v & = & u-\bar{u}_{B_r(x_0)}, & \text{on} \quad \partial B_r(x_0), \end{array}\right. \tag{3.5}$$

and we note that the existence of weak solution v of (3.5) follows from the standard theory in calculus of variation. Therefore, it remains to prove the estimates (3.2), and (3.4). Since $v-[u-\bar{u}_{B_r(x_0)}] \in W_0^{1,p}(B_r(x_0))$, we can take it as a test function for the equation (3.3), we obtain

$$\int_{B_r(x_0)}\langle \mathbf{A}(x,\lambda u,\nabla v),\nabla u-\nabla v\rangle dx = 0.$$

Similarly, we can use $v-[u-\bar{u}_{B_r(x_0)}]$ as a test function for the equation for (3.1) to see that

$$\int_{B_r(x_0)}\langle \mathbf{A}(x,\lambda u,\nabla u),\nabla u-\nabla v\rangle dx = \int_{B_r(x_0)}\langle |\mathbf{F}|^{p-2}\mathbf{F},\nabla u-\nabla v\rangle dx.$$

It then follows from these two identies that

$$\int_{B_r(x_0)}\langle \mathbf{A}(x,\lambda u,\nabla u)-\mathbf{A}(x,\lambda u,\nabla v),\nabla v-\nabla u\rangle dx = \int_{B_r(x_0)}\langle |\mathbf{F}|^{p-2}\mathbf{F},\nabla u-\nabla v\rangle dx. \tag{3.6}$$

We only need to consider the case $1<p<2$ because the case $p \ge 2$ is similar, and simpler. It follows from (i) of Lemma 2.10, Remark 2.1, and (3.6), that for each $\tau \in (0,1)$,

$$\begin{aligned} &\int_{B_r(x_0)}|\nabla u-\nabla v|^p dx \\ &\le \tau\int_{B_r(x_0)}|\nabla u|^p dx + C(\Lambda,\tau,p)\int_{B_r(x_0)}\langle \mathbf{A}(x,\lambda u,\nabla u)-\mathbf{A}(x,\lambda u,\nabla v),\nabla v-\nabla u\rangle dx \\ &\le \tau\int_{B_r(x_0)}|\nabla u|^p dx + C(\Lambda,\tau,p)\int_{B_r(x_0)}|\langle |\mathbf{F}|^{p-2}\mathbf{F},\nabla u-\nabla v\rangle| dx \\ &\le \tau\int_{B_r(x_0)}|\nabla u|^p dx + \frac{1}{2}\int_{B_r(x_0)}|\nabla u-\nabla v|^p dx + C(\Lambda,\tau,p)\int_{B_r(x_0)}|\mathbf{F}|^p dx, \end{aligned}$$

where in the last step, we have used Hölder's inequality and Young's inequality. Hence, by cancelling similar terms, we obtain

$$\fint_{B_r(x_0)}|\nabla u-\nabla v|^p dx \le 2\tau\fint_{B_r(x_0)}|\nabla u|^p dx + C(\Lambda,\tau,p)\fint_{B_r(x_0)}|\mathbf{F}|^p dx.$$

Now, choose $\tau = \epsilon^p\kappa^n/4$, and then choose $\delta_1 = \delta_1(\epsilon,\Lambda,n,p,\kappa) \in (0,\epsilon)$ sufficiently small such that $C(\Lambda,\tau,p)\delta^p < \epsilon^p\kappa^n/2$, the estimate (3.2) follows. It remains to prove (3.4). By Poincaré's inequality, we see that

$$\begin{aligned} \left(\fint_{B_r(x_0)}|v|^p dx\right)^{1/p} &\le C(p)\left[\left(\fint_{B_r(x_0)}|v-[u-\bar{u}_{B_r(x_0)}]|^p dx\right)^{1/p} + \left(\fint_{B_r(x_0)}|u-\bar{u}_{B_r(x_0)}|^p dx\right)^{1/p}\right] \\ &\le C(n,p)\left[r\left(\fint_{B_r(x_0)}|\nabla v-\nabla u|^p dx\right)^{1/p} + \left(\fint_{B_r(x_0)}|u-\bar{u}_{B_r(x_0)}|^p dx\right)^{1/p}\right]. \end{aligned}$$

From this and since $\kappa \in (0,1)$, it follows that

$$\lambda\left(\fint_{B_r(x_0)} |v|^p dx\right)^{1/p} \le C(n,p)[M + r\lambda\epsilon\kappa^{\frac{n}{p}}]$$

as desired. □

Next, we approximate the solution u by the solution w of the following equation whose principal part is a vector field that is independent on w and has small oscillation with respect to x-variable

$$\begin{cases} \operatorname{div}[\mathbf{A}(x, \lambda\bar{u}_{B_{\kappa r}(x_0)}, \nabla w)] &= 0, & \text{in} \quad B_{\kappa r}(x_0), \\ w &= v, & \text{on} \quad \partial B_{\kappa r}(x_0), \end{cases} \tag{3.7}$$

where v is the weak solution of (3.3) and $\kappa \in (0, 1/3)$ sufficiently small to be determined. Our next result is in the same fashion as that of Lemma 3.1.

Lemma 3.2. *Let $\Lambda, M > 0$, $p > 1$ and $\alpha \in (0,1]$ be fixed, and let $\epsilon \in (0,1)$. There exist positive, sufficiently small numbers $\kappa = \kappa(\epsilon, \Lambda, M, p, n, \alpha) \in (0, 1/3)$ and $\delta_2 = \delta_2(\epsilon, \Lambda, M, n, \alpha, p) \in (0, \epsilon)$ such that the following holds. Assume that $\mathbf{A} : B_{2R} \times \mathbb{K} \times \mathbb{R}^n \to \mathbb{R}^n$ satisfies* (1.2)–(1.4) *with some $R > 0$ and some open interval $\mathbb{K} \subset \mathbb{R}$, and assume that $\mathbf{F} \in L^p(B_{2R}, \mathbb{R}^n)$ and*

$$\fint_{B_r(x_0)} |\mathbf{F}|^p dx \le \delta_2^p,$$

some $x_0 \in B_R$ and some $r \in (0, R/2)$. Then, for every $\lambda > 0$, if $u \in W^{1,p}(B_{2R})$ is a weak solution of (3.1) *satisfying*

$$\fint_{B_{2\kappa r}(x_0)} |\nabla u|^p dx \le 1, \quad \fint_{B_r(x_0)} |\nabla u|^p dx \le 1, \quad \text{and} \quad [[\lambda u]]_{\mathrm{BMO}(B_R,R)} \le M,$$

it holds that

$$\left(\fint_{B_{\kappa r}(x_0)} |\nabla u - \nabla w|^p dx\right)^{1/p} \le \epsilon, \quad \text{and} \quad \left(\fint_{B_{\kappa r}} |\nabla w|^p dx\right)^{1/p} \le C_0(n,p). \tag{3.8}$$

where w is the weak solution of (3.7).

