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The Smith-Walley Interpretation of Subjective Probability: An Appreciation

Abstract. The right interpretation of subjective probability is implicit in the theories of upper and lower odds, and upper and lower previsions, developed, respectively, by Cedric Smith (1961) and Peter Walley (1991). On this interpretation you are free to assign contingent events the probability 1 (and thus to employ conditionalization as a method of probability revision) without becoming vulnerable to a weak Dutch book.

Keywords: Dutch book, coherence, strict coherence, subjective probability, probability kinematics.

1. Introduction

On the standard betting account of subjective probability (de Finetti, 1931; Ramsey, 1931), conforming your probability assessments with the axioms of finitely additive probability theory protects you against accepting what Ramsey calls a *Dutch Book*, i.e., a finite sequence of bets on which you are certain to sustain a net loss. Probability assessments immune to Dutch books, termed *coherent* by de Finetti, need not, however, protect you against accepting bets on which a net gain is impossible and a net loss is possible.

Indeed, you become vulnerable to such a *weak Dutch book* whenever you assign a contingent event E either of the extreme probabilities 0 or 1. For in the former case you agree, for no consideration whatsoever, to pay 1 unit of linear utility if E occurs, and in the latter you agree to pay 1 unit simply to have your stake returned if E occurs. Since E is neither impossible nor certain, it is logically possible to sustain a net loss in each case, and impossible to do better than break even.

Richard Jeffrey confessed to being unperturbed by such a prospect:

> *What if it IS logically possible to lose a bet on E when your $p(E) = 1$? Aren't you sure you won't lose*?

(Jeffrey, personal communication, 16 October 2000)

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Others, notably Abner Shimony (1955), have found such a prospect unacceptable, and have argued that reasonable subjective probability assessments should be *strictly coherent*, in the sense of being immune even to weak Dutch books. In particular, strict coherence dictates that only the certain event may be assigned the probability 1 (equivalently, only the impossible event may be assigned the probability 0).

Thus, as John Earman (1992) noted in his influential book on Bayesian epistemology, strict coherence is incompatible with conditionalization, the central tool of the Bayesian enterprise. One response to this dilemma might be to abandon conditionalization in favor of probability kinematics (Jeffrey 1983). As Earman points out, however, one would have to reproduce the results of Bayesian confirmation theory, and about merger of opinion and convergence to certainty (heretofore based on classical conditionalization) in a probability-kinematical framework for this gambit to have more than empty formal appeal.

Some results along these lines are known. There is, for example, a straightforward probability-kinematical version of the hypothetico-deductive principle (Wagner 1997), and of Jeffrey's (1991, 1995) solution to the old evidence problem (Wagner 1997, 1999, 2001). Moreover, Brian Skyrms (1994) has formulated a convergence theorem for probability kinematics under certain dynamic coherence assumptions.

The pursuit of further probability-kinematical versions of what Earman calls the "success stories" of Bayesian confirmation theory would certainly be worthwhile. But successful completion of this program is not indispensable to reconciling that theory with the avoidance of a weak Dutch book. As we shall see, all that is required to effect such a reconcilation is a small modification in the set of wagers that you are understood to accept in virtue of assigning event E the probability $p(E)$.

In the next section we give a formal account of coherence and strict coherence under the classical interpretation of subjective probability, and review de Finetti's characterization of coherence and its consequences. In section 3 we develop parallel accounts of these concepts under the aforementioned slight modification of the classical interpretation, and show that, under this modification, coherence and strict coherence coincide with classical coherence.

2. The Classical Interpretation

Suppose that Ω is a set of possible states of the world, assumed to be mutually exclusive and exhaustive, and $\mathbf{E}$ is an arbitrary nonempty family of subsets of Ω. Members of $\mathbf{E}$ are called *events*, and the event $E \in \mathbf{E}$ is said to *occur* if the true state of the world is among the elements of E. An event is *contingent* if it is a proper, nonempty subset of Ω.

A *probability assessment* p on $\mathbf{E}$ is simply a real-valued function with domain $\mathbf{E}$. On the classical betting interpretation of subjective probability, your assigning event E the probability $p(E)$ signals your commitment

(B) to pay $p(E)$ units of linear utility for the chance to receive 1 unit if E occurs, and nothing otherwise; and

(S) to accept $p(E)$ units of linear utility in exchange for obligating yourself to pay 1 unit if E occurs, and nothing otherwise.

