Newton's Inequality and a Test for Imaginary Roots

For my aunt, Ethel Groh Brunner, in her thirtieth year as a teacher.

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1. Introduction. Given a polynomial

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \tag{1.1}$$

of degree $n \geqslant 2$ with real coefficients, how can one determine whether $f(x)$ has imaginary (i.e., non-real) roots? If $n = 2$ the question is settled merely by checking the sign of the discriminant $a_1^2 - 4a_2a_0$. For $n > 2$, however, it may be necessary to calculate (by repeated applications of Sturm's Theorem [1, p. 87]) the exact number of real roots of $f(x)$, multiplicities counted, and compare this number with n. In certain cases, of course, the simpler data derived from Descartes' Rule of Signs [1, p. 48] will settle the question. For example, the following sufficient condition for the existence of imaginary roots may be derived from the Rule of Signs:

Theorem 1. *Let $f(x)$ be given by* (1.1), *where $a_0 \neq 0$ and $n \geqslant 3$. If, for some $k \in [2, n-1]$, $a_{k-1} = a_k = 0$, then $f(x)$ has imaginary roots.*

Proof. Consider the subsequence $a_{n_0} = a_0, a_{n_1}, a_{n_2}, \ldots, a_{n_r} = a_n$ of $a_0, a_1, a_2, \ldots, a_n$ consisting of all nonzero elements of the latter sequence. By the Rule of Signs, each sign change between a_{n_i} and $a_{n_{i+1}}$ allows at most one positive real root and each sign change between $(-1)^{n_i}a_{n_i}$ and $(-1)^{n_{i+1}}a_{n_{i+1}}$ allows at most one negative real root. If $n_i \not\equiv n_{i+1} \pmod 2$, there is a sign change between a_{n_i} and $a_{n_{i+1}}$ if and only if there is not a sign change between $(-1)^{n_i}a_{n_i}$ and $(-1)^{n_{i+1}}a_{n_{i+1}}$. If, however, $n_i \equiv n_{i+1} \pmod 2$, it is possible to have a sign change between a_{n_i} and $a_{n_{i+1}}$, and also between $(-1)^{n_i}a_{n_i}$ and $(-1)^{n_{i+1}}a_{n_{i+1}}$. In this latter case, there is at least one zero coefficient between a_{n_i} and $a_{n_{i+1}}$ in the original coefficient sequence, i.e., $n_{i+1} - n_i \geqslant 2$. Let $j = \text{card } \{i : 0 \leqslant i \leqslant r-1 \text{ and } n_i \equiv n_{i+1} \pmod 2\}$. Then card $\{i : 0 \leqslant i \leqslant r-1 \text{ and } n_i \not\equiv n_{i+1} \pmod 2\} = r - j$. Hence, there are, by previous remarks, at most $2j + (r - j) = j + r$ positive or negative roots of $f(x)$, multiplicities counted.

But by hypothesis, there is at least one $i \in [0, r-1]$, such that $n_{i+1} - n_i \geqslant 3$. If $n_i \not\equiv n_{i+1} \pmod 2$, $r \leqslant n - j - 2$, and so $j + r \leqslant n - 2$. If $n_i \equiv n_{i+1} \pmod 2$, $n_{i+1} - n_i \geqslant 4$ and $r \leqslant n - (j-1) - 3 = n - j - 2$, and so $j + r \leqslant n - 2$. Since, by hypothesis, zero is not a root of $f(x)$, we conclude that the number of real roots of $f(x)$, multiplicities counted, is at most $n - 2$. Hence $f(x)$ has imaginary roots.

This note introduces a rather general sufficient condition for the existence of imaginary roots, based on the above theorem and an inequality due to Newton.

