DIFFERENTIABILITY IN LOCAL FIELDS OF PRIME CHARACTERISTIC

CARL G. WAGNER

1. Introduction. In 1958 Mahler proved that every continuous p-adic function defined on the ring of p-adic integers is the uniform limit of an interpolation series of binomial form, and he exhibited a necessary and sufficient condition for such a function to be differentiable [2]. In [3] we showed that each continuous linear operator on the ring V of formal power series over a finite field (regarded as a vector space over that field) may be expanded in what may also be termed an interpolation series, and also characterized the differentiable operators. In [4] we dropped the linearity hypothesis of [3] and exhibited an interpolation series for each continuous function on V, and a sufficient condition for the differentiability of such a function. In the present paper we show (Theorem 4.1) that this condition is also necessary and thus obtain a complete characterization (Theorem 4.2) of differentiable functions of an "x-adic" variable.

2. Preliminaries. Denote by F the field of formal power series

$$\alpha = \sum_{i=-\infty}^{\infty} a_i x^i, \tag{2.1}$$

where the a_i are elements of the finite field $GF(q)$ of characteristic p, and all but a finite number of the a_i's vanish for $i < 0$. Let b be any real number such that $0 < b < 1$, and define an absolute value $|\ |$ on F by $|0| = 0$ and $|\alpha| = b^n$, where n is the least integer such that $a_n \neq 0$ for a nonzero α given by (2.1). As is familiar, F is complete with respect to the discrete non-archimedean absolute value $|\ |$ and, equipped with the metric topology induced by $|\ |$, F is a totally disconnected, locally compact topological field. In particular, polynomials over F give rise to continuous functions on F.

The valuation ring V of F consists of all formal power series of the form (2.1) where $a_i = 0$ for $i < 0$. V is compact and contains as a dense subring the ring $GF[q, x]$ of polynomials over $GF(q)$. Similarly, the quotient field of $GF[q, x]$, denoted $GF(q, x)$, is dense in F.

A polynomial $p(t)$ over $GF(q, x)$ is called *integral valued* if $p(m) \in GF[q, x]$ for all $m \in GF[q, x]$. A polynomial $p(t)$ over F is called *integral valued* (mod x) if $p(\alpha) \in V$ for all $\alpha \in V$. Since polynomials give rise to continuous functions

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in F and $GF[q, x]$ is dense in V, it follows that integral valued polynomials are integral valued (mod x). The integral valued polynomials constitute a $GF[q, x]$-module, and the polynomials which are integral valued (mod x) a V-module. Certain ordered bases, $(G_n(t)/g_n)$ and $(G_n'(t)/g_n)$, of these modules constructed by Carlitz [1], [4, 404–5] play an important role in the construction of interpolation series for continuous functions from V to F. Indeed, let $G_n(t)$, $G_n'(t)$, and g_n be defined as in [4, pp. 204–5] (note that $G_n'(t)$ is *not* the derivative of $G_n(t)$), and suppose that $f : V \to F$ is any continuous function. For each $i \geq 0$ set

$$A_i = (-1)^r \sum_{\deg m<r} \frac{G_{q^r-1-i}'(m)}{g_{q^r-1-i}} f(m), \tag{2.2}$$

where $i < q^r$ and $m \in GF[q, x]$. Then [4, Theorem B] $\lim_{i\to\infty} A_i = 0$ in F and

$$\sum_{i=0}^{\infty} A_i \frac{G_i(t)}{g_i} \tag{2.3}$$

converges uniformly to $f(t)$ for all $t \in V$. (Moreover, the coefficients A_i defined in (2.4) yield the *only* series of the form (2.3) with this property.) The sequence $(G_i(t)/g_i)$ plays here the same role as the sequence of Newton polynomials $(t(t-1)\cdots(t-i+1)/i!)$ in Mahler's work [2].

