



Periodicity and Parity Theorems for a Statistic on r -Mino Arrangements

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Abstract

We study polynomial generalizations of the r -Fibonacci and r -Lucas sequences which arise in connection with a certain statistic on linear and circular r -mino arrangements, respectively. By considering special values of these polynomials, we derive periodicity and parity theorems for this statistic on the respective structures.

1 Introduction

If $r \geq 2$, the r -Fibonacci numbers $F_n^{(r)}$ are defined by $F_0^{(r)} = F_1^{(r)} = \dots = F_{r-1}^{(r)} = 1$, with $F_n^{(r)} = F_{n-1}^{(r)} + F_{n-r}^{(r)}$ if $n \geq r$. The r -Lucas numbers $L_n^{(r)}$ are defined by $L_1^{(r)} = L_2^{(r)} = \dots = L_{r-1}^{(r)} = 1$ and $L_r^{(r)} = r + 1$, with $L_n^{(r)} = L_{n-1}^{(r)} + L_{n-r}^{(r)}$ if $n \geq r + 1$. If $r = 2$, the $F_n^{(r)}$ and $L_n^{(r)}$ reduce, respectively, to the classical Fibonacci and Lucas numbers (parametrized as in Wilf [12], by $F_0 = F_1 = 1$, etc., and $L_1 = 1$, $L_2 = 3$, etc.).

Polynomial generalizations of F_n and/or L_n have arisen as generating functions for statistics on binary words [1], lattice paths [5], and linear and circular domino arrangements [8]. Generalizations of $F_n^{(r)}$ and/or $L_n^{(r)}$ have arisen similarly in connection with statistics on Morse code sequences [4] as well as on linear and circular r -mino arrangements [9].

Cigler [3] introduces and studies a new class of q -Fibonacci polynomials, generalizing the classical sequence, which arise in connection with a certain statistic on Morse code sequences in which the dashes have length 2. The same statistic, which we'll denote by π , applied more generally to linear r -mino arrangements, leads to the polynomial generalization

$$F_n^{(r)}(q, t) := \sum_{0 \leq k \leq \lfloor n/r \rfloor} q^{\binom{k+1}{2}} \binom{n - (r-1)k}{k}_q t^k \quad (1.1)$$

of $F_n^{(r)}$. A natural extension of this π statistic to circular r -mino arrangements leads to the new polynomial generalization

$$L_n^{(r)}(q, t) := \sum_{0 \leq k \leq \lfloor n/r \rfloor} q^{\binom{k+1}{2}} \left[\frac{(r-1)k_q + (n - (r-1)k)_q}{(n - (r-1)k)_q} \right] \binom{n - (r-1)k}{k}_q t^k \quad (1.2)$$

of $L_n^{(r)}$.

In addition to deriving the above closed forms for $F_n^{(r)}(q, t)$ and $L_n^{(r)}(q, t)$, we present both algebraic and combinatorial evaluations of $F_n^{(r)}(-1, t)$ and $L_n^{(r)}(-1, t)$, as well as determine when the sequences $F_n^{(r)}(-1, 1)$ and $L_n^{(r)}(-1, 1)$ are periodic. Our algebraic proofs make frequent use of the identity [11, pp. 201–202]

$$\sum_{n \geq 0} \binom{n}{k}_q x^n = \frac{x^k}{(1-x)(1-qx) \cdots (1-q^k x)}, \quad k \in \mathbb{N}. \quad (1.3)$$

Our combinatorial proofs are based on the fact that $F_n^{(r)}(q, t)$ and $L_n^{(r)}(q, t)$ are bivariate generating functions for a pair of statistics defined, respectively, on linear and circular arrangements of r -minos. We also describe some variants of the π statistic on circular domino arrangements which lead to additional polynomial generalizations of the Lucas sequence.

In what follows, \mathbb{N} and \mathbb{P} denote, respectively, the nonnegative and positive integers. Empty sums take the value 0 and empty products the value 1, with $0^0 := 1$. If q is an indeterminate, then $0_q := 0$, $n_q := 1 + q + \cdots + q^{n-1}$ for $n \in \mathbb{P}$, $0_q! := 1$, $n_q! := 1_q 2_q \cdots n_q$ for $n \in \mathbb{P}$, and

$$\binom{n}{k}_q := \begin{cases} \frac{n_q!}{k_q! (n-k)_q!}, & \text{if } 0 \leq k \leq n; \\ 0, & \text{if } k < 0 \text{ or } 0 \leq n < k. \end{cases} \quad (1.4)$$

A useful variation of (1.4) is the well known formula [10, p. 29]

$$\binom{n}{k}_q = \sum_{\substack{d_0 + d_1 + \cdots + d_k = n-k \\ d_i \in \mathbb{N}}} q^{0d_0 + 1d_1 + \cdots + kd_k} = \sum_{t \geq 0} p(k, n-k, t) q^t, \quad (1.5)$$

where $p(k, n-k, t)$ denotes the number of partitions of the integer t with at most $n-k$ parts, each no larger than k .

2 Linear r -Mino Arrangements

Let $\mathcal{R}_{n,k}^{(r)}$ denote the set of coverings of the numbers $1, 2, \dots, n$ arranged in a row by k indistinguishable r -minos and $n - rk$ indistinguishable squares, where pieces do not overlap, an r -mino, $r \geq 2$, is a rectangular piece covering r numbers, and a square is a piece covering

a single number. Each such covering corresponds uniquely to a word in the alphabet $\{r, s\}$ comprising k r 's and $n - rk$ s 's so that

$$|\mathcal{R}_{n,k}^{(r)}| = \binom{n - (r-1)k}{k}, \quad 0 \leq k \leq \lfloor n/r \rfloor, \quad (2.1)$$

for all $n \in \mathbb{P}$. (If we set $\mathcal{R}_{0,0}^{(r)} = \{\emptyset\}$, the ‘‘empty covering,’’ then (2.1) holds for $n = 0$ as well.) In what follows, we will identify coverings c with such words $c_1c_2 \cdots$ in $\{r, s\}$. With

$$\mathcal{R}_n^{(r)} := \bigcup_{0 \leq k \leq \lfloor n/r \rfloor} \mathcal{R}_{n,k}^{(r)}, \quad n \in \mathbb{N}, \quad (2.2)$$

it follows that

$$|\mathcal{R}_n^{(r)}| = \sum_{0 \leq k \leq \lfloor n/r \rfloor} \binom{n - (r-1)k}{k} = F_n^{(r)}, \quad (2.3)$$

where $F_0^{(r)} = F_1^{(r)} = \cdots = F_{r-1}^{(r)} = 1$, with $F_n^{(r)} = F_{n-1}^{(r)} + F_{n-r}^{(r)}$ if $n \geq r$. Note that

$$\sum_{n \geq 0} F_n^{(r)} x^n = \frac{1}{1 - x - x^r}. \quad (2.4)$$

Given a covering $c = c_1c_2 \cdots$, let

$$\pi(c) := \sum_{i:c_i=r} i; \quad (2.5)$$

note that $\pi(c)$ gives the total resulting when one counts the number of pieces preceding each r -mino, inclusive, and adds up these numbers.