Proof. For a given sufficiently small $\epsilon > 0$, let $\epsilon' \in (0, \epsilon/2)$ and $\kappa \in (0, 1/3)$ both sufficiently small and depending on $\epsilon, \Lambda, M, n, \alpha, p$ which will be determined. Then, let $\delta_2 = \delta_1(\epsilon', \Lambda, n, p, \kappa) > 0$, where δ_1 is defined as in Lemma 3.1. Let v be the solution of (3.3). By using Lemma 3.1, we see that

$$\fint_{B_r(x_0)} |\nabla u - \nabla v|^p dx \le (\epsilon')^p \kappa^n, \quad \text{and} \quad \lambda\left(\fint_{B_r(x_0)} |v|^p dx\right)^{1/p} \le C(n,p)[r\epsilon'\lambda\kappa^{\frac{n}{p}} + M]. \tag{3.9}$$

Observe also that the first inequality in (3.9), the assumption in the lemma, and the fact that both ϵ and κ are small imply that

$$\begin{aligned} \left(\fint_{B_{2\kappa r}(x_0)} |\nabla v|^p dx\right)^{1/p} &\le \left(\fint_{B_{2\kappa r}(x_0)} |\nabla u - \nabla v|^p dx\right)^{1/p} + \left(\fint_{B_{2\kappa r}(x_0)} |\nabla u|^p dx\right)^{1/p} \\ &\le \left(\frac{1}{2^n\kappa^n} \fint_{B_r(x_0)} |\nabla u - \nabla v|^p dx\right)^{1/p} + \left(\fint_{B_{2\kappa r}(x_0)} |\nabla u|^p dx\right)^{1/p} \le \frac{\epsilon'}{2^{n/p}} + 1 \le 2. \end{aligned} \tag{3.10}$$

On the other hand, from the Caccioppli's type estimate in Lemma 2.11, (3.9) and $\kappa \in (0, 1/3)$, we also see that

$$\begin{aligned} \left(\frac{1}{|B_{2\kappa r}(x_0)|} \int_{B_{2\kappa r}(x_0)} |\nabla v|^p dx\right)^{1/p} &\le \frac{C(\Lambda, n, p)}{(1-2\kappa) r\kappa^{\frac{n}{p}}} \left(\frac{1}{|B_r(x_0)|} \int_{B_r(x_0)} |v|^p dx\right)^{1/p} \\ &\le C(\Lambda, n, p)\Big[\epsilon' + M(\lambda\kappa^{\frac{n}{p}} r)^{-1}\Big]. \end{aligned} \tag{3.11}$$

Now, let w be the weak solution of (3.7). As in the proof of Lemma 3.1, the existence of w is assured. Therefore, it remains to prove the estimate (3.8). Take $w - v \in W_0^{1,p}(B_{\kappa r}(x_0))$ as a test function for the equation (3.7) and the equation (3.3), we obtain

$$\int_{B_{\kappa r}(x_0)} \langle \mathbf{A}(x, \lambda u, \nabla v), \nabla w - \nabla v\rangle dx = \int_{B_{\kappa r}(x_0)} \langle \mathbf{A}(x, \lambda \bar{u}_{B_{\kappa r}(x_0)}, \nabla w), \nabla w - \nabla v\rangle dx = 0. \tag{3.12}$$

Again, we only need to consider the case $1 < p < 2$, as $p \geq 2$ can be done similarly using (ii) of Lemma 2.10. From now on, for simplicity, we write $\hat{u} = u - \bar{u}_{B_{\kappa r}(x_0)}$. We can use (i) of Lemma 2.10, the condition (1.4), and (3.12) to obtain with some $\tau > 0$ sufficiently small to be determined,

$$\begin{aligned}
&\int_{B_{\kappa r}(x_0)} |\nabla v - \nabla w|^p dx \\
&\leq \tau \int_{B_{\kappa r}(x_0)} |\nabla v|^p dx + C(\Lambda, p)\tau^{\frac{p-2}{p}} \int_{B_{\kappa r}(x_0)} \langle \mathbf{A}(x, \lambda \bar{u}_{B_{\kappa r}(x_0)}, \nabla v) - \mathbf{A}(x, \lambda \bar{u}_{B_{\kappa r}(x_0)}, \nabla w), \nabla v - \nabla w\rangle dx \\
&\leq \tau \int_{B_{\kappa r}(x_0)} |\nabla v|^p dx + C(\Lambda, p)\tau^{\frac{p-2}{p}} \int_{B_{\kappa r}(x_0)} \langle \mathbf{A}(x, \lambda \bar{u}_{B_{\kappa r}(x_0)}, \nabla v) - \mathbf{A}(x, \lambda u, \nabla v), \nabla v - \nabla w\rangle dx \\
&\leq \tau \int_{B_{\kappa r}(x_0)} |\nabla v|^p dx + C(\Lambda, p)\tau^{\frac{p-2}{p}} \int_{B_{\kappa r}(x_0)} |\lambda \hat{u}|^{\alpha} |\nabla v|^{p-1} |\nabla v - \nabla w| dx \\
&\leq \frac{1}{2} \int_{B_{\kappa r}(x_0)} |\nabla v - \nabla w|^p dx + \tau \int_{B_{\kappa r}(x_0)} |\nabla v|^p dx + C(\Lambda, p)\tau^{\frac{p-2}{p-1}} \int_{B_{\kappa r}(x_0)} |\lambda \hat{u}|^{\frac{\alpha p}{p-1}} |\nabla v|^p dx,
\end{aligned}$$

where in the last step, we have used the Hölder's inequality and Young's inequality. Hence, by cancelling similar terms, we obtain

$$\begin{aligned}
&\frac{1}{|B_{\kappa r}(x_0)|} \int_{B_{\kappa r}(x_0)} |\nabla v - \nabla w|^p dx \\
&\leq \frac{2\tau}{|B_{\kappa r}(x_0)|} \int_{B_{\kappa r}(x_0)} |\nabla v|^p dx + \frac{C(\Lambda, p)\tau^{\frac{p-2}{p-1}}}{|B_{\kappa r}(x_0)|} \int_{B_{\kappa r}(x_0)} |\lambda \hat{u}|^{\frac{\alpha p}{p-1}} |\nabla v|^p dx.
\end{aligned}$$