In what follows, we refer to (B) as *buying* I_E *for* $p(E)$ and to (S) as *selling* I_E *for* $p(E)$, where I_E is the *characteristic function* of the set E, i.e.,

$$I_E(\omega) = \begin{cases} 1, & \text{if } \omega \in E; \\ 0, & \text{if } \omega \notin E. \end{cases} \tag{1}$$

To pay (respectively, receive) a negative sum is to receive (respectively, pay) the absolute value of that sum.

Suppose that p is your subjective probability, that $E_1, \ldots, E_r$ and $F_1, \ldots, F_s$ are events (here, and henceforth, r and s denote nonnegative integers, at least one of which is positive), and that you buy each I_{E_i} for $p(I_{E_i})$ and sell each I_{F_j} for $p(I_{F_j})$. If $\omega \in \Omega$ turns out to be the true state of the world, then your net payoff on this sequence of wagers is

$$N\left[p, (E_i), (F_j), \omega\right] := \sum_i I_{E_i}(\omega) - p(E_i) + \sum_j p(F_j) - I_{F_j}(\omega). \tag{2}$$

Hence p is incoherent on $\mathbf{E}$ (i.e., vulnerable to a Dutch book) if and only if

DB[p] : There exist r, s, and events $E_1, \ldots, E_r$ and $F_1, \ldots, F_s$ in $\mathbf{E}$ such that $N\left[p, (E_i), (F_j), \omega\right] < 0$ for all $\omega \in \Omega$.

and thus coherent on $\mathbf{E}$ if and only if

C[p] : For all r, s, and all events $E_1, \ldots, E_r$ and $F_1, \ldots, F_s$ in $\mathbf{E}$ $N\left[p, (E_i), (F_j), \omega\right] \geqslant 0$ for some $\omega \in \Omega$.

A family $\mathbf{A}$ of subsets of Ω is an *algebra* if the impossible event $\varnothing$ and the certain event Ω are members of $\mathbf{A}$, and $\mathbf{A}$ is closed under finite unions and intersections and under complementation. A real-valued function p defined on an algebra $\mathbf{A}$ is a *finitely additive probability measure* if $0 \leqslant p(E) \leqslant 1$ for all $E \in \mathbf{A}$, $p(\varnothing) = 0$, $p(\Omega) = 1$, and $p(E \cup F) = p(E) + p(F)$ whenever $E, F \in \mathbf{A}$ and $E \cap F = \varnothing$. If $\mathbf{E}$ is an arbitrary family of subsets of Ω, the *algebra* $\mathbf{A}_\mathbf{E}$ *generated by* $\mathbf{E}$ is the smallest algebra of subsets of Ω that contains each $E \in \mathbf{E}$.

The following elegant characterization of coherence is due to de Finetti (1972, pp. 77-79):

THEOREM 1. *The probability assessment p is coherent on the family $\mathbf{E}$ if and only if p can be extended to a finitely additive probability measure on the algebra $\mathbf{A}_\mathbf{E}$ generated by $\mathbf{E}$.*[1]

It follows from Theorem 1 that if p is coherent on $\mathbf{E}$, then

(i) if $E \in \mathbf{E}$, then $0 \leqslant p(E) \leqslant 1$,

(ii) if $\varnothing \in \mathbf{E}$, then $p(\varnothing) = 0$,

(iii) if $\Omega \in \mathbf{E}$, then $p(\Omega) = 1$, and

(iv) if E, F, $E \cup F \in \mathbf{E}$ and $E \cap F = \varnothing$, then $p(E \cup F) = p(E) + p(F)$.

Note that, unless $\mathbf{E}$ is already an algebra, conditions (i) – (iv) are *not* sufficient for coherence. This can readily be seen from the example $\Omega = \{1, 2\}$, $\mathbf{E} = \{\{1\}, \{2\}\}$ and $p(\{1\}) = p(\{2\}) = 1$, which satisfies (i) – (iv), the latter three conditions being satisfied vacuously.

As noted in the previous section, coherence of p on $\mathbf{E}$ does not protect you against a weak Dutch book, defined formally by

WDB[p] : There exist r, s, and events $E_1, \ldots, E_r$ and $F_1, \ldots, F_s$ in $\mathbf{E}$ such that $N[p, (E_i), (F_j), \omega] \leqslant 0$ for all $\omega \in \Omega$, and $N[p, (E_i), (F_j), \omega] < 0$ for some $\omega \in \Omega$.