2. Newton's Inequality. If the polynomial $f(x)$ of (1.1) has no imaginary roots, then by Rolle's Theorem [1, p. 45] the same is true of all nonzero derivatives of $f(x)$. In addition, the *reciprocal polynomial* $Rf(x)$ of $f(x)$, defined by

$$Rf(x) = x^n f\left(\frac{1}{x}\right) = a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n,$$

clearly has no imaginary roots. These observations may be used to prove the following theorem, known as Newton's Inequality. The proof which we present is essentially that of Hardy, Littlewood, and Pólya [2, p. 104], but employs the aforementioned fact about reciprocal polynomials in order to avoid their use of partial differentiation.

Theorem 2. *Let* $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ *be a polynomial of degree* $n \geqslant 2$ *with real coefficients. If* $f(x)$ *has no imaginary roots, then for all* $k \in [1, n-1]$,

$$a_k^2 \geqslant \left(\frac{k+1}{k}\right)\left(\frac{n-k+1}{n-k}\right)a_{k-1}a_{k+1}. \tag{2.1}$$

Proof. Since $f(x)$ has no imaginary roots, it follows that

$$D^{k-1}f(x) = \frac{n!}{(n-k+1)!}a_n x^{n-k-1} + \cdots + \frac{(k+1)!}{2}a_{k+1}x^2 + k!a_k x + (k-1)!a_{k-1}$$

has no imaginary roots. Hence

$$RD^{k-1}f(x) = (k-1)!a_{k-1}x^{n-k+1} + k!a_k x^{n-k} + \frac{(k+1)!}{2}a_{k+1}x^{n-k-1} + \cdots + \frac{n!}{(n-k+1)!}a_n$$

has no imaginary roots, and the same is true of

$$\begin{aligned} h(x) &= D^{n-k-1}RD^{k-1}f(x) \\ &= \frac{(k-1)!\,(n-k+1)!}{2}a_{k-1}x^2 + k!\,(n-k)!a_k x \\ &\quad + \frac{(k+1)!\,(n-k-1)!}{2}a_{k+1}. \end{aligned}$$

If $a_{k-1} = 0$, (2.1) holds trivially. If not, $h(x)$ is a quadratic with real roots and (2.1) follows from the nonnegativity of the discriminant of $h(x)$.

We may combine Theorem 2 with Theorem 1 to obtain the following variant of Theorem 2:

Theorem 3. *Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial of degree $n \geqslant 2$ with real coefficients and suppose that $a_0 \neq 0$. If $f(x)$ has no imaginary roots, then for all $k \in [1, n-1]$*

$$a_k^2 > a_{k-1}a_{k+1}. \tag{2.2}$$

Proof. If $a_{k-1}a_{k+1} > 0$, (2.2) follows from (2.1) and the fact that $(k+1)/k > 1$ and $(n-k+1)/(n-k) > 1$. If $a_{k-1}a_{k+1} < 0$, (2.2) follows immediately since $a_k^2 \geqslant 0$. If either $a_{k-1} = 0$ or $a_{k+1} = 0$, then by Theorem 1 $a_k \neq 0$ and so $a_k^2 > 0 = a_{k-1}a_{k+1}$.

The contrapositive of Theorem 3 furnishes the following simple sufficient condition for the existence of imaginary roots:

Theorem 4. *Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ be a polynomial of degree $n \geqslant 2$ with real coefficients and suppose that $a_0 \neq 0$. If there exists a $k \in [1, n-1]$ such that $a_k^2 \leqslant a_{k-1}a_{k+1}$, then $f(x)$ has imaginary roots.*

In particular, for $f(x)$ as above, if $a_{k-1} = a_k = a_{k+1}$ for some $k \in [1, n-1]$, then $f(x)$ has imaginary roots.

It should be noted that the sufficient condition stated in Theorem 4 is not necessary. Indeed, any quadratic polynomial $x^2 + bx + c$ for which $b^2 - 4c < 0$ and $b^2 > c$ (e.g., $x^2 + 2x + 2$) will have imaginary roots, but not satisfy the condition of Theorem 4.

REFERENCES

1. N. Conkwright, Introduction to the Theory of Equations, Ginn, New York, 1941.
2. G. Hardy, J. Littlewood, and G. Pólya, Inequalities, 2nd ed., Cambridge University Press, London, 1967.