Let

$$F_n^{(r)}(q, t) := \sum_{c \in \mathcal{R}_n^{(r)}} q^{\pi(c)} t^{v(c)}, \quad n \in \mathbb{N}, \quad (2.6)$$

where $v(c) :=$ the number of r -minos in the covering c .

Categorizing linear covers of $1, 2, \dots, n$ according to whether the piece covering n is a square or r -mino yields the recurrence relation

$$F_n^{(r)}(q, t) = F_{n-1}^{(r)}(q, t) + q^{n-r+1} t F_{n-r}^{(r)}(q, t/q^{r-1}), \quad n \geq r, \quad (2.7)$$

with $F_i^{(r)}(q, t) = 1$ if $0 \leq i \leq r-1$, since the total number of pieces in $c \in \mathcal{R}_m^{(r)}$ is $m - (r-1)v(c)$. Categorizing covers of $1, 2, \dots, n$ according to whether the piece covering 1 is a square or r -mino yields

$$F_n^{(r)}(q, t) = F_{n-1}^{(r)}(q, qt) + qt F_{n-r}^{(r)}(q, qt), \quad n \geq r. \quad (2.8)$$

By combining relations (2.7) and (2.8), one gets a recurrence for $F_n^{(r)}(q, t)$ for each number q and t . For example when $r = 3$, this is

$$\begin{aligned} F_n^{(3)}(q, t) = F_{n-1}^{(3)}(q, t) + q^{n-2} t F_{n-5}^{(3)}(q, t) &+ q^{n-3} (1+q) t^2 F_{n-7}^{(3)}(q, t) \\ &+ q^{n-3} t^3 F_{n-9}^{(3)}(q, t). \end{aligned} \quad (2.9)$$

The $F_n^{(r)}(q, t)$ have the following explicit formula.

Theorem 2.1. For all $n \in \mathbb{N}$,

$$F_n^{(r)}(q, t) = \sum_{0 \leq k \leq \lfloor n/r \rfloor} q^{\binom{k+1}{2}} \binom{n - (r-1)k}{k}_q t^k. \quad (2.10)$$

Proof. It clearly suffices to show that

$$\sum_{c \in \mathcal{R}_{n,k}^{(r)}} q^{\pi(c)} = q^{\binom{k+1}{2}} \binom{n - (r-1)k}{k}_q.$$

Each $c \in \mathcal{R}_{n,k}^{(r)}$ corresponds uniquely to a sequence (d_0, d_1, \dots, d_k) , where d_0 is the number of squares following the k^{th} r -mino (counting from left to right) in the covering c , d_k is the number of squares preceding the first r -mino, and, for $0 < i < k$, d_{k-i} is the number of squares between the i^{th} and $(i+1)^{\text{st}}$ r -mino. Then $\pi(c) = (d_k + 1) + (d_k + d_{k-1} + 2) + \dots + (d_k + d_{k-1} + \dots + d_1 + k) = \binom{k+1}{2} + kd_k + (k-1)d_{k-1} + \dots + 1d_1$ so that

$$\begin{aligned} \sum_{c \in \mathcal{R}_{n,k}^{(r)}} q^{\pi(c)} &= q^{\binom{k+1}{2}} \sum_{\substack{d_0 + d_1 + \dots + d_k = n - rk \\ d_i \in \mathbb{N}}} q^{0d_0 + 1d_1 + \dots + kd_k} \\ &= q^{\binom{k+1}{2}} \binom{n - (r-1)k}{k}_q, \end{aligned}$$

by (1.5). □

Theorem 2.2. The ordinary generating function of the sequence $(F_n^{(r)}(q, t))_{n \geq 0}$ is given by

$$\sum_{n \geq 0} F_n^{(r)}(q, t) x^n = \sum_{k \geq 0} \frac{q^{\binom{k+1}{2}} t^k x^{rk}}{(1-x)(1-qx) \dots (1-q^k x)}. \quad (2.11)$$

Proof. By (2.10) and (1.3),

$$\begin{aligned} \sum_{n \geq 0} F_n^{(r)}(q, t) x^n &= \sum_{n \geq 0} \left(\sum_{0 \leq k \leq \lfloor n/r \rfloor} q^{\binom{k+1}{2}} \binom{n - (r-1)k}{k}_q t^k \right) x^n \\ &= \sum_{k \geq 0} q^{\binom{k+1}{2}} t^k x^{(r-1)k} \sum_{n \geq kr} \binom{n - (r-1)k}{k}_q x^{n - (r-1)k} \\ &= \sum_{k \geq 0} q^{\binom{k+1}{2}} t^k x^{(r-1)k} \cdot \frac{x^k}{(1-x)(1-qx) \dots (1-q^k x)}. \end{aligned}$$

□

Note that $F_n^{(r)}(1, 1) = F_n^{(r)}$, whence (2.11) generalizes (2.4). Setting $q = 1$ and $q = -1$ in (2.11) yields

Corollary 2.2.1. *The ordinary generating function of the sequence $(F_n^{(r)}(1, t))_{n \geq 0}$ is given by*

$$\sum_{n \geq 0} F_n^{(r)}(1, t)x^n = \frac{1}{1 - x - tx^r}. \quad (2.12)$$

and

Corollary 2.2.2. *The ordinary generating function of the sequence $(F_n^{(r)}(-1, t))_{n \geq 0}$ is given by*

$$\sum_{n \geq 0} F_n^{(r)}(-1, t)x^n = \frac{1 + x - tx^r}{1 - x^2 + t^2x^{2r}}. \quad (2.13)$$

When $r = 2$ and $t = 1$ in (2.13), we get

$$\sum_{n \geq 0} F_n^{(2)}(-1, 1)x^n = \frac{1 + x - x^2}{1 - x^2 + x^4} = \frac{(1 + x + x^3 - x^4)(1 - x^6)}{1 - x^{12}}, \quad (2.14)$$

which implies

Corollary 2.2.3. *The sequence $(F_n^{(2)}(-1, 1))_{n \geq 0}$ is periodic with period 12; namely, if $a_n := F_n^{(2)}(-1, 1)$ for $n \geq 0$, then $a_0 = 1$, $a_1 = 1$, $a_2 = 0$, $a_3 = 1$, $a_4 = -1$, and $a_5 = 0$ with $a_{n+6} = -a_n$, $n \geq 0$.*

(We call a sequence $(b_n)_{n \geq 0}$ *periodic with period d* if $b_{n+d} = b_n$ for all $n \geq m$ for some $m \in \mathbb{N}$.)