For $q_1 > p$ and sufficiently close to p depending only on Λ, p, we write $q_1 = \frac{\alpha p p_1}{(p-1)(p_1-p)} > p$. Then, using the Hölder's inequality, the self-improving regularity estimate (i.e. Lemma 2.9), and (3.10), we obtain

$$\begin{aligned}
&\frac{1}{|B_{\kappa r}(x_0)|} \int_{B_{\kappa r}(x_0)} |\nabla v - \nabla w|^p dx \\
&\leq \frac{2\tau}{|B_{\kappa r}(x_0)|} \int_{B_{\kappa r}(x_0)} |\nabla v|^p dx \\
&\qquad + C(\Lambda, p)\tau^{\frac{p-2}{p-1}} \left(\frac{1}{|B_{\kappa r}(x_0)|} \int_{B_{\kappa r}(x_0)} |\lambda \hat{u}|^{q_1} \right)^{\frac{p_1-p}{p_1}} \left(\frac{1}{|B_{\kappa r}(x_0)|} \int_{B_{\kappa r}(x_0)} |\nabla v|^{p_1} dx \right)^{\frac{p}{p_1}} \\
&\leq C(\Lambda, n, p) \left[2\tau + \tau^{\frac{p-2}{p-1}} \left(\frac{1}{|B_{\kappa r}(x_0)|} \int_{B_{\kappa r}(x_0)} |\lambda \hat{u}|^{q_1} dx \right)^{\frac{p_1-p}{p_1}} \right] \left(\frac{1}{|B_{2\kappa r}(x_0)|} \int_{B_{2\kappa r}(x_0)} |\nabla v|^p dx \right).
\end{aligned}$$

Now, from the well-known John-Nirenberg's theorem, we further write

$$\begin{aligned}\frac{1}{|B_{\kappa r}(x_0)|}\int_{B_{\kappa r}(x_0)}|\lambda\hat{u}|^{q_1}dx &= \frac{1}{|B_{\kappa r}(x_0)|}\int_{B_{\kappa r}(x_0)}|\lambda\hat{u}|^{p/2}|\lambda\hat{u}|^{q_1-p/2}dx\\ &\le \left(\frac{1}{|B_{\kappa r}(x_0)|}\int_{B_{\kappa r}(x_0)}|\lambda\hat{u}|^{p}dx\right)^{1/2}\left(\frac{1}{|B_{\kappa r}(x_0)|}\int_{B_{\kappa r}(x_0)}|\lambda\hat{u}|^{2q_1-p}dx\right)^{1/2}\\ &\le C(n,\alpha,p)[[\lambda u]]^{q_1-\frac{p}{2}}_{\mathrm{BMO}(B_R,R)}\left(\frac{1}{|B_{\kappa r}(x_0)|}\int_{B_{\kappa r}(x_0)}|\lambda\hat{u}|^{p}dx\right)^{1/2}\\ &= C(n,M,\alpha,p)\left(\frac{1}{|B_{\kappa r}(x_0)|}\int_{B_{\kappa r}(x_0)}|\lambda\hat{u}|^{p}dx\right)^{1/2}.\end{aligned}$$

Therefore,

$$\begin{aligned}&\frac{1}{|B_{\kappa r}(x_0)|}\int_{B_{\kappa r}(x_0)}|\nabla v-\nabla w|^p dx\\ &\le C(\Lambda,n,\alpha,p)\left[2\tau+\tau^{\frac{p-2}{p-1}}\left(\frac{1}{|B_{\kappa r}(x_0)|}\int_{B_{\kappa r}(x_0)}|\lambda\hat{u}|^{p}dx\right)^{\frac{p_1-p}{2p_1}}\right]\left(\frac{1}{|B_{2\kappa r}(x_0)|}\int_{B_{2\kappa r}(x_0)}|\nabla v|^p dx\right).\end{aligned}\tag{3.13}$$

From (3.11) and $[[\lambda u]]_{\mathrm{BMO}(B_{2R},R)}\le M$, we can take $\tau=1/2$ in (3.13) to particularly obtain

$$\begin{aligned}\left(\frac{1}{|B_{\kappa r}(x_0)|}\int_{B_{\kappa r}(x_0)}|\nabla v-\nabla w|^p dx\right)^{1/p} &\le C(\Lambda,M,n,\alpha,p)\left(\frac{1}{|B_{2\kappa r}(x_0)|}\int_{B_{2\kappa r}(x_0)}|\nabla v|^p dx\right)^{1/p}\\ &\le C_1(\Lambda,M,n,p)\Big[\epsilon'+M(r\kappa^{\frac{n}{p}}\lambda)^{-1}\Big].\end{aligned}$$

Hence, if $\frac{\epsilon\kappa^{\frac{n}{p}}\lambda r}{4MC_1(\Lambda,M,n,p)}\ge 1$, we choose ϵ' sufficiently small so that

$$C_1(\Lambda,n,p)\epsilon'<\epsilon/4.$$

Then,

$$\left(\frac{1}{|B_{\kappa r}(x_0)|}\int_{B_{\kappa r}(x_0)}|\nabla v-\nabla w|^p dx\right)^{1/p}\le\epsilon/2.$$

From this, the first estimate in (3.9) and the triangle inequality, the first estimate of (3.8) follows. Therefore, it remains to consider the case

$$\lambda\kappa^{\frac{n}{p}}r\epsilon\le 4MC_1(\Lambda,M,n,p).\tag{3.14}$$

In this case, we first note that from our choice that $\epsilon'\le\epsilon$, we particularly have

$$\lambda\kappa^{\frac{n}{p}}\epsilon' r\le C(\Lambda,M,n,p).$$

Then, it follows from the second estimate in (3.9) that

$$\lambda\left(\frac{1}{|B_r(x_0)|}\int_{B_r(x_0)}|v|^p dx\right)^{1/p}\le C(\Lambda,M,n,p).$$

On the other hand, from (3.3), and the scaling invariances discussed in Subsection 2.1, we observe that $\tilde{v}(x)=\lambda v(x-x_0)$ is a weak solution of

$$\operatorname{div}[\hat{\mathbf{A}}_0(x,\nabla\tilde{v})]=0,\quad\text{in}\quad B_r,$$

where $\hat{\mathbf{A}}_0(x,\xi)=\lambda^{p-1}\mathbf{A}(x-x_0,\lambda u(x-x_0),\lambda^{-1}\xi)$ for all $x\in B_r,\xi\in\mathbb{R}^n$. From this and Remark 2.1, we can apply the Hölder's regularity theory in Lemma 2.8 for the solution $\tilde{v}$ to find that there is $\beta\in(0,1)$ depending only on Λ,M,n,p such that

$$\lambda\|v\|_{L^\infty(B_{5r/6}(x_0))}\le C(\Lambda,M,n,p),\quad\text{and}\quad\lambda|v(x)-v(y)|\le C(\Lambda,M,p,n)\kappa^{\beta},\quad\forall\ x,y\in\overline{B}_{\kappa r}(x_0).\tag{3.15}$$

