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Characterizations of Monotone and 2-Monotone Capacities

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Given a capacity c and a probability measure p on a finite set, there is a natural way to combine c and p to produce a measure. For fixed c, these measures are probability measures for all p precisely when c is monotone, and dominate c for all p precisely when c is 2-monotone.

KEY WORDS: Capacity; lower probability; conditionalization; belief function.

1. INTRODUCTION

A *capacity* on a finite set X is a mapping $c: 2^X \to [0, 1]$ such that $c(\varnothing) = 0$ and $c(X) = 1$. A capacity c is *monotone* if $A \subset B \Rightarrow c(A) \leqslant c(B)$, *superadditive* if $A \cap B = \varnothing \Rightarrow c(A \cup B) \geqslant c(A) + c(B)$, and r-monotone if, for every sequence $A_1,..., A_r$ of subsets of X,

$$C(A_1 \cup \cdots \cup A_r) \geqslant \sum_{\substack{I \subset \{1,...,r\} \\ I \neq \varnothing}} (-1)^{|I|-1} c\left(\bigcap_{i \in I} A_i\right) \tag{1.1}$$

Two-monotonicity is also called *convexity*, a term justified in Shapley.[7] A capacity that is r-monotone for all $r \geqslant 2$ is called a *belief function*, a term due to Shafer,[6] or an *infinitely monotone capacity*, a term due to Choquet.[2]

A probability measure q is said to *dominate* a capacity c on X if $q(A) \geqslant c(A)$ for all $A \subset X$. There may of course be no such dominating probabilities, even if c is superadditive (see Papamarcou and Fine[5]). Shapley[7] has proved, however, that 2-monotonicity of c is sufficient (though not necessary) for the set of probability measures dominating c to be nonempty.

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A useful tool for studying a capacity c is its *Möbius transform* m, defined for all $E \subset X$ by

$$m(E) = \sum_{H \subset E} (-1)^{|E-H|}\, c(H) \tag{1.2}$$

Clearly, $m: 2^X \to \mathbf{R}$, $m(\emptyset) = 0$, and for all $A \subset X$,

$$\begin{aligned}\sum_{E \subset A} m(E) &= \sum_{E \subset A} \sum_{H \subset E} (-1)^{|E-H|}\, c(H) \\ &= \sum_{H \subset A} c(H) \sum_{H \subset E \subset A} (-1)^{|E-H|} \\ &= \sum_{H \subset A} c(H) \sum_{i=0}^{|A-H|} (-1)^i \binom{|A-H|}{i} = c(A)\end{aligned} \tag{1.3}$$

In particular,

$$\sum_{E \subset X} m(E) = c(X) = 1 \tag{1.4}$$

From the Möbius transform of any capacity we can construct a measure q as follows. Take any "weight function" $\lambda: X \times 2^X \to [0, 1]$ such that (i) $x \notin E \Rightarrow \lambda(x, E) = 0$, and (ii) $\sum_x \lambda(x, E) = 1$ for all nonempty $E \subset X$, and, for all $A \subset X$, let

$$q(A) = \sum_{x \in A} \sum_{E \subset X} \lambda(x, E)\, m(E) \tag{1.5}$$

We call such a set function q a *smear of m*. Clearly $q(\emptyset) = 0$, $q(X) = 1$, and $A \cap B = \emptyset \Rightarrow q(A \cup B) = q(A) + q(B)$. Hence q is a probability measure if and only if q is nonnegative.

Chateauneuf and Jaffray[1] have proved that if c is a capacity, then every smear of its Möbius transform m is a probability measure if and only if

$$m(\{x\}) + \sum_{\substack{E \supset \{x\} \\ E \neq \{x\}}} \min\{m(E), 0\} \geq 0 \qquad \text{for all } x \in X \tag{1.6}$$

Note that monotonicity of c is necessary (though not sufficient) for (1.6). There is of course no guarantee that such smears of m will dominate c. However, using the fact that a capacity is infinitely monotone if and only if its Möbius transform is nonnegative, it is easy to prove that if c is a capacity, then every smear of m is a probability measure that dominates c if and only if c is infinitely monotone (see Dempster,[3] Shafer,[6] and Chateauneuf and Jaffray[1]).