Remark. Corollary 2.2.3 is the $q = -1$ case of the well known formula

$$\sum_{0 \leq k \leq \lfloor n/2 \rfloor} (-1)^k q^{\binom{k}{2}} \binom{n-k}{k}_q = \begin{cases} (-1)^{\lfloor n/3 \rfloor} q^{n(n-1)/6}, & \text{if } n \equiv 0, 1 \pmod{3}; \\ 0, & \text{if } n \equiv 2 \pmod{3}. \end{cases}$$

See, e.g., Cigler [5], Ekhad and Zeilberger [6], and Kupershmidt [7].

We now show that the periodic behavior of $F_n^{(r)}(-1, 1)$ seen when $r = 2$ is restricted to that case. The following lemma is established in [9]. We include its proof here for completeness.

Lemma 2.3. *If $r \geq 3$, then $g_r(x) := 1 - x + x^r$ does not divide any polynomial of the form $1 - x^m$, where $m \in \mathbb{P}$.*

Proof. We first describe the roots of unity that are zeros of $g_r(x)$, where $r \geq 2$. If z is such a root of unity, let $y = z^{r-1}$. Since $z(1 - z^{r-1}) = 1$ and z is a root of unity, it follows that both y

and $1 - y$ are roots of unity. In particular, $|y| = |1 - y| = 1$. Therefore, $1 - 2\operatorname{Re}(y) + |y|^2 = 1$, so $\operatorname{Re}(y) = \frac{1}{2}$. This forces y , and hence $1 - y$, to be primitive 6^{th} roots of unity. But $1 - y = \frac{1}{z}$, so z is also a primitive 6^{th} root of unity.

This implies that the only possible roots of unity which are zeros of g_r are the primitive 6^{th} roots of unity. Since the derivative of g_r has no roots of unity as zeros, these 6^{th} roots of unity can only be simple zeros of g_r . In particular, if every root of g_r is a root of unity, then $r = 2$. \square

Theorem 2.4. *The sequence $(F_n^{(r)}(-1, 1))_{n \geq 0}$ is never periodic for $r \geq 3$.*

Proof. By (2.13) at $t = 1$, we must show that $1 - x^2 + x^{2r}$ does not divide the product $(1 - x^m)(1 + x - x^r)$ for any $m \in \mathbb{P}$ whenever $r \geq 3$. First note that the polynomials $1 - x^2 + x^{2r}$ and $1 + x - x^r$ cannot share a zero; for if t_0 is a common zero, then $t_0^2 - 1 = t_0^{2r} = (t_0 + 1)^2$, i.e., $t_0 = -1$, which isn't a zero of either polynomial. Observe next that $1 - x^2 + x^{2r} = g_r(x^2)$, where $g_r(x)$ is as in Lemma 2.3, so that $1 - x^2 + x^{2r}$ fails to divide $1 - x^m$ for any $m \in \mathbb{P}$, since $g_r(x)$ fails to, which completes the proof. \square

Iterating (2.7) or (2.8) yields $F_{-i}^{(r)}(q, t) = 0$ if $1 \leq i \leq r - 1$, which we'll take as a convention.

Theorem 2.5. *Let $m \in \mathbb{N}$. If m and r have the same parity, then*

$$F_m^{(r)}(-1, t) = F_{\lfloor m/2 \rfloor}^{(r)}(1, -t^2) - tF_{(m-r)/2}^{(r)}(1, -t^2), \quad (2.15)$$

and if m and r have different parity, then

$$F_m^{(r)}(-1, t) = F_{\lfloor m/2 \rfloor}^{(r)}(1, -t^2). \quad (2.16)$$

Proof. Taking the even and odd parts of both sides of (2.13) followed by replacing x with $x^{1/2}$ yields

$$\sum_{n \geq 0} F_{2n}^{(r)}(-1, t)x^n = \frac{1 - tx^{r/2}}{1 - x + t^2x^r}$$

and

$$\sum_{n \geq 0} F_{2n+1}^{(r)}(-1, t)x^n = \frac{1}{1 - x + t^2x^r},$$

when r is even, and

$$\sum_{n \geq 0} F_{2n}^{(r)}(-1, t)x^n = \frac{1}{1 - x + t^2x^r}$$

and

$$\sum_{n \geq 0} F_{2n+1}^{(r)}(-1, t)x^n = \frac{1 - tx^{(r-1)/2}}{1 - x + t^2x^r},$$

when r is odd, from which (2.15) and (2.16) now follow from (2.12) upon putting together cases.

For a combinatorial proof of (2.15) and (2.16), we first assign to each r -mino arrangement $c \in \mathcal{R}_m^{(r)}$ the weight $w_c := (-1)^{\pi(c)} t^{v(c)}$, where t is an indeterminate. Let $\mathcal{R}_m^{(r)'}$ consist of those $c = c_1 c_2 \cdots$ in $\mathcal{R}_m^{(r)}$ satisfying the conditions $c_{2i-1} = c_{2i}$, $i \geq 1$. Suppose $c \in \mathcal{R}_m^{(r)} - \mathcal{R}_m^{(r)'}$, with i_0 being the smallest value of i for which $c_{2i-1} \neq c_{2i}$. Exchanging the positions of the $(2i_0 - 1)^{st}$ and $(2i_0)^{th}$ pieces within c produces a π -parity changing involution of $\mathcal{R}_m^{(r)} - \mathcal{R}_m^{(r)'}$ which preserves v .