The estimate (3.15), (3.10), and (3.13) imply that

$$\begin{aligned}&\frac{1}{|B_{\kappa r}(x_0)|}\int_{B_{\kappa r}(x_0)}|\nabla v-\nabla w|^p dx\\&\le C(\Lambda,M,n,\alpha,p)\left[2\tau+\tau^{\frac{p-2}{p-1}}\left(\frac{1}{|B_{\kappa r}(x_0)|}\int_{B_{\kappa r}(x_0)}|\lambda\hat{u}|^p dx\right)^{\frac{p_1-p}{2p_1}}\right].\end{aligned}\tag{3.16}$$

On the other hand, for $v'=v+\bar{u}_{B_{\kappa r}}$, we can write

$$\begin{aligned}&\frac{1}{|B_{\kappa r}(x_0)|}\int_{B\kappa r(x_0)}|\lambda\hat{u}|^p dx\\&\le C(p)\Bigg[\frac{1}{|B_{\kappa r}(x_0)|}\int_{B_{\kappa r}(x_0)}|\lambda(u-v')|^p dx+\frac{1}{|B_{\kappa r}(x_0)|}\int_{B_{\kappa r}(x_0)}|\lambda(v'-\bar{v'}_{B_{\kappa r}(x_0)})|^p dx\\&\quad+\frac{1}{|B_{\kappa r}(x_0)|}\int_{B_{\kappa r}(x_0)}|\lambda(\bar{u}_{B_{\kappa r}(x_0)}-\bar{v'}_{B_{\kappa r}(x_0)})|^p dx\\&\le C(n,p)\Bigg[\frac{1}{\kappa^n|B_r(x_0)|}\int_{B_r(x_0)}|\lambda(u-v')|^p dx+\frac{1}{|B_{\kappa r}(x_0)|}\int_{B_{\kappa r}(x_0)}|\lambda(v-\bar{v}_{B_{\kappa r}(x_0)})|^p dx\Bigg].\end{aligned}$$

Since $u-v'\in W_0^{1,2}(B_r(x_0))$, we can use the Poincaré's inequality for the first term in the right hand side of the last estimate to obtain

$$\begin{aligned}&\left(\frac{1}{|B_{\kappa r}(x_0)|}\int_{B_{\kappa r}(x_0)}|\lambda\hat{u}|^p dx\right)^{1/p}\\&\le C(\Lambda,n,p)\left[\frac{\lambda r}{\kappa^{\frac{n}{p}}}\left(\frac{1}{|B_r(x_0)|}\int_{B_r(x_0)}|\nabla u-\nabla v|^p dx\right)^{1/p}+\lambda\sup_{x,y\in\overline{B}_{\kappa r}(x_0)}|v(x)-v(y)|\right],\end{aligned}$$

From this estimate, (3.9), and (3.15), we infer that

$$\left(\frac{1}{|B_{\kappa r}(x_0)|}\int_{B_{\kappa r}(x_0)}|\lambda\hat{u}|^p dx\right)^{1/p}\le C(\Lambda,p,n)\Big[\lambda r\epsilon'+\kappa^\beta\Big].$$

From this, we can control the estimate in (3.16) as

$$\frac{1}{|B_{\kappa r}(x_0)|}\int_{B_{\kappa r}(x_0)}|\nabla v-\nabla w|^p dx\le C(\Lambda,M,n,\alpha,p)\left[2\tau+\tau^{\frac{p-2}{p-1}}\left(\lambda r\epsilon'+\kappa^\beta\right]\right)^{\frac{p(p_1-p)}{2p_1}}\Bigg].$$

Then, combining this last estimate with (3.14), we obtain

$$\frac{1}{|B_{\kappa r}(x_0)|}\int_{B_{\kappa r}(x_0)}|\nabla v-\nabla w|^p dx\le C_2(\Lambda,M,\alpha,p,n)\left(\tau+\tau^{\frac{p-2}{p-1}}\Big[\frac{\epsilon'}{\epsilon\kappa^{\frac{n}{p}}}+\kappa^\beta\Big]^{\frac{p(p_1-p)}{2p_1}}\right).$$

We firstly choose $\tau>0$ so that

$$C_2(\Lambda,M,n,\alpha,p)\tau=\frac{1}{2}\left(\frac{\epsilon}{2}\right)^p.$$

Next, we choose κ sufficiently small depending only on Λ,n,α,p and ϵ so that

$$\kappa^\beta\le\frac{1}{2}\left[\frac{(\epsilon/2)^p}{4C_2(\Lambda,M,p,\alpha,n)\tau^{\frac{p-2}{p-1}}}\right]^{\frac{2p_1}{p(p_1-p)}}$$

and finally we choose $\epsilon'\in(0,\epsilon/2)$ and sufficiently small so that

$$\epsilon'\le\frac{\kappa^{\frac{n}{p}}\epsilon}{2}\left[\frac{(\epsilon/2)^p}{4C_2(\Lambda,M,p,\alpha,n)\tau^{\frac{p-2}{p-1}}}\right]^{\frac{2p_1}{p(p_1-p)}}.$$

From these choices, it follows that

$$\left(\frac{1}{|B_{\kappa r}(x_0)|}\int_{B_{\kappa r}(x_0)}|\nabla v-\nabla w|^p dx\right)^{1/p}\le \epsilon/2.$$

The first estimate (3.8) then holds thanks to this estimate, the first estimate in (3.9), and the triangle inequality.

Finally, to complete the proof, it remains to verify the second estimate of (3.8). By using the triangle inequality, the assumption of the lemma and the fact that $\epsilon\in(0,1)$, we see that

$$\begin{aligned}\left(\fint_{B_{\kappa r}(x_0)}|\nabla w|^p dx\right)^{1/p}&\le\left(\fint_{B_{\kappa r}(x_0)}|\nabla w-\nabla u|^p dx\right)^{1/p}+\left(\fint_{B_{\kappa r}(x_0)}|\nabla u|^p dx\right)^{1/p}\\&\le\epsilon+\left(2^n\fint_{B_{2\kappa r}(x_0)}|\nabla u|^p dx\right)^{1/p}\le\epsilon+2^{\frac{n}{p}}\le 1+2^{\frac{n}{p}}=C_0(n,p).\end{aligned}$$

The proof is therefore complete. □

Summerizing the efforts, we can state and prove the main result of the section.