3. Preliminaries on Differentiability. Our goal in the remainder of the paper is to characterize differentiability of a function f at a point $u \in V$ in terms of the coefficients A_i in the interpolation series (2.3) for f. We begin by introducing a certain sequence of auxiliary polynomials over $GF(q, x)$:

Let $H_0(t) = 1$ and for all $i \geq 1$ set

$$H_i(t) = \frac{G_{i+1}(t)}{tg_i} \tag{3.1}$$

Then by [4, (2.8)] $H_i(t)$ is a polynomial of degree i over $GF(q, x)$ with leading coefficient $1/g_i$, so that $(H_i(t))$ is an ordered basis of the $GF(q, x)$-vector space of polynomials over $GF(q, x)$. In fact, the following stronger assertion is true.

LEMMA 3.1. *The sequence $(H_i(t))$ for $i \geq 0$ is an ordered basis of the $GF[q, x]$-module of integral valued polynomials over $GF(q, x)$.*

Proof. By [4, Proposition 2], for all $i \geq 1$

$$H_{i-1}(t) = \frac{G_{q^{e(i)}-1}'(t)G_{i-q^{e(i)}}(t)}{g_{q^{e(i)}-1}g_{i-q^{e(i)}}} \tag{3.2}$$

where $q^{e(i)} \mid i$ and $q^{e(i)+1} \nmid i$. Hence [1; 503] $H_i(t)$ is integral valued for all $i \geq 0$.

Thus it remains only to show that if f is an integral valued polynomial of arbitrary degree n over $GF(q, x)$ and $f(t) = \sum_{i=0}^{n} \alpha_i H_i(t)$, then $\alpha_i \in GF[q, x]$

for each i. Clearly, it suffices to show this for the integral valued polynomials $G_i(t)/g_i$ for $0 \leq i \leq n$. For each such i we have

$$H_i(t) = \sum_{j=0}^{i} c(i, j) \frac{G_j(t)}{g_j}$$

where, by previous remarks, $c(i, j) \in GF[q, x]$ and $c(i, i) = 1$. Solving this triangular system by Cramer's rule yields the desired result.

Let $f : V \to F$ be a continuous function with interpolation series (2.3). As usual, we say that f is differentiable at $u \in V$ if $\lim D(t)$ exists as $t \to 0$ where $D(t) = (f(t + u) - f(u))/t$. By [4, Theorem C] (the stronger hypothesis $f : V \to V$ appearing in the statement of Theorem C is not really used in its proof), we have

$$D(t) = \sum_{j=1}^{\infty} \frac{A_j(u)}{L_{e(j)}} H_{j-1}(t) \qquad (t \in V - \{0\}) \tag{3.3}$$

where

$$A_j(u) = \sum_{k=0}^{\infty} \binom{j + k}{j} A_{j+k} \frac{G_k(u)}{g_k}, \tag{3.4}$$

$q^{e(j)} \mid j$ but $q^{e(j)+1} \nmid j$, and L_n is given by [4, (2.6)]. We remark that $\lim A_j(u) = 0$ as $j \to \infty$. If, in addition $\lim A_j(u)/L_{e(j)} = 0$ as $j \to \infty$, then [4, Theorem C] f is differentiable at u and

$$f'(u) = \sum_{j=1}^{\infty} \frac{A_j(u)}{L_{e(j)}} H_{j-1}(0) = \sum_{n=0}^{\infty} (-1)^n \frac{A_{q^n}(u)}{L_n}. \tag{3.5}$$

As we shall see (Theorem 4.1), the condition $\lim A_j(u)/L_{e(j)} = 0$ as $j \to \infty$ is also necessary for the differentiability of f at u. Before proving this, however, we prove a partial converse of Theorem C which is of independent interest in that it yields the formula for $f'(u)$ directly from the hypothesis of differentiability. We require first the following lemma.

