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THE CORROBORATION PARADOX

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- p = a probability measure on an algebra generated by events E and H.
- E is *p-positively relevant to* H if

$p(H|E) > p(H)$ ($\Leftrightarrow$ $p(H|E) > p(H|E^c)$)

- Positive relevance is <u>symmetric</u> and <u>non-transitive</u>.

- THE DETERMINANT TEST

	H	H^c
E	$a = p(EH)$	$b = p(EH^c)$
E^c	$c = p(E^cH)$	$d = p(E^cH^c)$

E and H are p-positively relevant to each other if and only if $ad - bc > 0$.

- One's naïve intuitions about positive relevance (based on regarding positive relevance as an attenuated implication relation) are almost invariably mistaken, and result in a plethora of *paradoxes of positive relevance.* The following explains in part the ubiquity of such paradoxes:
- $E_1, \ldots, E_n$ = evidentiary events bearing on hypothesis H, all subsets of some set of possible states of the world Ω.
- If I is a subset of $[n] := \{1, \ldots, n\}$,

$$E_I := \cap_{i \varepsilon I} E_i, \text{ with } E_\varnothing = \Omega$$

$$E_I^{\#} := E_I \cap (\cap_{i \varepsilon [n]-I} E_i^c)$$

e.g., $E_\varnothing^{\#} = E_1^c \cap \ldots \cap E_n^c$ and

$$E_{[n]}^{\#} = E_{[n]} = E_1 \cap \ldots \cap E_n$$

• $E_1,\ldots, E_n$ and H are *qualitatively independent* (Rényi) if, for every subset I of [n], the atomic events $H \cap E_I^{\#}$ and $H^c \cap E_I^{\#}$ are nonempty.

THE MOTHER OF ALL (well, at least many) PARADOXES OF POSITIVE RELEVANCE.

Theorem 1. If $E_1,\ldots, E_n$ and H are qualitatively independent and $\{c_I\}$ is *any* family of real numbers in the open interval (0,1) indexed on subsets I of [n], then there exists a probability measure p on the algebra $\mathcal{A}$ generated by $E_1,\ldots, E_n$ and H such that $p(H|E_I) = c_I$ for every I.

EXAMPLE: Corroboration Paradoxes

• Suppose that each of the evidentiary events E_i ($i = 1,\ldots,n$) is p-positively relevant to H. These events are *mutually corroborating with respect to* H if, for all subsets J and I of [n], where J properly contains I, $p(H|E_J) > p(H|E_I)$.

• A *corroboration paradox* occurs whenever events E_i, each positively relevant to some H, fail to be mutually corroborating with respect to H.

•How troubling is the possibility of encountering such paradoxes?

A paradox is *not* a contradiction. Still, there is something unsettling about the possibility that, say, $p(H|E_i) > p(H)$, $i=1,2$, but $p(H|E_1E_2) \leq p(H|E_1)$ or $p(H|E_1E_2) \leq p(H|E_2)$, or even worse, $p(H|E_1E_2) < p(H)$.

JOHN POLLOCK'S RESPONSE:

Not to worry! It is *defeasibly reasonable* to assume that two items of evidence, each positively relevant to some hypothesis, are mutually corroborating.

Why?

Because if $p(H) = a$, $p(H|E_1) = r$, and $p(H|E_2) = s$, where $0 < r, s, a < 1$ and $r, a, s \in \mathbb{Q}$, it is defeasibly reasonable to assume that

$$p(H|E_1E_2) = rs(1 - a) / \{a(1 - r - s) + rs\}$$

$$\uparrow$$

$$Y(r,s|a)$$

and it is easy to verify the following

Theorem 2. If $r > a$, then $Y(r,s|a) > s$, and if $s > a$, then $Y(r,s|a) > r$.

(result is also true if $>$ is replaced by $<$)

• Where does the Y-function come from?

1. Ω = a finite set of possible states of the world, equipped with the <u>uniform</u> probability measure p: $p(E) = |E| / |\Omega|$.

2. Suppose the family $\mathcal{F}(\Omega, a, r, s) := \{(H, E_1, E_2): p(H) = a,\ p(H|E_1) = r,\ \text{and}\ p(H|E_2) = s\}$ is nonempty.

3. P = the uniform probability on $\mathcal{F}(\Omega, a, r, s)$.

Pollock's Theorems:

Theorem 4. For all $\delta > 0$ and all $\varepsilon > 0$ there exists an infinite sequence of finite sets Ω of increasing cardinality such that

$$P\{|p(H|E_1E_2) - Y(r,s,a)| \leq \delta\} \geq 1 - \varepsilon .$$

Theorem 5. The P-probability of the subset of $\mathscr{F}(\Omega, a, r, s)$ consisting of those triples (H, E_1, E_2) for which E_1 and E_2 are mutually p-corroborating with respect to H can be made as close to 1 as we wish on an infinite sequence of finite sets Ω of increasing cardinality.

Note that Pollock invokes the *principle of insufficient reason* in his analysis. We will see that a differently structured application of this principle issues a very different verdict on the frequency with which one may expect to encounter corroboration paradoxes.

JONATHAN COHEN'S APPROACH

Given

(1) $p(H|E_1) > p(H)$ and

(2) $p(H|E_2) > p(H)$,

find supplementary conditions which, along with (1) and (2), imply that

(3) $p(H|E_1E_2) > p(H|E_1)$ and

(4) $p(H|E_1E_2) > p(H|E_2)$.