If m and r have the same parity, then

$$\begin{aligned}
F_m^{(r)}(-1, t) &= \sum_{c \in \mathcal{R}_m^{(r)}} w_c = \sum_{c \in \mathcal{R}_m^{(r)'}} w_c = \sum_{\substack{c \in \mathcal{R}_m^{(r)'} \\ v(c) \text{ even}}} w_c + \sum_{\substack{c \in \mathcal{R}_m^{(r)'} \\ v(c) \text{ odd}}} w_c \\
&= \sum_{\substack{c \in \mathcal{R}_m^{(r)'} \\ v(c) \text{ even}}} (-1)^{v(c)/2} t^{v(c)} - t \sum_{\substack{c \in \mathcal{R}_m^{(r)'} \\ v(c) \text{ even}}} (-1)^{v(c)/2} t^{v(c)} \\
&= \sum_{z \in \mathcal{R}_{\lfloor m/2 \rfloor}^{(r)}} (-1)^{v(z)} t^{2v(z)} - t \sum_{z \in \mathcal{R}_{(m-r)/2}^{(r)}} (-1)^{v(z)} t^{2v(z)} \\
&= F_{\lfloor m/2 \rfloor}^{(r)}(1, -t^2) - t F_{(m-r)/2}^{(r)}(1, -t^2),
\end{aligned}$$

which gives (2.15), since each pair of consecutive r -minos in $c \in \mathcal{R}_m^{(r)'}$ contributes a factor of -1 towards the sign $(-1)^{\pi(c)}$ and since members of $\mathcal{R}_m^{(r)'}$ for which $v(c)$ is odd end in a single r -mino. If m and r differ in parity, then

$$F_m^{(r)}(-1, t) = \sum_{c \in \mathcal{R}_m^{(r)}} w_c = \sum_{c \in \mathcal{R}_m^{(r)'}} w_c = \sum_{z \in \mathcal{R}_{\lfloor m/2 \rfloor}^{(r)}} (-1)^{v(z)} t^{2v(z)} = F_{\lfloor m/2 \rfloor}^{(r)}(1, -t^2),$$

which gives (2.16), since members of $\mathcal{R}_m^{(r)'}$ must contain an even number of r -minos. \square

The involution of the previous theorem in the case $r = 2$ can be extended to account for the periodicity in Corollary 2.2.3 as follows. If $n \geq 6$, let $\mathcal{R}_n^{(2)*} \subseteq \mathcal{R}_n^{(2)'}$ consist of those domino arrangements $c = c_1 c_2 \cdots$ that contain at least $4\lfloor n/6 \rfloor$ pieces satisfying the conditions

$$c_{4i-3} c_{4i-2} c_{4i-1} c_{4i} = ssdd, \quad 1 \leq i \leq \lfloor n/6 \rfloor; \quad (2.17)$$

if $0 \leq n \leq 5$, then let $\mathcal{R}_n^{(2)*} = \mathcal{R}_n^{(2)'}$.

A π -parity changing involution of $\mathcal{R}_n^{(2)'}$ when $n \geq 6$ is given by the pairing

$$(ssdd)^k ssssu \longleftrightarrow (ssdd)^k ddu,$$

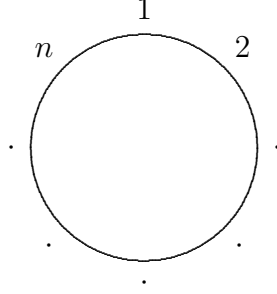
where $k \geq 0$ and u is some (non-empty) word in $\{d, s\}$. If $n = 6m + i$, where $m \geq 1$ and $0 \leq i \leq 5$, then

$$\begin{aligned}
F_n^{(2)}(-1, 1) &= \sum_{c \in \mathcal{R}_n^{(2)}} (-1)^{\pi(c)} = \sum_{c \in \mathcal{R}_n^{(2)'}} (-1)^{\pi(c)} = \sum_{c \in \mathcal{R}_n^{(2)*}} (-1)^{\pi(c)} \\
&= (-1)^m \sum_{c \in \mathcal{R}_i^{(2)*}} (-1)^{\pi(c)} = (-1)^m F_i^{(2)}(-1, 1),
\end{aligned}$$

which implies Corollary 2.2.3, upon checking directly the cases $0 \leq n \leq 5$, as each *ssdd* unit in $c \in \mathcal{R}_n^{(2)*}$ contributes a factor of -1 towards the sign $(-1)^{\pi(c)}$.

3 Circular r -Mino Arrangements

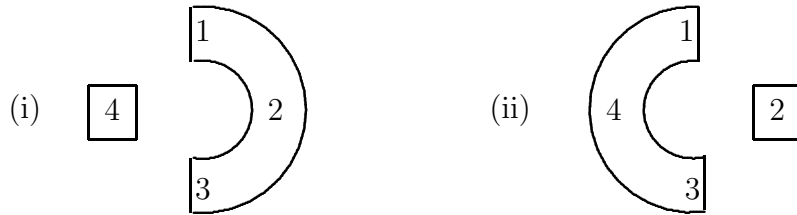
If $n \in \mathbb{P}$ and $0 \leq k \leq \lfloor n/r \rfloor$, let $\mathcal{C}_{n,k}^{(r)}$ denote the set of coverings by k r -minos and $n - rk$ squares of the numbers $1, 2, \dots, n$ arranged clockwise around a circle:



By the *initial segment* of an r -mino occurring in such a cover, we mean the segment first encountered as the circle is traversed clockwise. Classifying members of $\mathcal{C}_{n,k}^{(r)}$ according as (i) 1 is covered by one of r segments of an r -mino or (ii) 1 is covered by a square, and applying (2.1), yields

$$\begin{aligned} |\mathcal{C}_{n,k}^{(r)}| &= r \binom{n - (r-1)k - 1}{k-1} + \binom{n - (r-1)k - 1}{k} \\ &= \frac{n}{n - (r-1)k} \binom{n - (r-1)k}{k}, \quad 0 \leq k \leq \lfloor n/r \rfloor. \end{aligned} \tag{3.1}$$

Below we illustrate two members of $\mathcal{C}_{4,1}^{(3)}$:



In covering (i), the initial segment of the 3-mino covers 1, and in covering (ii), the initial segment covers 3.

With

$$\mathcal{C}_n^{(r)} := \bigcup_{0 \leq k \leq \lfloor n/r \rfloor} \mathcal{C}_{n,k}^{(r)}, \quad n \in \mathbb{P}, \tag{3.2}$$

it follows that

$$|\mathcal{C}_n^{(r)}| = \sum_{0 \leq k \leq \lfloor n/r \rfloor} \frac{n}{n - (r-1)k} \binom{n - (r-1)k}{k} = L_n^{(r)}, \quad (3.3)$$

where $L_1^{(r)} = L_2^{(r)} = \dots = L_{r-1}^{(r)} = 1$, $L_r^{(r)} = r + 1$, and $L_n^{(r)} = L_{n-1}^{(r)} + L_{n-r}^{(r)}$ if $n \geq r + 1$. Note that

$$\sum_{n \geq 1} L_n^{(r)} x^n = \frac{x + rx^r}{1 - x - x^r}. \quad (3.4)$$

We'll associate to each $c \in \mathcal{C}_n^{(r)}$ a word $u_c = u_1 u_2 \dots$ in the alphabet $\{r, s\}$, where

$$u_i := \begin{cases} r, & \text{if the } i^{\text{th}} \text{ piece of } c \text{ is an } r\text{-mino;} \\ s, & \text{if the } i^{\text{th}} \text{ piece of } c \text{ is a square,} \end{cases}$$

and one determines the i^{th} piece of c by starting with the piece covering 1 and proceeding clockwise from that piece. Note that for each word starting with r , there are exactly r associated members of $\mathcal{C}_n^{(r)}$, while for each word starting with s , there is only one associated member.