Correspondingly, strict coherence (i.e., immunity to WDB), is formally defined by

SC[p] : For all r, s, and all events $E_1, \ldots, E_r$ and $F_1, \ldots, F_s$ in $\mathbf{E}$
$N[p, (E_i), (F_j), \omega] > 0$ for some $\omega \in \Omega$, or
$N[p, (E_i), (F_j), \omega] \geqslant 0$ for all $\omega \in \Omega$.

[1]It can be shown using the Hahn-Banach theorem that a finitely additive probability measure (fapm) on an algebra $\mathbf{A}$ of subsets of Ω can always be extended to a fapm on 2^Ω, the set of all subsets of Ω. Hence Theorem 1 is equivalent to the assertion that p is coherent on $\mathbf{E}$ if and only if it can be extended to a fapm on 2^Ω.

We have already noted the dilemma posed by SC for Bayesian epistemology. In the next section we shall see how this dilemma may be easily resolved.

3. The Smith-Walley Interpretation

The following interpretation of subjective probability is implicit in the theories of upper and lower odds, and of upper and lower previsions, developed, respectively, by Cedric Smith (1961) and Peter Walley (1991). On this interpretation, your assigning event E the probability $p(E)$ signals your commitment

(B^*) to buy I_E for any price *strictly less than* $p(E)$; and

(S^*) to sell I_E for any price *strictly greater than* $p(E)$.

We shall refer to the above as the *threshold interpretation* of p.[2] In particular, setting $p(E) = 0$ no longer commits you to selling I_E for no consideration, but only for each $\alpha > 0$. Similarly, setting $p(E) = 1$ no longer commits you to buying I_E for 1, but only for each $\beta < 1$. In both cases it is possible for you to experience a net gain. Hence on the Smith-Walley interpretation of subjective probability there are no problems associated with assigning contingent events either of the extreme probabilities 0 or 1.

Introducing modifications into a theory in order to remedy problematic features of that theory may of course have the unintended consequence of creating a new set of problems. As we now show, no such consequences detract from the Smith-Walley modification of classical subjective probability.

Suppose that p is your probability assessment, that $E_1, \ldots, E_r$ and $F_1, \ldots, F_s$ are events, and that $\epsilon > 0$. On the threshold interpretation, you are committed to buying each I_{E_i} for $p(E_i) - \epsilon$, and to selling each I_{F_j} for $p(F_j) + \epsilon$. If $\omega \in \Omega$ turns out to be the true state of the world, then your net payoff on this sequence of wagers is

$$N_\epsilon\,[p, (E_i), (F_j), \omega] := \sum_i I_{E_i}(\omega) - (p(E_i) - \epsilon) + \sum_j (p(F_j) + \epsilon) - I_{F_j}(\omega). \quad (3)$$

[2]Note that under either the classical or the threshold interpretation of p, you will assign *the same numerical value* to $p(E)$. For if you will buy or sell I_E for α, you will surely, for all $\epsilon > 0$, buy I_E for $\alpha - \epsilon$ and sell I_E for $\alpha + \epsilon$, and this can be true for no $\alpha' \neq \alpha$. Suppose, conversely, that α is your threshold probability for E. While this entails no commitment to buy or sell I_E for α, should you undertake, either voluntarily or involuntarily, to specify a single price for which you would buy or sell I_E, that price must coincide with α.

On the threshold interpretation your probability assessment p on $\mathbf{E}$ is vulnerable to a Dutch book if and only if

$$\mathbf{DB}^*[\mathbf{p}]: \quad \text{There exist } r,\ s, \text{ events } E_1, \ldots, E_r \text{ and } F_1, \ldots, F_s \text{ in } \mathbf{E}, \text{ and } \epsilon > 0 \text{ such that } N_\epsilon\left[p, (E_i), (F_j), \omega\right] < 0 \text{ for all } \omega \in \Omega.$$

and thus coherent on $\mathbf{E}$ if and only if

$$\mathbf{C}^*[\mathbf{p}]: \quad \text{For all } r,\ s, \text{ all events } E_1, \ldots, E_r \text{ and } F_1, \ldots, F_s \text{ in } \mathbf{E}, \text{ and all } \epsilon > 0,\ N_\epsilon\left[p, (E_i), (F_j), \omega\right] \geqslant 0 \text{ for some } \omega \in \Omega.$$