Proposition 3.3. *Let $\Lambda>0, p>1$ and $\alpha\in(0,1]$ be fixed. Then, for every $\epsilon\in(0,1)$, there exist sufficiently small numbers $\kappa=\kappa(\epsilon,\Lambda,M,p,n,\alpha)\in(0,1/2]$ and $\delta=\delta(\epsilon,\Lambda,M,\alpha,n,p)\in(0,\epsilon)$ such that the following holds. Assume that $\mathbf{A}:B_{2R}\times\mathbb{K}\times\mathbb{R}^n\to\mathbb{R}^n$ such that* (1.2)–(1.4) *and* (1.8) *hold for some $R>0$ and some open interval $\mathbb{K}\subset\mathbb{R}$, and assume that*

$$\fint_{B_{2r}(x_0)}|\mathbf{F}|^p dx\le\delta^p,$$

for some $x_0\in\overline{B}_R$ and some $r\in(0,R/2)$. Then, for every $\lambda\ge 0$, if $u\in W^{1,p}(B_{2R})$ is a weak solution of (3.1) *satisfying*

$$\fint_{B_{4\kappa r}(x_0)}|\nabla u|^p dx\le 1,\quad \fint_{B_{2r}(x_0)}|\nabla u|^p dx\le 1,\quad \text{and}\quad [[\lambda u]]_{\mathrm{BMO}(B_R,R)}\le M,$$

then there is $h\in W^{1,p}(B_{7\kappa r/4}(x_0))$ such that the following estimate holds

$$\fint_{B_{7\kappa r/4}(x_0)}|\nabla u-\nabla h|^p dx\le\epsilon^p,\quad \|\nabla h\|_{L^\infty(B_{3\kappa r/2}(x_0))}\le C(\Lambda,n,p). \tag{3.17}$$

Proof. For given ϵ, let

$$\delta=\min\{\delta_0(\epsilon/[2C_0(n,p)],\Lambda,n,p),\delta_2(\epsilon/2,\Lambda,M,\alpha,p)\},$$

where δ_0 is defined in Lemma 2.12, δ_2 is defined in Lemma 3.2, and $C_0(n,p)>1$ is a constant defined in (3.8). We now prove our Proposition 3.2 with this choice of δ,κ. Note that since both numbers $\hat{\delta}_0,\delta_2$ are independent on λ, so do δ,κ. If $\lambda=0$, then our proposition follows directly from Lemma 2.12 with $\mathbf{G}$ replaced by $\mathbf{F}$ and for $\kappa=1/2$. Also, when $\lambda>0$, let κ be a number defined as in Lemma 3.2. Then, our proposition follows directly by applying Lemma 3.2 with r replaced by $2r$, Lemma 2.12 with $\mathbf{G}=0$ and r replaced by $2\kappa r$, and the triangle inequality. □

4. Level set estimates and proof of Theorem 1.1

4.1. **Level set estimates.** Recall that the Hardy-Littlewood maximal function $\mathcal{M}(f)$ is defined in (2.3), and $\mathcal{M}_U(f)=\mathcal{M}(f\chi_U)$ for an open set U and its characteristic function χ_U. Our first result of this subsection is the following important lemma on the density of the level sets of solution u of (3.1).

Lemma 4.1. *Let Λ,M be positive numbers, $p,\gamma>1,\alpha\in(0,1]$, and let $\epsilon>0$ sufficiently small. Then there exist sufficiently large number $N=N(\Lambda,n,p)\ge 1$ and there exist two positive sufficiently small numbers $\kappa=\kappa(\epsilon,\Lambda,M,p,n,\gamma,\alpha)\in(0,1/2]$ and $\delta=\delta(\epsilon,\Lambda,M,p,n,\gamma,\alpha)\in(0,\epsilon)$ such that the following statement holds. Suppose that $\mathbf{A}:B_{2R}\times\mathbb{K}\times\mathbb{R}^n\to\mathbb{R}^n$ such that* (1.2)–(1.4) *and* (1.8) *hold for some $R>0$*

and some open interval $\mathbb{K} \subset \mathbb{R}$. *Suppose also that* $u \in W^{1,p}(B_{2R})$ *is a weak solution of* (3.1) *satisfying* $[[\lambda u]]_{\mathrm{BMO}(B_R,R)} \le M$ *with some* $\lambda \ge 0$. *If* $y \in B_R$ *and* $\rho \in (0, \kappa_0)$ *such that*

$$B_\rho(y) \cap \Big\{B_R : \mathcal{M}_{B_{2R}}(|\nabla u|^p) \le 1\Big\} \cap \Big\{B_R : \mathcal{M}_{B_{2R}}(|\mathbf{F}|^p) \le \delta^p\Big\} \neq \emptyset,$$

for $\kappa_0 = \min\Big\{1, R\Big\}\kappa/6$, *then*

$$\omega\Big(\Big\{x \in B_R : \mathcal{M}_{B_{2R}}(|\nabla u|^p) > N\Big\} \cap B_\rho(y)\Big) \le \epsilon\omega(B_\rho(y)), \tag{4.1}$$

for $\omega \in A_q$ *with* $[\omega]_{A_q} \le \gamma$ *and* $q > 1$.

Proof. The proof is standard using Proposition 3.3. However, as Proposition 3.3 is stated differently compared to the other similar approximation estimates in the literature, details of the proof of this lemma is required. For a given $\epsilon > 0$, let $\epsilon' > 0$ be a positive number to be determined depending only on ϵ, Λ, n, p and γ. Then, let $\kappa = \kappa(\epsilon', \Lambda, M, p, n, \alpha)$ and $\delta = \delta(\epsilon', \Lambda, M, p, n, \alpha)$ be the numbers defined in Proposition 3.3. We prove the lemma with this choice of δ, κ. By the assumption, we can find

$$x_0 \in B_\rho(y) \cap \Big\{B_R : \mathcal{M}_{B_{2R}}(|\nabla u|^p) \le 1\Big\} \cap \Big\{B_R : \mathcal{M}_{B_{2R}}(|\mathbf{F}|^p) \le \delta^p\Big\}. \tag{4.2}$$

Let $r = \kappa^{-1}\rho \in (0, R/6)$. Since $\rho \in (0, \kappa_0)$ and κ is sufficiently small, $B_{4r}(y) \subset B_{5r}(x_0) \subset B_{2R}$. From this and (4.2), it follows that

$$\fint_{B_{4r}(y)} |\nabla u|^p dx \le \frac{|B_{5r}(x_0)|}{|B_{4r}(y)|} \fint_{B_{5r}(x_0)} |\nabla u|^p dx \le \left(\frac{5}{4}\right)^n,$$
$$\fint_{B_{4r}(y)} |\mathbf{F}|^p dx \le \frac{|B_{5r}(x_0)|}{|B_{4r}(y)|} \fint_{B_{5r}(x_0)} |\mathbf{F}|^p dx \le \left(\frac{5}{4}\right)^n \delta^p.$$