Given $c \in \mathcal{C}_n^{(r)}$ and its associated word $u_c = u_1 u_2 \dots$, let

$$\pi(c) := \sum_{i: u_i = r} i; \quad (3.5)$$

note that $\pi(c)$ gives the sum of the numbers gotten by counting the number of pieces preceding each r -mino, inclusive (counting back each time counterclockwise to the piece covering 1).

Let

$$L_n^{(r)}(q, t) := \sum_{c \in \mathcal{C}_n^{(r)}} q^{\pi(c)} t^{v(c)}, \quad n \in \mathbb{P}, \quad (3.6)$$

where $v(c) :=$ the number of r -minos in the covering c .

Categorizing circular covers c of $1, 2, \dots, n$ according to whether the last letter in u_c is an s or r yields the recurrence relation

$$L_n^{(r)}(q, t) = L_{n-1}^{(r)}(q, t) + q^{n-r+1} t L_{n-r}^{(r)}(q, t/q^{r-1}), \quad n \geq r + 1, \quad (3.7)$$

with $L_i^{(r)}(q, t) = 1$ if $1 \leq i \leq r - 1$ and $L_r^{(r)}(q, t) = 1 + rqt$, as seen upon removing the final piece of c , sliding the remaining pieces together to form a circle, and renumbering (if necessary) so that 1 corresponds to the same position as before. The $L_n^{(r)}(q, t)$, though, do not seem to satisfy a recurrence like (2.8). The following theorem gives an explicit formula for $L_n^{(r)}(q, t)$.

Theorem 3.1. *For all $n \in \mathbb{P}$,*

$$L_n^{(r)}(q, t) = \sum_{0 \leq k \leq \lfloor n/r \rfloor} q^{\binom{k+1}{2}} \left[\frac{(r-1)k_q + (n - (r-1)k)_q}{(n - (r-1)k)_q} \right] \binom{n - (r-1)k}{k}_q t^k. \quad (3.8)$$

Proof. It suffices to show that

$$\sum_{c \in \mathcal{C}_{n,k}^{(r)}} q^{\pi(c)} = q^{\binom{k+1}{2}} \left[\frac{(r-1)k_q + (n - (r-1)k)_q}{(n - (r-1)k)_q} \right] \binom{n - (r-1)k}{k}_q.$$

Partitioning $\mathcal{C}_{n,k}^{(r)}$ into the categories employed above in deriving (3.1), and applying (2.10), yields

$$\begin{aligned} \sum_{c \in \mathcal{C}_{n,k}^{(r)}} q^{\pi(c)} &= r q^{k-1+1} \cdot q^{\binom{k}{2}} \binom{n - (r-1)k - 1}{k-1}_q \\ &\quad + q^k \cdot q^{\binom{k+1}{2}} \binom{n - (r-1)k - 1}{k}_q \\ &= q^{\binom{k+1}{2}} \left[\frac{rk_q + q^k(n - rk)_q}{(n - (r-1)k)_q} \right] \binom{n - (r-1)k}{k}_q \\ &= q^{\binom{k+1}{2}} \left[\frac{(r-1)k_q + (n - (r-1)k)_q}{(n - (r-1)k)_q} \right] \binom{n - (r-1)k}{k}_q, \end{aligned} \tag{3.9}$$

which completes the proof. \square

Note that $L_n^{(r)}(1, 1) = L_n^{(r)}$. By (3.8) and (2.10), the $L_n^{(r)}(q, t)$ are related to the $F_n^{(r)}(q, t)$ by the formula

$$L_n^{(r)}(q, t) = F_n^{(r)}(q, t) + (r-1)qt F_{n-r}^{(r)}(q, qt), \quad n \geq 1, \tag{3.10}$$

which reduces to

$$L_n^{(r)} = F_n^{(r)} + (r-1)F_{n-r}^{(r)}, \quad n \geq 1, \tag{3.11}$$

when $q = t = 1$. Formula (3.10) can also be realized by considering the way in which 1 is covered in $c \in \mathcal{C}_n^{(r)}$, the first term representing those c for which 1 is covered by a square or an initial segment of an r -mino and the second term representing the remaining $r-1$ possibilities.

Theorem 3.2. *The ordinary generating function of the sequence $(L_n^{(r)}(q, t))_{n \geq 1}$ is given by*

$$\sum_{n \geq 1} L_n^{(r)}(q, t) x^n = \frac{x}{1-x} + \sum_{k \geq 1} \frac{q^{\binom{k+1}{2}} t^k x^{rk} [r - (r-1)q^k x]}{(1-x)(1-qx) \cdots (1-q^k x)}. \tag{3.12}$$

Proof. From (3.9),

$$\begin{aligned}
\sum_{n \geq 1} L_n^{(r)}(q, t) x^n &= \sum_{n \geq 1} x^n \sum_{0 \leq k \leq \lfloor n/r \rfloor} \left(q^{k + \binom{k+1}{2}} t^k \binom{n - (r-1)k - 1}{k} \right)_q \\
&\quad + r q^{\binom{k+1}{2}} t^k \binom{n - (r-1)k - 1}{k-1} \Big)_q \\
&= \frac{x}{1-x} + \sum_{k \geq 1} q^{k + \binom{k+1}{2}} t^k \sum_{n \geq rk+1} \binom{n - (r-1)k - 1}{k} x^n \\
&\quad + r \sum_{k \geq 1} q^{\binom{k+1}{2}} t^k \sum_{n \geq rk} \binom{n - (r-1)k - 1}{k-1} x^n \\
&= \frac{x}{1-x} + \sum_{k \geq 1} q^{k + \binom{k+1}{2}} t^k x^{(r-1)k+1} \cdot \frac{x^k}{(1-x)(1-qx) \cdots (1-q^k x)} \\
&\quad + r \sum_{k \geq 1} q^{\binom{k+1}{2}} t^k x^{(r-1)k+1} \cdot \frac{x^{k-1}}{(1-x)(1-qx) \cdots (1-q^{k-1} x)} \\
&= \frac{x}{1-x} + \sum_{k \geq 1} q^{\binom{k+1}{2}} t^k \cdot \frac{x^{rk} [q^k x + r(1-q^k x)]}{(1-x)(1-qx) \cdots (1-q^k x)},
\end{aligned}$$

by (1.3). □

Note that (3.12) reduces to (3.4) when $q = t = 1$. Setting $q = 1$ and $q = -1$ in (3.12) yields