Our aim here is to prove analogous results for the restricted class of probability-based smears. Specifically, suppose that p is a probability measure on X such that $p(E)>0$ for all non-empty $E \subset X$, and define $\lambda: X \times 2^X \to [0, 1]$ by $\lambda(x, \varnothing)=0$ and $\lambda(x, E)=p(x \mid E)$ when $E \neq \varnothing$. (Here, and subsequently, we omit curly brackets from our notation for a singleton set if no confusion arises thereby.) With this p-based weight function, (1.5) takes the nice form

$$q(A)=\sum_{\substack{E \subset X \\ E \neq \varnothing}} m(E)\, p(A \mid E) \tag{1.7}$$

We call q (generically) a *probability smear of m* and (specifically) the *p-smear of m*. We shall prove that if c is a capacity, then every probability smear of its Möbius transform is a probability measure if and only if c is monotone, and that all probability smears of m dominate c if and only if c is 2-monotone.

2. PRELIMINARIES

In this section we establish several lemmata used in the proofs of our main results.

Lemma 1. If A is a finite set, p is a measure on A, and $\phi: 2^A \to \mathbf{R}$, then

$$\sum_{\substack{C \subset A \\ C \neq \varnothing}} p(C)\, \phi(C)=\sum_{a \in A} p(a) \sum_{\substack{C \subset A \\ a \in C}} \phi(C) \tag{2.1}$$

Proof. Replace $p(C)$ by $\sum_{a \in C} p(a)$ on the left-hand side of (2.1), and then interchange summation. □

Lemma 2. If S is a finite set and ϕ, $\psi: 2^S \to \mathbf{R}$, then

$$\sum_{C \subset S} \phi(C)\, \psi(C)=\sum_{C \subset S}\left(\sum_{E \subset C} \phi(E)\right)\left(\sum_{F \subset S-C}(-1)^{|F|}\, \psi(F \cup C)\right) \tag{2.2}$$

Proof. The right-hand side of (2.2) is clearly equal to

$$\begin{aligned}
&\sum_{C \subset S}\left(\sum_{E \subset C} \phi(E)\right)\left(\sum_{C \subset G \subset S}(-1)^{|G-C|}\, \psi(G)\right) \\
&\quad=\sum_{G \subset S} \psi(G) \sum_{E \subset G} \phi(E) \sum_{E \subset C \subset G}(-1)^{|G-C|} \\
&\quad=\sum_{G \subset S} \psi(G)\, \phi(G) \\
&\quad=\sum_{C \subset S} \phi(C)\, \psi(C)
\end{aligned}$$

□

Lemma 3. If $u>0$ and $p_i \geqslant 0$ for all $i \in [n] := \{1,..., n\}$, then

$$\sum_{I \subset [n]} (-1)^{|I|} \left(u + \sum_{i \in I} p_i\right)^{-1} \geqslant 0 \tag{2.3}$$

Proof. If $x>0$, $\int_0^\infty e^{-xt}\,dt = x^{-1}$, and so the left-hand side of (2.3) is equal to

$$\int_0^\infty \left(\sum_{I \subset [n]} (-1)^{|I|} e^{-\sum_{i \in I} p_i t}\right) e^{-ut}\,dt$$

$$= \int_0^\infty \left(\sum_{I \subset [n]} \prod_{i \in I} (-e^{-p_i t})\right) e^{-ut}\,dt$$

$$= \int_0^\infty \left(\prod_{i=1}^{n} (1 - e^{-p_i t})\right) e^{-ut}\,dt \geqslant 0$$

□

Lemma 4. A capacity c on X is 2-monotone if and only if its Möbius transform m satisfies

$$\sum_{\{a,b\} \subset E \subset A} m(E) \geqslant 0 \qquad \text{for all } a, b \in X \text{ and all } A \subset X \text{ such that } a, b \in A$$

Proof. See Chateauneuf and Jaffray.[1]

3. MAIN RESULTS

In this section c denotes a capacity on the finite set X, and m its Möbius transform, as defined by (1.2). A probability smear of m is a mapping q defined by (1.7), where p is a probability measure on X such that $p(E)>0$ for all $E \subset X$.

Theorem 1. If c is a capacity, then every probability smear q of m is a probability measure if and only if c is monotone.