Moreover, we also have $B_{8\rho}(y) \subset B_{9\rho}(x_0) \subset B_{2R}$ and therefore

$$\fint_{B_{8\kappa r}(y)} |\nabla u|^p dx = \fint_{B_{8\rho}(y)} |\nabla u|^p dx \le \frac{|B_{9\rho}(x_0)|}{|B_{8\rho}(y)|} \fint_{B_{9\rho}(x_0)} |\nabla u|^p dx \le \left(\frac{9}{8}\right)^n.$$

Hence, all conditions in Proposition 3.3 are satisfied with some suitable scaling. From this, and our choice of κ, δ, we can apply Proposition 3.3 to find a function $h \in W^{1,p}(B_{\frac{7\rho}{2}}(y))$ satisfying

$$\fint_{B_{\frac{7\rho}{2}}(y)} |\nabla u - \nabla h|^p dx \le (\epsilon')^p \left(\frac{3}{2}\right)^n, \quad \|\nabla h\|_{L^\infty(B_{3\rho}(y))} \le C_*(\Lambda, n, p),$$

where in the above estimates, we have used the fact that $\kappa r = \rho$. Let us now denote

$$N = \max\Big\{2^p C_*^p, 2^n\Big\},$$

and we will prove (4.1) with this choice of N. To this end, we will firstly prove that

$$\Big\{x \in B_\rho(y) : \mathcal{M}_{B_{\frac{7\rho}{2}}(y)}(|\nabla u - \nabla h|^p)(x) \le C_*^p\Big\} \subset \Big\{x \in B_\rho(y) : \mathcal{M}_{B_{2R}}(|\nabla u|^p)(x) \le N\Big\}. \tag{4.3}$$

To prove this statement, let x be a point in the set on the left side of (4.3), and we shall verify that

$$\mathcal{M}_{B_{2R}}(|\nabla u|^p)(x) \le N. \tag{4.4}$$

Let $\rho' > 0$ be any number. If $\rho' < 2\rho$, then $B_{\rho'}(x) \subset B_{3\rho}(y) \subset B_{2R}$, and it follows that

$$\begin{aligned}\left(\fint_{B_{\rho'}(x)} |\nabla u(z)|^p dz\right)^{1/p} &\le \left(\fint_{B_{\rho'}(x)} |\nabla u(z) - \nabla h(z)|^p dz\right)^{1/p} + \left(\fint_{B_{\rho'}(x)} |\nabla h(z)|^p dz\right)^{1/p} \\ &\le \Big(\mathcal{M}_{B_{7\rho/2}(y)}(|\nabla u - \nabla h|^p)(x)\Big)^{1/p} + \|\nabla h\|_{L^\infty(B_{3\rho}(y))} \le 2C_* \le N^{1/p}.\end{aligned}$$

On the other hand, if $\rho' \geq 2\rho$, we note that $B_{\rho'}(x) \subset B_{2\rho'}(x_0)$, and it follows from this and (4.2) that

$$\frac{1}{|B_{\rho'}(x)|}\int_{B_{\rho'}(x)\cap B_{2R}} |\nabla u(z)|^p dz \leq \frac{|B_{2\rho'}(x_0)|}{|B_{\rho'}(x)|}\frac{1}{|B_{2\rho'}(x_0)|}\int_{B_{2\rho'}(x_0)\cap B_{2R}} |\nabla u(z)|^p dz \leq 2^n \leq N.$$

Hence, (4.4) is verified and therefore (4.3) is proved. Observe that (4.3) is in fact equivalent to

$$\Big\{x \in B_\rho(y) : \mathcal{M}_{B_{2R}}(|\nabla u|^p)(x) > N\Big\} \subset E := \Big\{x \in B_\rho(y) : \mathcal{M}_{B_{7\rho/2}(y)}(|\nabla u - \nabla h|^p)(x) > C_*^p\Big\}. \tag{4.5}$$

On the other hand, from the weak type (1,1) estimate of Hardy-Littlewood maximal function, see Lemma 2.6, it is true that

$$\frac{|E|}{|B_\rho(y)|} \leq \frac{C(n)}{C_*^p} \fint_{B_{7\rho/2}(y)} |\nabla u - \nabla h|^p dz \leq C_1(\Lambda, n, p)(\epsilon')^p.$$

From this and the doubling property of A_q-weights as in (ii) of Lemma 2.4, it follows

$$\frac{\omega(E)}{\omega(B_\rho(y))} \leq C(n,\gamma)\left(\frac{|E|}{|B_\rho(y)|}\right)^\beta \leq C'(\Lambda, n, p, \gamma)(\epsilon')^{p\beta},$$

for some $\beta = \beta(\gamma, n) > 0$. Therefore, by choosing ϵ' depending on $\epsilon, \Lambda, n, p, \gamma$ such that

$$C'(\Lambda, n, p, \gamma)(\epsilon')^{p\beta} = \epsilon,$$

we obtain

$$\omega(E) \leq \epsilon\omega(B_\rho(y)).$$

From this estimate and the definition of E in (4.5), the estimate (4.1) follows and the proof is complete. □

The following level set estimate is a direct corollary of Lemma 4.1 and Lemma 2.7, which is also the main result of the subsection.

Lemma 4.2. *Let Λ, M be positive numbers, $p, \gamma > 1, \alpha \in (0,1]$, and let $\epsilon > 0$ be sufficiently small. Then there exists a sufficiently large number $N = N(\Lambda, n, p) \geq 1$, and there exists a sufficiently small number $\delta = \delta(\epsilon, \Lambda, M, p, n, \alpha) \in (0, \epsilon)$ such that the following statement holds. Assume that $\mathbf{A} : B_{2R} \times \mathbb{K} \times \mathbb{R}^n \to \mathbb{R}^n$ such that* (1.2)–(1.4) *and* (1.8) *hold for some $R > 0$ and some open interval $\mathbb{K} \subset \mathbb{R}$. Suppose also that for any $\lambda \geq 0$, if $u \in W^{1,p}(B_{2R})$ is a weak solution of* (3.1) *satisfying*

$$[[\lambda u]]_{\mathrm{BMO}(B_R,R)} \leq M, \quad \omega(\{B_R : \mathcal{M}_{B_{2R}}(|\nabla u|^p) > N\}) \leq \epsilon\omega(B_{\kappa_0}(y)), \quad \forall\, y \in \overline{B}_R, \tag{4.6}$$

for some $\omega \in A_q$, for some $q > 1$ and $[\omega]_{A_q} \leq \gamma$. Then with ϵ_1 defined in Lemma 2.7,

$$\begin{aligned} &\omega\Big(\Big\{B_R : \mathcal{M}_{B_{2R}}(|\nabla u|^p) > N\Big\}\Big) \\ &\leq \epsilon_1\left[\omega\Big(\Big\{B_R : \mathcal{M}_{B_{2R}}(|\nabla u|^p) > 1\Big\}\Big) + \omega\Big(\Big\{B_R : \mathcal{M}_{B_{2R}}(|\mathbf{F}|^p) > \delta^p\Big\}\Big)\right], \end{aligned} \tag{4.7}$$

where κ_0 is defined in Lemma 4.1.