Corollary 3.2.1. *The ordinary generating function of the sequence $(L_n^{(r)}(1, t))_{n \geq 1}$ is given by*

$$\sum_{n \geq 1} L_n^{(r)}(1, t) x^n = \frac{x + r t x^r}{1 - x - t x^r}. \quad (3.13)$$

and

Corollary 3.2.2. *The ordinary generating function of the sequence $(L_n^{(r)}(-1, t))_{n \geq 1}$ is given by*

$$\sum_{n \geq 1} L_n^{(r)}(-1, t) x^n = \frac{x + x^2 - r t x^r - (r-1) t x^{r+1} - r t^2 x^{2r}}{1 - x^2 + t^2 x^{2r}}. \quad (3.14)$$

When $r = 2$ and $t = 1$ in (3.14), we get

$$\sum_{n \geq 1} L_n^{(2)}(-1, 1) x^n = \frac{x - x^2 - x^3 - 2x^4}{1 - x^2 + x^4} = \frac{(x - x^2 - 3x^4 - x^5 - 2x^6)(1 - x^6)}{1 - x^{12}}, \quad (3.15)$$

which implies

Corollary 3.2.3. *The sequence $(L_n^{(2)}(-1, 1))_{n \geq 1}$ is periodic with period 12; namely, if $a_n := L_n^{(2)}(-1, 1)$ for $n \geq 1$, then $a_1 = 1$, $a_2 = -1$, $a_3 = 0$, $a_4 = -3$, $a_5 = -1$, and $a_6 = -2$ with $a_{n+6} = -a_n$, $n \geq 1$.*

This periodic behavior is again restricted to the case $r = 2$.

Theorem 3.3. *The sequence $(L_n^{(r)}(-1, 1))_{n \geq 1}$ is never periodic for $r \geq 3$.*

Proof. By (3.14) at $t = 1$, we must show that $f(x) := 1 - x^2 + x^{2r}$ does not divide the product $(1 - x^m)h(x)$, where $h(x) := x + x^2 - rx^r - (r-1)x^{r+1} - rx^{2r}$, for any $m \in \mathbb{P}$ whenever $r \geq 3$. By the proof of Theorem 2.4, it suffices to show that f and h are relatively prime. Suppose, to the contrary, that t_0 is a common zero of f and h so that $t_0(1+t_0) + r(1-t_0^2) = t_0(1+t_0) - rt_0^{2r} = t_0^r[r + (r-1)t_0]$. Squaring, substituting $t_0^{2r} = t_0^2 - 1$, and noting $t_0 \neq -1$ implies that t_0 must then be a root of the equation $(x+1)[(r-1)x-r]^2 = (x-1)[(r-1)x+r]^2$, which reduces to $(r^2-1)x^2 = r^2$. But $t_0 = \pm \frac{r}{\sqrt{r^2-1}}$ is a zero of neither f nor h after all, which implies f and h are relatively prime and completes the proof. \square

Recall that $F_{-i}^{(r)}(q, t) = 0$ if $1 \leq i \leq r-1$, by convention.

Theorem 3.4. *Let $m \in \mathbb{P}$. If r is even, then*

$$L_{2m}^{(r)}(-1, t) = L_m^{(r)}(1, -t^2) - rtF_{m-\frac{r}{2}}^{(r)}(1, -t^2) \quad (3.16)$$

and

$$L_{2m-1}^{(r)}(-1, t) = F_{m-1}^{(r)}(1, -t^2) - (r-1)tF_{m-\frac{r}{2}-1}^{(r)}(1, -t^2), \quad (3.17)$$

and if r is odd, then

$$L_{2m}^{(r)}(-1, t) = L_m^{(r)}(1, -t^2) - (r-1)tF_{m-\frac{r+1}{2}}^{(r)}(1, -t^2) \quad (3.18)$$

and

$$L_{2m-1}^{(r)}(-1, t) = F_{m-1}^{(r)}(1, -t^2) - rtF_{m-\frac{r+1}{2}}^{(r)}(1, -t^2). \quad (3.19)$$

Proof. Taking the even and odd parts of both sides of (3.14) followed by replacing x with $x^{1/2}$ yields

$$\sum_{m \geq 1} L_{2m}^{(r)}(-1, t)x^m = \frac{x - rtx^{\frac{r}{2}} - rt^2x^r}{1 - x + t^2x^r}$$

and

$$\sum_{m \geq 1} L_{2m-1}^{(r)}(-1, t)x^m = \frac{x - (r-1)tx^{\frac{r}{2}+1}}{1 - x + t^2x^r},$$

when r is even, and

$$\sum_{m \geq 1} L_{2m}^{(r)}(-1, t)x^m = \frac{x - (r-1)tx^{\frac{r+1}{2}} - rt^2x^r}{1 - x + t^2x^r}$$

and

$$\sum_{m \geq 1} L_{2m-1}^{(r)}(-1, t)x^m = \frac{x - rtx^{\frac{(r+1)}{2}}}{1 - x + t^2x^r},$$

when r is odd, from which (3.16)–(3.19) now follow from (3.13) and (2.12).

For a combinatorial proof of (3.16)–(3.19), we first assign to each covering $c \in \mathcal{C}_n^{(r)}$ the weight $w_c := (-1)^{\pi(c)}t^{v(c)}$, where t is an indeterminate. Let $\mathcal{C}_n^{(r)'}$ consist of those c in $\mathcal{C}_n^{(r)}$ whose associated word $u_c = u_1u_2 \cdots$ satisfies the conditions $u_{2i} = u_{2i+1}$, $i \geq 1$. Suppose $c \in \mathcal{C}_n^{(r)} - \mathcal{C}_n^{(r)'}$, with i_0 being the smallest value of i for which $u_{2i} \neq u_{2i+1}$. Exchanging the positions of the $(2i_0)^{\text{th}}$ and $(2i_0 + 1)^{\text{st}}$ pieces within c produces a π -parity changing, v -preserving involution of $\mathcal{C}_n^{(r)} - \mathcal{C}_n^{(r)'}$.