Proof. Sufficiency. As remarked in Section 1, it suffices to show that $q(a) \geqslant 0$ for all $a \in X$. By (1.7) and (1.2),

$$q(a) = \sum_{\substack{E \subset X \\ E \neq \varnothing}} m(E)\, p(a \mid E) = p(a) \sum_{\substack{E \subset X \\ a \in E}} \frac{m(E)}{p(E)}$$

$$= p(a) \sum_{\substack{E \subset X \\ a \in E}} \frac{1}{p(E)} \sum_{H \subset E} (-1)^{|E-H|} c(H)$$

$$= p(a) \sum_{H \subset X} c(H) \sum_{E \supset H \cup a} (-1)^{|E-H|} \frac{1}{p(E)}$$

$$= p(a) \sum_{H \subset X - a} (c(H \cup a) - c(H)) \sum_{E \supset H \cup a} (-1)^{|E-(H \cup a)|} \frac{1}{p(E)}$$

which is nonnegative by monotonicity of c, and by Lemma 3, with $u = p(H \cup a)$ and $p_i = p(x_i)$, where $X - (H \cup a) = \{x_1, ..., x_n\}$.

Necessity. If c is not monotone, there exists a set $A \subset X$, with $|A| \geqslant 2$, and $a \in A$ such that $c(A) - c(A - a) < 0$, and so by (1.3),

$$\sum_{E \subset A} m(E) - \sum_{E \subset A - a} m(E) = \sum_{\substack{E \subset A \\ a \in E}} m(E) < 0 \tag{3.1}$$

We show that there exists a probability measure p such that $q(a) < 0$, where q is the p-smear of m. First note that for any probability measure p, if q is the p-smear of m, we have from (1.7) that

$$q(a) = \sum_{\substack{E \subset A \\ E \neq \varnothing}} m(E)\, p(a \mid E) + \sum_{\substack{E \subset X \\ E \not\subset A}} m(E)\, p(a \mid E) \tag{3.2}$$

Suppose first that $A = X$. Writing q in (3.2) as q_ε and setting $p = p_\varepsilon$, where $p_\varepsilon(x) = \varepsilon/(|X| - 1)$ for all $x \neq a$, and $p_\varepsilon(a) = 1 - \varepsilon$, it is easy to check that

$$\lim_{\varepsilon \to 0} q_\varepsilon(a) = \sum_{\substack{E \subset A \\ a \in E}} m(E) \tag{3.3}$$

Since the right-hand side of (3.3) is negative by (3.1), there exists an $\varepsilon > 0$ such that $q_\varepsilon(a) < 0$.

If A is a proper subset of X, again write q in (3.2) as q_ε and set $p = p_\varepsilon$, where now $p_\varepsilon(a) = \varepsilon$, $p_\varepsilon(x) = \varepsilon^2$ for all $x \in A - a$, and $p_\varepsilon(x) = (1 - p_\varepsilon(A))/|X - A|$ for all $x \in X - A$. It is again easy to check that (3.3) holds in this case, and so $q_\varepsilon(a) < 0$ for some $\varepsilon > 0$. □

Theorem 2. If c is a capacity, then every probability smear q of m is a probability measure that dominates c if and only if c is 2-monotone.

Proof. Sufficiency. It suffices to show that $q(A) \geqslant c(A)$ for every non-empty subset A of X. By (1.7) and (1.2), with $\bar{A} := X—A$,

$$q(A) = \sum_{\substack{C \subset A \\ C \neq \varnothing}} \sum_{D \subset \bar{A}} m(C \cup D)\, p(C \mid C \cup D)$$

$$= \sum_{\substack{C \subset A \\ C \neq \varnothing}} \sum_{D \subset \bar{A}} \sum_{H \subset D} (-1)^{|D - H|} \sum_{G \subset C} (-1)^{|C - G|} c(G \cup H)\, p(C \mid C \cup D) \tag{3.4}$$