Proof. Let N, κ_0, δ be defined as in Lemma 4.1. We apply Lemma 2.7 with

$$C = \Big\{x \in B_R : \mathcal{M}_{B_{2R}}(|\nabla u|^p)(x) > N\Big\},$$

and

$$D = \Big\{x \in B_R : \mathcal{M}_{B_{2R}}(|\nabla u|^p)(x) > 1\Big\} \cup \Big\{x \in B_R : \mathcal{M}_{B_{2R}}(|\mathbf{F}|^p)(x) > \delta^p\Big\}.$$

Observe that by the second condition in (4.6), (i) of Lemma 2.7 is satisfied. On the other hand, by Lemma 4.1, (ii) of Lemma 2.7 holds true. Therefore, both conditions of Lemma 2.7 are valid, and (4.7) follows directly from Lemma 2.7. □

4.2. **Proof of the interior $W^{1,q}$-regularity estimates.** From the Lemma 4.2 and an iterating procedure, we obtain the following lemma

Lemma 4.3. *Let $\Lambda, M, p, \alpha, \epsilon, N, \delta, \kappa, \kappa_0$ be as in Lemma 4.2. Also, let $\mathbf{A}, R$ be as in Lemma 4.2. Then, for any $\lambda \geq 0$, if $u \in W^{1,p}(B_{2R})$ is a weak solution of* (3.1) *satisfying*

$$[[\lambda u]]_{\mathrm{BMO}(B_R,R)} \leq M, \quad \text{and} \quad \omega(\{B_R : \mathcal{M}_{B_{2R}}(|\nabla u|^p) > N\}) \leq \epsilon\omega(B_{\kappa_0}(y)), \quad \forall\, y \in \overline{B}_R,$$

for some $\omega \in A_q$ with $q > 1$ and $[\omega]_{A_q} \leq \gamma$, then with ϵ_1 defined as in Lemma 2.7, and for any $k \in \mathbb{N}$, the following estimate holds

$$\begin{aligned}\omega\Big(\Big\{B_R : \mathcal{M}_{B_{2R}}(|\nabla u|^p) > N^k\Big\}\Big) &\leq \epsilon_1^k \omega\Big(\Big\{B_R : \mathcal{M}_{B_{2R}}(|\nabla u|^p) > 1\Big\}\Big)\\ &\quad + \sum_{i=1}^{k} \epsilon_1^i \omega\Big(\Big\{B_R : \mathcal{M}_{B_{2R}}(|\mathbf{F}|^p) > \delta^p N^{k-i}\Big\}\Big).\end{aligned} \tag{4.8}$$

Proof. The proof is based induction on $k \in \mathbb{N}$, and an iteration of Lemma 4.2. See, for example, [35, Lemma 4.10]. □

We now can complete the proof of Theorem 1.1.

Proof of Theorem 1.1. The proof now is quite standard. However, we include it here for completeness, and for the transparency regarding the role of the scaling parameter λ. Let $N = N(\Lambda, p, n)$ be defined as in Lemma 4.3. For $q > 1$, we denote choose $\epsilon > 0$ and sufficiently small and depending only on Λ, n, p, q and γ such that

$$\epsilon_1 N^q = 1/2,$$

where ϵ_1 is defined in Lemma 4.3. With this ϵ, we can now choose

$$\delta = \delta(\epsilon, \Lambda, M, p, q, n, \alpha), \quad \kappa = \kappa(\epsilon, \Lambda, M, p, q, n, \gamma, \alpha), \quad \kappa_0 = \min\Big\{1, R\Big\}\kappa/6$$

as determined by Lemma 4.3. Assume that the assumptions of Theorem 1.1 hold with this choice of δ. For $\lambda \geq 0$, let us assume that u is a weak solution of (3.1) satisfying $[[\lambda u]]_{\mathrm{BMO}(B_R)} \leq M$, and let

$$E = E(\lambda, N) = \Big\{B_R : \mathcal{M}_{B_{2R}}(|\nabla u|^p) > N\Big\}. \tag{4.9}$$

We now prove the estimate in Theorem 1.1 with the following additional assumption that

$$\omega(E) \leq \epsilon\omega(B_{\kappa_0}(y)), \quad \forall\, y \in \overline{B}_R. \tag{4.10}$$

Let us now consider the sum

$$S = \sum_{k=1}^{\infty} N^{qk} \omega\Big(\Big\{B_R : \mathcal{M}_{B_{2R}}(|\nabla u|^p) > N^k\Big\}\Big).$$

From (4.10), we can apply Lemma 4.3 to obtain

$$S \leq \sum_{k=1}^{\infty} N^{kq} \sum_{i=1}^{k} \epsilon_1^i \omega\Big(\Big\{B_R : \mathcal{M}_{B_{2R}}(|\mathbf{F}|^p) > \delta^p N^{k-i}\Big\}\Big) + \sum_{k=1}^{\infty} \Big(N^q \epsilon_1\Big)^k \omega\Big(\Big\{B_R : \mathcal{M}_{B_{2R}}(|\nabla u|^p) > 1\Big\}\Big).$$

By Fubini's theorem, the above estimate can be rewritten as

$$S \leq \sum_{j=1}^{\infty} (N^q \epsilon_1)^j \sum_{k=j}^{\infty} N^{q(k-j)} \omega\Big(\Big\{B_R : \mathcal{M}_{B_{2R}}(|\mathbf{F}|^p) > \delta^p N^{k-j}\Big\}\Big) + \sum_{k=1}^{\infty} \Big(N^q \epsilon_1\Big)^k \omega\Big(\Big\{B_R : \mathcal{M}_{B_{2R}}(|\nabla u|^p) > 1\Big\}\Big). \tag{4.11}$$