If r is even and $n = 2m$, then

$$\begin{aligned} L_{2m}^{(r)}(-1, t) &= \sum_{c \in \mathcal{C}_{2m}^{(r)}} w_c = \sum_{c \in \mathcal{C}_{2m}^{(r)'}} w_c = \sum_{\substack{c \in \mathcal{C}_{2m}^{(r)'} \\ v(c) \text{ even}}} w_c + \sum_{\substack{c \in \mathcal{C}_{2m}^{(r)'} \\ v(c) \text{ odd}}} w_c \\ &= \sum_{\substack{c \in \mathcal{C}_{2m}^{(r)'} \\ v(c) \text{ even}}} (-1)^{v(c)/2} t^{v(c)} - rt \sum_{\substack{c \in \mathcal{R}_{2m-r}^{(r)'} \\ v(c) \text{ even}}} (-1)^{v(c)/2} t^{v(c)} \\ &= \sum_{z \in \mathcal{C}_m^{(r)}} (-1)^{v(z)} t^{2v(z)} - rt \sum_{z \in \mathcal{R}_{m-\frac{r}{2}}^{(r)}} (-1)^{v(z)} t^{2v(z)} \\ &= L_m^{(r)}(1, -t^2) - rt F_{m-\frac{r}{2}}^{(r)}(1, -t^2), \end{aligned}$$

which gives (3.16), where $\mathcal{R}_n^{(r)'}$ is as in the proof of Theorem 2.5, since members of $\mathcal{C}_{2m}^{(r)'}$ with $v(c)$ even must begin and end with the same type of piece, while members with $v(c)$ odd must have $u_1 = r$ in u_c with r possibilities for the position of its initial segment. Similarly, if r is odd and $n = 2m - 1$, then

$$\begin{aligned} L_{2m-1}^{(r)}(-1, t) &= \sum_{c \in \mathcal{C}_{2m-1}^{(r)'}} w_c = \sum_{\substack{c \in \mathcal{C}_{2m-1}^{(r)'} \\ u_1=s}} w_c + \sum_{\substack{c \in \mathcal{C}_{2m-1}^{(r)'} \\ u_1=r}} w_c \\ &= \sum_{c \in \mathcal{R}_{2m-2}^{(r)'}} w_c - rt \sum_{c \in \mathcal{R}_{2m-r-1}^{(r)'}} w_c \\ &= F_{m-1}^{(r)}(1, -t^2) - rt F_{m-\frac{(r+1)}{2}}^{(r)}(1, -t^2), \end{aligned}$$

which gives (3.19).

For the cases that remain, let $\mathcal{C}_n^{(r)*} \subseteq \mathcal{C}_n^{(r)'}$ such that $\mathcal{C}_n^{(r)'}$ – $\mathcal{C}_n^{(r)*}$ comprises those c which satisfy the following additional conditions:

- (i) c contains an even number of pieces in all;
- (ii) $u_1 \neq u_p$ in $u_c = u_1u_2 \cdots u_p$;

(iii) if $u_1 = r$, then 1 corresponds to the initial segment of the r -mino covering it.

Pair members of $\mathcal{C}_n^{(r)'} - \mathcal{C}_n^{(r)*}$ of opposite π -parity as follows: given $c \in \mathcal{C}_n^{(r)'} - \mathcal{C}_n^{(r)*}$, let c' be the covering resulting when $u_c = u_1 u_2 \cdots u_p$ is read backwards.

If r is even and $n = 2m - 1$, then

$$\begin{aligned}
L_{2m-1}^{(r)}(-1, t) &= \sum_{c \in \mathcal{C}_{2m-1}^{(r)}} w_c = \sum_{c \in \mathcal{C}_{2m-1}^{(r)*}} w_c = \sum_{\substack{c \in \mathcal{C}_{2m-1}^{(r)*} \\ u_1 = s}} w_c + \sum_{\substack{c \in \mathcal{C}_{2m-1}^{(r)*} \\ u_1 = r}} w_c \\
&= \sum_{\substack{c \in \mathcal{R}_{2m-2}^{(r)'} \\ v(c) \text{ even}}} w_c - (r-1)t \sum_{\substack{c \in \mathcal{R}_{2m-r-2}^{(r)'} \\ v(c) \text{ even}}} w_c \\
&= F_{m-1}^{(r)}(1, -t^2) - (r-1)t F_{m-\frac{r}{2}-1}^{(r)}(1, -t^2),
\end{aligned}$$

which gives (3.17), since members of $\mathcal{C}_{2m-1}^{(r)*}$ with $u_1 = s$ must end in a double letter, while those with $u_1 = r$ must end in a single s with 1 not corresponding to the initial segment of the r -mino covering it. Similarly, if r is odd and $n = 2m$, then

$$\begin{aligned}
L_{2m}^{(r)}(-1, t) &= \sum_{c \in \mathcal{C}_{2m}^{(r)*}} w_c = \sum_{\substack{c \in \mathcal{C}_{2m}^{(r)*} \\ u_1 = u_p}} w_c + \sum_{\substack{c \in \mathcal{C}_{2m}^{(r)*} \\ u_1 \neq u_p}} w_c \\
&= \sum_{\substack{c \in \mathcal{C}_{2m}^{(r)*} \\ u_1 = u_p}} (-1)^{v(c)/2} t^{v(c)} - (r-1)t \sum_{c \in \mathcal{R}_{2m-r-1}^{(r)'}} (-1)^{v(c)/2} t^{v(c)} \\
&= \sum_{z \in \mathcal{C}_m^{(r)}} (-1)^{v(z)} t^{2v(z)} - (r-1)t \sum_{z \in \mathcal{R}_{m-\left(\frac{r+1}{2}\right)}^{(r)}} (-1)^{v(z)} t^{2v(z)} \\
&= L_m^{(r)}(1, -t^2) - (r-1)t F_{m-\left(\frac{r+1}{2}\right)}^{(r)}(1, -t^2),
\end{aligned}$$

which gives (3.18). □

4 Variants of the π Statistic

Modifying the π statistic of the previous section in different ways yields additional polynomial generalizations of $L_n^{(r)}$. In this section, we look at some specific variants of the π statistic on circular r -mino arrangements, taking $r = 2$ for simplicity. We'll use the notation $\mathcal{C}_n = \mathcal{C}_n^{(2)}$, $\mathcal{C}_{n,k} = \mathcal{C}_{n,k}^{(2)}$, and $F_n(q, t) = F_n^{(2)}(q, t)$.

We first partition \mathcal{C}_n as follows: let $\overrightarrow{\mathcal{C}}_n$ comprise those coverings in which 1 is covered by a square or by an initial segment of a domino and let $\overleftarrow{\mathcal{C}}_n$ comprise those coverings in which 1 is covered by the second segment of a domino.