Cohen's conditions:

(5) $p(E_1|E_2H) \geq p(E_1|H)$

($\Leftrightarrow$ $p(E_2|E_1H) \geq p(E_2|H)$) and

(6) $p(E_1|E_2H^c) \leq p(E_1|H^c)$

($\Leftrightarrow$ $p(E_2|E_1H^c) \leq p(E_2|H^c)$)

Theorem 6. (1)&(2)&(5)&(6) => (3)&(4)

Remark. (5) and (6) are generalizations of the conditional independence of E_1 and E_2, given H, and given H^c. Given (1) and (2), conditions (5) and (6) are sufficient, *but not necessary*, to ensure mutual corroboration.

Nevertheless, Cohen claims that mutual corroboration "will not normally" occur unless (5) and (6) hold.

Like Pollock's theorems, this is a *second-order* probability assertion, though not as explicitly articulated.

A FRAMEWORK FOR ASSESSING PROBABILITY PARADOXES

- $\mathcal{A}$ = an algebra of propositions
- $\prod_{\mathcal{A}}$ = the set of all probability measures p on $\mathcal{A}$.
- $\mathcal{C}$, $\mathcal{D}$, and $\mathcal{S}$ denote subsets of $\prod_{\mathcal{A}}$, or, alternatively, predicates on $\prod_{\mathcal{A}}$. Write $\mathcal{C}$(p) when p ε $\mathcal{C}$, etc.

The general form of probability paradoxes: Naïve intuition mistakenly suggests that condition $\mathcal{C}$(p) implies some "desirable" condition $\mathcal{D}$(p).

Natural response to discovering that this is not the case: Find a "supplementary" condition $\mathcal{S}$(p) such that
$\mathcal{C}$(p) & $\mathcal{S}$(p) => $\mathcal{D}$(p)

"PROBABLE PROBABILITIES"

- $\sum$ = a sigma algebra of subsets of $\prod \mathscr{A}$, with P = a probability measure on $\sum$.
- $\mathscr{C}$, $\mathscr{D}$, and $\mathscr{S}$ ε $\sum$; P($\mathscr{C}$), P($\mathscr{D}$), P($\mathscr{S}$) > 0.

- $P(\mathscr{D}^c | \mathscr{C})$ = the *prevalence* of the paradox, relative to P.
- $P(\mathscr{S} | \mathscr{C})$ = the *incidence of* $\mathscr{S}$ in $\mathscr{C}$, relative to P.
- $P(\mathscr{S} | \mathscr{D} \cap \mathscr{C})$ = the $\mathscr{S}$- *provenance of* $\mathscr{D}$ in $\mathscr{C}$, relative to P.
- Since $\mathscr{C} \cap \mathscr{S}$ is a subset of $\mathscr{D}$,

$$P(\mathscr{S} | \mathscr{C}) = P(\mathscr{D} | \mathscr{C}) \times P(\mathscr{S} | \mathscr{D} \cap \mathscr{C})$$

"incidence = (1-prevalence)×provenance

Application to corroboration paradox:

- $\mathcal{C}(p) \Leftrightarrow p(H|E_i) > p(H)$, i = 1,2
- $\mathcal{D}(p) \Leftrightarrow p(H|E_1E_2) > p(H|E_i)$, i = 1,2
- $\mathcal{S}(p) \Leftrightarrow p(E_1|E_2H) \geq p(E_1|H)$

  $\& \; p(E_1|E_2H^c) \leq p(E_1|H^c)$
- $\mathcal{A}$ = the algebra of propositions generated by E_1, E_2, and H.
- Identify $\prod_{\mathcal{A}}$ with

  $\mathbb{T} := \{(x_1,\ldots,x_7): x_i \geq 0 \; \& \; x_1+\ldots+x_7 \leq 1\}$
- $\sum$ = the set of Lebesgue measurable subsets of $\mathbb{T}$, and P = the uniform (i.e., normalized Lebesgue) measure on $\sum$.
- By Monte Carlo simulation,

$P(\mathcal{D}^c|\mathcal{C}) \approx 0.37$, $P(\mathcal{S}|\mathcal{D}\mathcal{C}) \approx 0.39$, and $P(\mathcal{S}|\mathcal{C}) \approx 0.25$

CONCLUSIONS

1. The result $P(\mathcal{D}^c|\mathcal{C}) \approx 0.37$ shows that a differently structured application of the principle of insufficient reason yields an estimate of the prevalence of the corroboration paradox quite different from Pollock's. No surprise—results of applying this principle are notoriously unstable.

2. The result $P(\mathcal{S}|\mathcal{D}\mathcal{C}) \approx 0.39$ is at odds with Cohen's assertion that E_1 and E_2 will "not normally" be mutually corroborating unless condition $\mathcal{S}$ holds.

N.B. I am <u>not</u> claiming that P is the "right" second order probability for assessing the corroboration paradox.

Rather, P (and the experiment constituting the Monte Carlo simulation of P) serves a cautionary role, highlighting the fact that the claims of Pollock and Cohen are overly broad and insufficiently supported.

- The prospects for rescuing…

(i) Pollock: dim

(ii) Cohen: much more encouraging—examine concrete examples in various fields of inquiry—law, medicine, etc.

•Supplementary conditions $\mathcal{S}$(p) play several important roles:

(1) In many experimental situations, we can “design in” condition $\mathcal{S}$(p), guaranteeing that $\mathcal{D}$(p) holds whenever $\mathcal{C}$(p) is observed. Example: testing a drug vs. placebo at multiple locations.

(2) If $\mathcal{C}$(p) holds, but not $\mathcal{D}$(p), we know that $\mathcal{S}$(p) fails, which can explain in part why $\mathcal{D}$(p) fails. Example: the Berkeley admissions case.

(3) Assessing qualitative probability relations when $\mathcal{C}$(p), $\mathcal{D}$(p), and $\mathcal{S}$(p) involve inequalities.