LEMMA 3.2. *For* $r \geq 1$ *and* $1 \leq j \leq q^r - 1$

$$\sum_{\substack{\deg m < r \\ m \neq 0}} H_{j-1}(m) = \begin{cases} (-1)^{k+1} & \text{if } j = q^k, \quad k = 0, 1, \cdots, r - 1 \\ 0 & \text{otherwise} \end{cases} \tag{3.6}$$

Proof. If $j = q^k$, where $k = 0, 1, \cdots, r - 1$, then by (3.1)

$$\begin{aligned} \sum_{\substack{\deg m < r \\ m \neq 0}} H_{j-1}(m) &= \sum_{\deg m < r} \frac{G_{q^k-1}'(m)}{g_{q^k-1}} - \frac{G_{q^k-1}'(0)}{g_{q^k-1}} \\ &= 0 - (-1)^k = (-1)^{k+1}, \end{aligned} \tag{3.7}$$

where the last line of (3.7) follows from [1, (5.12)] and [4, (5.6)]. If $j \neq q^k$, then by [4, (5.6)], (3.2), and [1, (5.12)]

$$(3.8)\qquad \sum_{\substack{\deg m<r\\ m\neq 0}} H_{j-1}(m) = \sum_{\deg m<r} \frac{G_{q^{e(j)}-1}{}'(m)G_{j-q^{e(j)}}(m)}{g_{q^{e(j)}-1}g_{j-q^{e(j)}}} = 0$$

THEOREM 3.3. *Let $f : V \to F$ be a continuous function with interpolation series* (2.3). *If f is differentiable at $u \in V$, then*

$$(3.9)\qquad f'(u) = \sum_{n=0}^{\infty} (-1)^n \frac{A_{q^n}(u)}{L_n},$$

and so $\lim A_{q^n}(u)/L_n = 0$ as $n \to \infty$.

Proof. Suppose that $f'(u) = \lambda$. Define $D(t)$ by (3.3) for $t \neq 0$ and set $D(0) = \lambda$. Then $D : V \to F$ continuously, and so by [4, Theorem B]

$$D(t) = \sum_{i=0}^{\infty} D_i \frac{G_i(t)}{g_i}$$

for all $t \in V$, where by (2.2)

$$(3.10)\qquad D_i = (-1)^r \sum_{\deg m<r} \frac{G_{q^r-1-i}{}'(m)}{g_{q^r-1-i}} D(m) \qquad (i < q^r).$$

Since (D_i) is a null sequence, so is its subsequence (D_{q^r-1}) for $r \geq 0$. But

$$\begin{aligned}(3.11)\qquad D_{q^r-1} &= (-1)^r\lambda + (-1)^r \sum_{\substack{\deg m<r\\ m\neq 0}} D(m)\\ &= (-1)^r\lambda + (-1)^r \sum_{\substack{\deg m<r\\ m\neq 0}} \sum_{j=1}^{q^r-1} \frac{A_j(u)}{L_{e(j)}} H_{j-1}(m)\\ &= (-1)^r\lambda + (-1)^r \sum_{k=0}^{r-1} (-1)^{k+1} \frac{A_{q^k}(u)}{L_k}\end{aligned}$$

by (3.10), (3.3), and Lemma 3.2. Since $\lim D_{q^r-1} = 0$ as $r \to \infty$, (3.9) follows immediately from the last line of (3.11).

4. We now prove the full converse of [4, Theorem C].

THEOREM 4.1. *Let $f : V \to F$ be a continuous function with interpolation series* (2.3). *If f is differentiable at $u \in V$, then*

$$(4.1)\qquad \lim_{j\to\infty} \frac{A_j(u)}{L_{e(j)}} = 0,$$

where $A_j(u)$, $e(j)$, and L_n are defined by (3.4) *and the immediately following text.*