On the other hand, by (1.2) and (1.3),

$$\sum_{\substack{C \subset A \\ C \neq \varnothing}} \sum_{D \subset \bar{A}} \sum_{H \subset D} (-1)^{|D-H|} \sum_{G \subset C} (-1)^{|C-G|} c(G)\, p(C \mid C \cup D)$$

$$= \sum_{\substack{C \subset A \\ C \neq \varnothing}} m(C)\, p(C \mid C) = c(A) \tag{3.5}$$

and (3.4) and (3.5) yield

$$\begin{aligned} q(A) - c(A) &= \sum_{\substack{C \subset A \\ C \neq \varnothing}} \sum_{D \subset \bar{A}} \sum_{H \subset D} (-1)^{|D-H|} \sum_{G \subset C} (-1)^{|C-G|} \\ &\quad \times (c(G \cup H) - c(G))\, p(C \mid C \cup D) \\ &= \sum_{\substack{C \subset A \\ C \neq \varnothing}} \sum_{H \subset \bar{A}} \sum_{G \subset C} (-1)^{|C-G|} (c(G \cup H) - c(G)) \\ &\quad \times \sum_{H \subset D \subset \bar{A}} (-1)^{|D-H|} \frac{p(C)}{p(C) + p(D)} \\ &= \sum_{H \subset \bar{A}} \sum_{K \subset \bar{A} - H} (-1)^{|K|} \sum_{\substack{C \subset A \\ C \neq \varnothing}} \sum_{G \subset C} (-1)^{|C-G|} \\ &\quad \times (c(G \cup H) - c(G)) \frac{p(C)}{p(C) + p(H) + p(K)} \end{aligned} \tag{3.6}$$

Applying Lemma 1 to the last line of (3.6) yields

$$\begin{aligned} q(A) - c(A) &= \sum_{H \subset \bar{A}} \sum_{K \subset \bar{A} - H} (-1)^{|K|} \sum_{a \in A} p(a) \\ &\quad \times \sum_{\substack{C \subset A \\ a \in C}} \frac{\sum_{G \subset C} (-1)^{|C-G|} (c(G \cup H) - c(G))}{p(C) + p(H) + p(K)} \end{aligned} \tag{3.7}$$

The part of (3.7) beginning with the fourth summation sign is clearly equal to

$$\sum_{C \subset A - a} \left\{ \sum_{G \subset C \cup a} (-1)^{|(C \cup a) - G|} (c(G \cup H) - c(G)) \right\}$$
$$\times \{ p(C \cup a) + p(H) + p(K) \}^{-1}$$

which by Lemma 2, with $S = A - a$, $\phi(C)$ equal to the first bracketed expression above, and $\psi(C) = \{ p(C \cup a) + p(H) + p(K) \}^{-1}$, is equal to

$$\sum_{C \subset A-a} \left\{ \sum_{E \subset C} \sum_{G \subset E \cup a} (-1)^{|(E \cup a)-G|} (c(G \cup H) - c(G)) \right\}$$
$$\times \left\{ \sum_{F \subset (A-a)-C} (-1)^{|F|} (p(F \cup C \cup a) + p(H) + p(K))^{-1} \right\}$$
$$= \sum_{\substack{C \subset A \\ a \in C}} \left\{ \sum_{\substack{E \subset C \\ a \in E}} \sum_{G \subset E} (-1)^{|E-G|} (c(G \cup H) - c(G)) \right\}$$
$$\times \left\{ \sum_{F \subset A-C} (-1)^{|F|} (p(F \cup C) + p(H) + p(K))^{-1} \right\} \tag{3.8}$$

The first bracketed expression in the preceding line is, by an interchange of summations, equal to

$$\sum_{\substack{G \subset C \\ a \in G}} \left[\sum_{G \subset E \subset C} (-1)^{|E-G|} \right] (c(G \cup H) - c(G))$$
$$+ \sum_{G \subset C-a} \left[\sum_{G \subset E \subset C-a} (-1)^{|E-G|+1} \right] (c(G \cup H) - c(G))$$
$$= c(C \cup H) - c(C) - c((C-a) \cup H) + c(C-a)$$
$$:= \alpha(a, C, H) \tag{3.9}$$

From (3.7), (3.8), (3.9), the fact that $p(F \cup C) + p(H) + p(K) = p(C \cup H) + p(F \cup K)$, and the substitution $G = F \cup K$, we have

$$q(A) - c(A) = \sum_{H \subset \bar{A}} \sum_{a \in A} p(a) \sum_{\substack{C \subset A \\ a \in C}} \alpha(a, C, H)\, \beta(C, H) \tag{3.10}$$

where

$$\beta(C, H) = \sum_{G \subset X-(C \cup H)} \frac{(-1)^{|G|}}{p(C \cup H) + p(G)} \tag{3.11}$$

Since $C \cap H = \varnothing$, the convexity of c ensures that $\alpha(a, C, H) \geqslant 0$. That $\beta(C, H) \geqslant 0$ follows from Lemma 3. Hence $q(A) - c(A) \geqslant 0$, as asserted.