Define the statistic π_1 on \mathcal{C}_n by

$$\pi_1(c) = \begin{cases} \pi(c), & \text{if } c \in \overrightarrow{\mathcal{C}}_n; \\ \pi(c) - 2v(c) + n, & \text{if } c \in \overleftarrow{\mathcal{C}}_n. \end{cases} \quad (4.1)$$

Note that $\pi_1(c)$ gives the sum of the numbers obtained by counting back counterclockwise the pieces from each domino to the piece covering 2 whenever $c \in \overleftarrow{\mathcal{C}}_n$.

Theorem 4.1. For all $n \in \mathbb{P}$,

$$\sum_{c \in \mathcal{C}_n} q^{\pi_1(c)} t^{v(c)} = \sum_{0 \leq k \leq \lfloor n/2 \rfloor} q^{\binom{k+1}{2}} \left[\frac{(n-k)_q + q^{n-2k} k_q}{(n-k)_q} \right] \binom{n-k}{k}_q t^k. \quad (4.2)$$

Proof. By (2.10) when $r = 2$,

$$\begin{aligned} \sum_{c \in \mathcal{C}_{n,k}} q^{\pi_1(c)} &= q^{\binom{k+1}{2}} \binom{n-k}{k}_q + q^{\binom{k}{2} + k + (n-2k)} \binom{n-k-1}{k-1}_q \\ &= q^{\binom{k+1}{2}} \left[\binom{n-k}{k}_q + q^{n-2k} \frac{k_q}{(n-k)_q} \binom{n-k}{k}_q \right] \\ &= q^{\binom{k+1}{2}} \left[\frac{(n-k)_q + q^{n-2k} k_q}{(n-k)_q} \right] \binom{n-k}{k}_q. \end{aligned}$$

□

If $\hat{L}_n(q, t)$ denotes the distribution polynomial in (4.2), then

$$\hat{L}_n(q, t) = F_n(q, t) + q^{n-1} t F_{n-2}(q, t/q), \quad n \geq 1, \quad (4.3)$$

by (4.2) and (2.10), or by considering whether or not c belongs to $\overrightarrow{\mathcal{C}}_n$. The $\hat{L}_n(q, t)$ satisfy the nice recurrence

$$\hat{L}_n(q, t) = \hat{L}_{n-1}(q, qt) + qt \hat{L}_{n-2}(q, qt), \quad n \geq 3, \quad (4.4)$$

with $\hat{L}_1(q, t) = 1$ and $\hat{L}_2(q, t) = 1 + 2qt$, the first term of (4.4) accounting for those $c \in \overrightarrow{\mathcal{C}}_n$ where 1 is covered by a square as well as those $c \in \overleftarrow{\mathcal{C}}_n$ where 2 is covered by a square and the second term accounting for the cases that remain.

Next define π_2 on \mathcal{C}_n by

$$\pi_2(c) = \begin{cases} \pi(c), & \text{if } c \in \overrightarrow{\mathcal{C}}_n; \\ \pi(c) - v(c), & \text{if } c \in \overleftarrow{\mathcal{C}}_n. \end{cases} \quad (4.5)$$

Theorem 4.2. For all $n \in \mathbb{P}$,

$$\sum_{c \in \mathcal{C}_n} q^{\pi_2(c)} t^{v(c)} = \sum_{0 \leq k \leq \lfloor n/2 \rfloor} q^{\binom{k}{2}} \frac{n_q}{(n-k)_q} \binom{n-k}{k}_q t^k. \quad (4.6)$$

Proof. By (2.10) when $r = 2$,

$$\begin{aligned}
\sum_{c \in \mathcal{C}_{n,k}} q^{\pi_2(c)} &= q^{\binom{k+1}{2}} \binom{n-k}{k}_q + q^{\binom{k}{2}+k-k} \binom{n-k-1}{k-1}_q \\
&= q^{\binom{k}{2}} \left[\frac{q^k(n-k)_q + k_q}{(n-k)_q} \right] \binom{n-k}{k}_q \\
&= q^{\binom{k}{2}} \frac{n_q}{(n-k)_q} \binom{n-k}{k}_q.
\end{aligned}$$

□

Theorem 4.2 provides a combinatorial interpretation of the generalized Lucas polynomials

$$Luc_n(x, t) := \sum_{0 \leq k \leq \lfloor n/2 \rfloor} q^{\binom{k}{2}} \frac{n_q}{(n-k)_q} \binom{n-k}{k}_q x^{n-2k} t^k, \quad (4.7)$$

studied by Cigler [2, 3]. Note that the joint distribution of π_2 and v on \mathcal{C}_n is $Luc_n(1, t)$, with the x variable of $Luc_n(x, t)$ recording the number of squares in $c \in \mathcal{C}_n$. Considering whether or not c belongs to $\vec{\mathcal{C}}_n$ leads directly to the relation (cf. [3])

$$Luc_n(1, t) = F_n(q, t) + tF_{n-2}(q, t), \quad n \geq 1. \quad (4.8)$$

The $Luc_n(1, t)$ do not seem to satisfy a two-term recurrence like (3.7) or (4.4).

Similar reasoning shows that $Luc_n(1, t)$ is also the joint distribution of the statistics π_3 and v on \mathcal{C}_n , where

$$\pi_3(c) = \begin{cases} \pi(c) - v(c), & \text{if } c \in \vec{\mathcal{C}}_n; \\ \pi(c) - 2v(c) + n, & \text{if } c \in \overleftarrow{\mathcal{C}}_n, \end{cases} \quad (4.9)$$

which yields the relation

$$Luc_n(1, t) = F_n(q, t/q) + q^{n-1}tF_{n-2}(q, t/q), \quad n \geq 1. \quad (4.10)$$

The π_2 statistic on \mathcal{C}_n can be generalized to $\mathcal{C}_n^{(r)}$ by letting $\pi_2(c) = \pi(c)$, if the number 1 is covered by a square or an initial segment of an r -mino, and letting $\pi_2(c) = \pi(c) - v(c)$, otherwise. Reasoning as in Theorem 4.2 with π_2 on $\mathcal{C}_n^{(r)}$ leads to

$$Luc_n^{(r)}(x, t) := \sum_{0 \leq k \leq \lfloor n/r \rfloor} q^{\binom{k}{2}} \left[\frac{(r-2)k_q + (n - (r-2)k)_q}{(n - (r-1)k)_q} \right] \binom{n - (r-1)k}{k}_q x^{n-rk} t^k, \quad (4.11)$$

which generalizes $Luc_n(x, t)$. The $Luc_n^{(r)}(x, t)$ are connected with the $F_n^{(r)}(q, t)$ by the simple relation

$$Luc_n^{(r)}(1, t) = F_n^{(r)}(q, t) + (r-1)tF_{n-r}^{(r)}(q, t), \quad n \geq 1, \quad (4.12)$$

which generalizes (4.8).

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