On the threshold interpretation, p is vulnerable to a weak Dutch book on $\mathbf{E}$ if and only if

$$\mathbf{WDB}^*[\mathbf{p}]: \quad \text{There exist } r,\ s, \text{ events } E_1, \ldots, E_r \text{ and } F_1, \ldots, F_s \text{ in } \mathbf{E}, \text{ and } \epsilon > 0 \text{ such that } N_\epsilon\left[p, (E_i), (F_j), \omega\right] \leqslant 0 \text{ for all } \omega \in \Omega, \text{ and } N_\epsilon\left[p, (E_i), (F_j), \omega\right] < 0 \text{ for some } \omega \in \Omega.$$

and thus strictly coherent on $\mathbf{E}$ if and only if

$$\mathbf{SC}^*[\mathbf{p}]: \quad \text{For all } r,\ s, \text{ all events } E_1, \ldots, E_r \text{ and } F_1, \ldots, F_s \text{ in } \mathbf{E}, \text{ and all } \epsilon > 0,\ N_\epsilon\left[p, (E_i), (F_j), \omega\right] > 0 \text{ for some } \omega \in \Omega, \text{ or } N_\epsilon\left[p, (E_i), (F_j), \omega\right] \geqslant 0 \text{ for all } \omega \in \Omega.$$

THEOREM 2. *If p is a probability assessment on the family $\mathbf{E}$, then*

$$SC[p] \Longrightarrow C[p] \Longleftrightarrow C^*[p] \Longleftrightarrow SC^*[p]. \tag{4}$$

Proof. Trivially, $SC[p] \Longrightarrow C[p]$ and $SC^*[p] \Longrightarrow C^*[p]$. Note that

$$N_\epsilon\left[p, (E_i), (F_j), \omega\right] = N\left[p, (E_i), (F_j), \omega\right] + (r+s)\epsilon. \tag{5}$$

To show that $C^*[p] \Longrightarrow C[p]$, we prove the contrapositive, $DB[p] \Longrightarrow DB^*[p]$: Suppose that $E_1, \ldots, E_r$ and $F_1, \ldots, F_s$ are events for which $N\left[p, (E_i), (F_j), \omega\right] < 0$ for all $\omega \in \Omega$. Even if Ω is infinite, $N\left[p, (E_i), (F_j), \omega\right]$ takes on only finitely many (indeed, at most 2^{r+s}) distinct vaules as ω runs through Ω. Let $-\nu$, where $\nu > 0$, be the largest of these values, and set $\epsilon = \nu/2(r+s)$. Then it follows from (5) that, for all $\omega \in \Omega$, $N_\epsilon\left[p, (E_i), (F_j), \omega\right] \leqslant -\nu/2 < 0$. Finally, $C[p] \Longrightarrow SC^*[p]$. For suppose that $E_1, \ldots, E_r$ and $F_1, \ldots, F_s$ are events. By $C[p]$, there exists $\omega \in \Omega$ such that $N\left[p, (E_i), (F_j), \omega\right] \geqslant 0$, whence, by (5), for all $\epsilon > 0$, $N_\epsilon\left[p, (E_i), (F_j), \omega\right] \geqslant (r+s)\epsilon > 0$, which establishes $SC^*[p]$. ∎

Combining Theorems 1 and 2 yields

THEOREM 3. *If* p *is a probability assessment on the family* $\mathbf{E}$*, then* $SC^*[p]$ *if and only if* p *can be extended to a finitely additive probability measure on the algebra* $\mathbf{A}_{\mathbf{E}}$ *generated by* $\mathbf{E}$*.*

So we see that vulnerability, under the classical interpretation, of an otherwise coherent probability to a weak Dutch book is no defect in the probability, but simply an (easily effaced) artifact of the gratuitous requirement that one must buy or sell at the "fair" (= threshold) price, not just at prices that one considers to be positively advantageous. With this observation we rest our case for adopting the Smith-Walley interpretation of subjective probability.[3]

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[3]Interestingly, the interpretation of subjective probability implicit in Smith's formulation of upper and lower odds and Walley's formulation of upper and lower previsions is not explicitly articulated by either author. This was of course not a matter of being unaware of this interpretation, but rather of having bigger fish to fry, namely, the promotion of statistical reasoning with imprecise probabilities.

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