Proof. Suppose that $f'(u) = \lambda$. Let $g(t) = f(t + u) - \lambda t - f(u)$. Then $g : V \to F$ continuously, and g is differentiable at 0 with $g'(0) = 0$. Let $h(t) = g(t)/t$ for $t \neq 0$ and $h(0) = 0$. Then $h : V \to F$ continuously and so by [4, Theorem B] there exists a null sequence (H_i) in F such that

$$h(t) = \sum_{i=0}^{\infty} H_i \frac{G_i(t)}{g_i}. \tag{4.2}$$

Now by [4, 5.14]

$$\begin{aligned} g(t) &= \sum_{j=1}^{\infty} A_j(u) \frac{G_j(t)}{g_j} - \lambda t \\ &= \sum_{j=1}^{\infty} A_j'(u) \frac{G_j(t)}{g_j}, \end{aligned} \tag{4.3}$$

where $A_1'(u) = A_1(u) - \lambda$ and $A_j'(u) = A_j(u)$ if $j \geq 2$. Now for each $i \geq 0$, $tG_i(t)/g_i$ is an integral valued polynomial, so by previous remarks, there exists $d(i, j) \in GF[q, x]$ such that

$$\frac{tG_i(t)}{g_i} = \sum_{j=1}^{i+1} d(i, j) \frac{G_j(t)}{g_j}. \tag{4.4}$$

But then since $g(t) = th(t)$ for all $t \in V$, we have by (4.2) and (4.4)

$$\begin{aligned} g(t) &= \sum_{i=0}^{\infty} H_i \frac{tG_i(t)}{g_i} \\ &= \sum_{i=0}^{\infty} H_i \sum_{j=1}^{i+1} d(i, j) \frac{G_j(t)}{g_j} \\ &= \sum_{j=1}^{\infty} \frac{G_j(t)}{g_j} \sum_{i=j-1}^{\infty} d(i, j)H_i, \end{aligned} \tag{4.5}$$

when the summation interchange is justified by the fact that (H_i) is a null sequence and $G_j(t)/g_j$ is integral valued (mod x). Thus by the previously mentioned uniqueness of interpolation series coefficients we have, comparing (4.3) and (4.5),

$$A_j'(u) = \sum_{i=j-1}^{\infty} d(i, j)H_i. \tag{4.6}$$

Since (H_i) is a null sequence it will follow (from the non-archimedean property of the absolute value in F) that $(A_j'(u)/L_{e(j)})$, and hence $(A_j(u)/L_{e(j)})$, is a null sequence if we can show that $d(i, j)/L_{e(j)} \in GF[q, x]$. But by (2.4), [4, (5.7)], and (3.1)

$$\begin{aligned} \frac{G_i(t)}{g_i} &= \sum_{j=1}^{i+1} d(i, j) \frac{G_j(t)}{tg_j} = \sum_{j=1}^{i+1} \frac{d(i, j)}{L_{e(j)}} \frac{G_j(t)}{tg_{j-1}} \\ &= \sum_{j=1}^{i+1} \frac{d(i, j)}{L_{e(j)}} H_{j-1}(t). \end{aligned} \tag{4.7}$$

Since $G_i(t)/g_i$ is integral valued, it follows from Lemma 3.3 that $d(i, j)/L_{e(j)} \in GF[q, x]$.

Combining Theorem 4.1 with [4, Theorem C] yields the following characterization of differentiability in V:

THEOREM 4.2. *Let $f : V \to F$ be a continuous function with interpolation series* (2.3). *Then f is differentiable at $u \in V$ if and only if*

$$\lim_{j\to\infty} \frac{A_j(u)}{L_{e(j)}} = 0. \tag{4.8}$$

in which case,

$$f'(u) = \sum_{n=0}^{\infty} (-1)^n \frac{A_{q^n}(u)}{L_n}. \tag{4.9}$$

We remark in conclusion that if the function f of Theorem 4.1 is a linear transformation from V to F, each regarded as $GF(q)$-vector spaces, then the interpolation series for f takes a particularly simple form [3, Theorem 4.2], as do the necessary and sufficient conditions for differentiability of the present paper [4, 2.10], [3, Theorems 5.1, 5.2]. An example of a continuous, nowhere differentiable linear operator on V may also be found in [3].

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DEPARTMENT OF MATHEMATICS, UNIVERSITY OF TENNESSEE, KNOXVILLE, TENNESSEE 37916