Observe that

$$\omega\Big(\Big\{B_R : \mathcal{M}_{B_{2R}}(|\nabla u|^p) > 1\Big\}\Big) \leq \omega\Big(B_R\Big).$$

From this, the choice of ϵ, and Lemma 2.5, and (4.11) it follows that

$$S \leq C\left[\left\|\mathcal{M}_{B_{2R}}(|\mathbf{F}|^p)\right\|^q_{L^q(B_R,\omega)} + \omega(B_R)\right].$$

Applying the Lemma 2.5 again, we infer that

$$\left\|\mathcal{M}_{B_{2R}}(|\nabla u|^p)\right\|^q_{L^q(B_R,\omega)} \le C\left[\left\|\mathcal{M}_{B_{2R}}(|\mathbf{F}|^p)\right\|^q_{L^q(B_{2R},\omega)} + \omega(B_R)\right].$$

Also, by the Lesbegue's differentiation theorem, it is true that

$$|\nabla u(x)|^p \le \mathcal{M}_{B_{2R}}(|\nabla u|^p)(x), \quad \text{a.e } x \in B_R.$$

Hence,

$$\|\nabla u\|^{pq}_{L^{pq}(B_R,\omega)} \le C\left[\left\|\mathcal{M}_{B_{2R}}(|\mathbf{F}|^p)\right\|^q_{L^q(B_R,\omega)} + \omega(B_R)\right].$$

From this and Lemma 2.6, it follows that

$$\|\nabla u\|_{L^{pq}(B_R,\omega)} \le C\left[\|\mathbf{F}\|_{L^{pq}(B_{2R},\omega)} + \omega(B_R)^{1/q}\right]. \tag{4.12}$$

Summarizing the efforts, we conclude that (4.12) holds true as long as u is a weak solution of (3.1) for $\lambda \ge 0$ and (4.10) holds.

It now remains to remove the additional assumption (4.10). To this end, assume all assumptions in Theorem 1.1 holds, and let u be a weak solution of (3.1) with some $\lambda \ge 0$. Let $\mu > 0$ sufficiently large to be determined, and let $\lambda' = \lambda\mu \ge 0$, $u_\mu = u/\mu$, and $\mathbf{F}_\mu = \mathbf{F}/\mu$. We note that u_μ is a weak solution of

$$\operatorname{div}[\hat{\mathbf{A}}(x, \lambda' u_\mu, \nabla u_\mu)] \quad = \quad \operatorname{div}[|\mathbf{F}_\mu|^{p-2}\mathbf{F}_\mu], \qquad \text{in} \quad B_{2R}, \tag{4.13}$$

where

$$\hat{\mathbf{A}}(x, z, \xi) = \frac{\mathbf{A}(x, z, \mu\xi)}{\mu^{p-1}}.$$

Note that by Remark 2.1, $\hat{\mathbf{A}}$ satisfies all (1.2)-(1.4) with the same constants Λ, p, α. Moreover, $\hat{\mathbf{A}}$ also satisfies (1.8). We then denote

$$E_\mu = \left\{B_R : \mathcal{M}_{B_{2R}}(|\nabla u_\mu|^p) > N\right\}.$$

and we assume that

$$K_0 = \left(\frac{1}{|B_{2R}|}\int_{B_{2R}} |\nabla u|^p dx\right)^{1/p} > 0. \tag{4.14}$$

We claim that we can choose $\mu = CK_0$ with some sufficiently large constant C depending only on Λ, M, p, q, n and R/κ_0 such that

$$\omega(E_M) \le \epsilon\omega(B_{\kappa_0}(y)), \quad \forall\, y \in \overline{B}_R. \tag{4.15}$$

If this holds, we can apply (4.12) for u_μ which is a weak solution of (4.13) to obtain

$$\left\|\nabla u_\mu\right\|_{L^{pq}(B_R,\omega)} \le C\left[\left\|\mathbf{F}_\mu\right\|_{L^{pq}(B_{2R},\omega)} + \omega(B_R)^{1/q}\right].$$

Then, by multiplying this equality with μ, we obtain

$$\|\nabla u\|_{L^{pq}(B_R,\omega)} \le C\left[\|\mathbf{F}\|_{L^{pq}(B_{2R},\omega)} + \omega(B_R)^{1/q}K_0\right].$$

The proof of Theorem 1.1 is therefore complete if we can prove (4.15). To this end, using the doubling property of $\omega \in A_q$ as in (i) of Lemma 2.4, we have

$$\frac{\omega(E_\mu)}{\omega(B_{\kappa_0}(y))} = \frac{\omega(E_\mu)}{\omega(B_{2R})}\frac{\omega(B_{2R})}{\omega(B_{\kappa_0}(y))} \le \gamma\frac{\omega(E_\mu)}{\omega(B_{2R})}\left(\frac{2R}{\kappa_0}\right)^{nq}.$$

From this, and using (ii) of Lemma 2.4 again, we can find $\beta = \beta(\gamma, n) > 0$ such that

$$\frac{\omega(E_\mu)}{\omega(B_{\kappa_0}(y))} \le C(\gamma, n)\left(\frac{2R}{\kappa_0}\right)^{nq}\left(\frac{|E_\mu|}{|B_{2R}|}\right)^{\beta/p}. \tag{4.16}$$

Now, by the definition of E_μ, and the weak type (1-1) estimate for maximal function, we see that

$$\begin{aligned}\frac{|E_\mu|}{|B_{2R}|} &= \left|\left\{B_R : \mathcal{M}_{B_{2R}}(|\nabla u|^p) > N\mu^p\right\}\right|/|B_{2R}| \\ &= \frac{C(n,p)}{N\mu^p}\frac{1}{|B_{2R}|}\int_{B_{2R}}|\nabla u|^p dx \le \frac{C(p,n)K_0^p}{N\mu^p},\end{aligned}$$

where K_0 is defined in (4.14). From this estimate and (4.16), it follows that

$$\frac{\omega(E_\mu)}{\omega(B_{\kappa_0}(y))} \le C^*(\Lambda,\gamma,p,n)\left(\frac{2R}{\kappa_0}\right)^{nq}\left(\frac{K_0}{\mu}\right)^{\beta}.$$

Now, we choose μ such that

$$\mu = K_0\left[\epsilon^{-1}C^*(\Lambda,\gamma,p,n)\left(\frac{2R}{\kappa_0}\right)^{nq}\right]^{1/\beta}$$

then it follows that

$$\omega(E_\mu) \le \epsilon\omega(B_{\kappa_0}(y)), \quad \forall\, y \in \overline{B}_R.$$

This proves (4.15) and completes the proof of Theorem 1.1. □

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