Necessity. If c is not 2-monotone, then by Lemma 4 there exist a set $A \subset X$ and $a, b \in A$ such that

$$\sum_{\{a,b\} \subset E \subset A} m(E) < 0 \tag{3.12}$$

If $a=b$, then $c(A)-c(A-a)<0$, and we argue as in the proof of necessity in Theorem 1. So suppose that $a \neq b$. We show that there exists a probability measure p such that $q(A-a)<c(A-a)$, where q is the p-smear of m. First note that for any probability measure p, if q is the p-smear of m, then from (1.7), (1.3), and the fact that $p(A-a \mid E)=1$ when $E \subset A-a$, it follows that

$$
\begin{aligned}
q(A-a)-c(A-a) &= \sum_{\substack{E \subset X \\ E \not\subset A-a}} m(E)\, p(A-a \mid E) \\
&= \sum_{\substack{E \subset A \\ a \in E}} m(E)\, p(A-a \mid E) \\
&\quad + \sum_{\substack{E \subset X \\ E \not\subset A}} m(E)\, p(A-a \mid E) \qquad (3.13)
\end{aligned}
$$

Suppose first that $A=X$. Writing $q=q_\varepsilon$ in (3.13) and setting $p=p_\varepsilon$, where $p_\varepsilon(a)=p_\varepsilon(b)=(1-\varepsilon)/2$ and $p_\varepsilon(x)=\varepsilon/(|X|-2)$ for all $x \in X-\{a, b\}$, it is easy to check that

$$
\lim_{\varepsilon \to 0} q_\varepsilon(A-a)-c(A-a)=\tfrac{1}{2} \sum_{\{a,b\} \subset E \subset A} m(E) \qquad (3.14)
$$

Since the right-hand side of (3.14) is negative by (3.12), there exists an $\varepsilon>0$ such that $q_\varepsilon(A-a)<c(A-a)$.

If A is a proper subset of X, write $q=q_\varepsilon$ in (3.13) and set $p=p_\varepsilon$, where $p_\varepsilon(a)=p_\varepsilon(b)=\varepsilon$, $p_\varepsilon(x)=\varepsilon^2$ for all $x \in A-\{a, b\}$, and $p_\varepsilon(x)=(1-p_\varepsilon(A))/|X-A|$ for all $x \in X-A$. Again, it is easy to check that (3.14) holds, so that $q_\varepsilon(A-a)<c(A-a)$ for some $\varepsilon>0$.

4. REMARKS

Note that, given a 2-monotone capacity c with Möbius transform m, the formula

$$
q(A)=\sum_{\substack{E \subset X \\ E \neq \varnothing}} m(E)\, p(A \mid E) \qquad (4.1)
$$

might, by Theorem 2, function as a rule for updating the prior probability p in the light of new evidence that bounds the possible revisions of p below by c. This proposal is examined in detail in the case where c is infinitely monotone in Wagner,[8] where a formal criterion for applying (4.1) is presented when c arises from a multivalued mapping from some probability space to X, as in Dempster.[3] We note also that if c is infinitely monotone (whence m is nonnegative, by an earlier remark) and $\mathscr{E} = \{E \subset X: m(E) > 0\}$ is a partition of X, then $q(E) = m(E)$ for all $E \in \mathscr{E}$ and (4.1) becomes

$$q(A) = \sum_{E \in \mathscr{E}} q(E)\, p(A \mid E) \tag{4.2}$$

the well-known conditionalization rule of Jeffrey,[4] whereby the posterior probability measure q is specified first on members of the partition $\mathscr{E}$, and then extended by (4.2) to arbitrary subsets A.

We remark in conclusion that the "Shapley value" q^* of a 2-monotone capacity c (which allocates benefits to cooperating parties in the convex game c—see Shapley[7]), defined for all $a \in X$ by

$$q^*(a) = \frac{1}{|X|!} \sum_{\substack{A \subset X \\ a \in A}} (|A| - 1)!\, |X - A|!\, (c(A) - c(A - a)) \tag{4.3}$$

is simply the p-smear of the Möbius transform m of c, where p is the uniform probability measure on X. We leave the proof as an interesting combinatorial exercise.

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