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A NON-INDUCTIVE PROOF OF THE PRINCIPLE OF INCLUSION AND EXCLUSION

Lemma. For all positive integers n,

(1) $$\sum_{k=o}^{n}(-1)^k\binom{n}{k}=0.$$

Proof. Let $\mathcal{E}$ denote the set of all subsets of [n] having even cardinality, and $\mathcal{O}$ the set of all subsets of [n] having odd cardinality. Formula (1) is equivalent to the assertion that $|\mathcal{E}| = |\mathcal{O}|$. It remains only to observe that the map from $\mathcal{E}$ to $\mathcal{O}$ defined by (i) $E \mapsto E - \{1\}$ if $1 \in E$, and (ii) $E \mapsto E \cup \{1\}$ if $1 \notin E$ is a bijection.

The Characteristic Function of a Set. Suppose that A and B are sets and $B \subset A$. The *characteristic function of* B, denoted χ_B, is defined for all $a \in A$ by (i) $\chi_B(a) = 1$ if $a \in B$, and (ii) $\chi_B(a) = 0$ if $a \notin B$. Note that if B is finite, then

(2) $$|B| = \sum_{a\in A}\chi_B(a).$$

Theorem (*Principle of Inclusion and Exclusion, a.k.a. the Sieve Formula*)**.**
Let $A_1, \ldots, A_n$ be a sequence of subsets of the finite set A. Then

(3) $$|A_1 \cup \ldots \cup A_n| = \sum_{\varnothing\neq I\subset[n]}(-1)^{|I|-1}\,|\cap_{i\in I}A_i\,|.$$

Proof. Let $A_I := \cap_{i\in I}A_i$. By (2), formula (3) is equivalent to

(4) $$\sum_{a\in A}\chi_{A_1\cup\ldots\cup A_n}(a)=\sum_{\varnothing\neq I\subset[n]}(-1)^{|I|-1}\sum_{a\in A}\chi_{A_I}(a)=\sum_{a\in A}\sum_{\varnothing\neq I\subset[n]}(-1)^{|I|-1}\chi_{A_I}(a).$$

And formula (4) holds if, for each $a \in A$,

(5) $$\chi_{A_1\cup\ldots\cup A_n}(a)=\sum_{\varnothing\neq I\subset[n]}(-1)^{|I|-1}\chi_{A_I}(a).$$

If a is an element of none of the sets A_i, then (5) holds in the form 0 = 0. Suppose then that $a \in A_i$ for precisely those $i \in J$, where $|J| = j > 0$. Then the left-hand side of (5) is equal to 1, and the right-hand side of (5) is equal to

(6) $$\sum_{\varnothing\neq I\subset J}(-1)^{|I|-1}=\sum_{i=1}^{j}(-1)^{i-1}\binom{j}{i}=1,$$

by the Lemma.