Logic and Proof I Carl Wagner

Logical Notation and Terminology

A. There is a confusing variety of notation and terminology appearing in textbooks on logic, proof, and discrete mathematics. As a guide to reading other texts and articles, I have listed below the logical notation employed in our text, along with alternative notations that you may encounter elsewhere:

Symbol employed in our text	Alternative notations	
1. $\wedge$	&	**AND** **.**
2. $\vee$		**OR**
3. $\rightarrow$	$\supset$	**IF ____ THEN**
4. $\leftrightarrow$	$\equiv$	**IFF**
5. $+$		**XOR**
6. $\mid$	$\uparrow$	**NAND**
7. $\downarrow$		**NOR**
8. $\subseteq$	$\subset$	
9. $\subset$	$\subsetneq$	
10. \	–	(I often use this symbol, which is easier to write.)
11. $\neg$	~	(I often use this symbol, which is easier to write.)

B. One unusual feature of our text is that no mention is ever made of the concept of the *complement* of a set. In our text, the notion of the *set difference* $B\backslash A := \{x \mid x \in B \wedge x \notin A\}$ is taken as fundamental. Then the complement of the set A is simply the set $U\backslash A = \{x \mid x \notin A\}$, where U is the universal set. Note that since *every* x under discussion is assumed to be an element of U, it is not necessary (although it would not be incorrect) to write $U\backslash A = \{x \mid x \in U \wedge x \notin A\}$.

The more standard treatment takes the notion of the complement of a set as fundamental, defining the complement A^c of the set A by: $A^c := \{x \mid x \notin A\}$, and then defining the set difference $B\backslash A$ (= B – A) for any sets A and B by: $B\backslash A := B \cap A^c$. It is nice to have the notation A^c available so that one can state the most familiar form of De Morgan's laws for sets (see Notes on Logic and Proof II, 1.). There are of course generalizations of these laws that hold for set differences in general, namely,

(i) $C - (A \cap B) = (C - A) \cup (C - B)$, and

(ii) $C - (A \cup B) = (C - A) \cap (C - B)$.

3. The symbols $\Rightarrow$ and $\Leftrightarrow$. In some discrete mathematics texts, the symbol $\Rightarrow$ is (incorrectly) used in place of $\rightarrow$, and the symbol $\Leftrightarrow$ is (incorrectly) used in place of $\leftrightarrow$. It represents a very serious conceptual error to do this. The symbols $\Rightarrow$ and $\Leftrightarrow$ do not denote connectives, but, rather, relations between statements.

If P and Q are propositions, $P \Leftrightarrow Q$, which is read "P is equivalent to Q," asserts that the propositions P and Q are true in exactly the same circumstances. The assertion $P \Leftrightarrow Q$ is either true or false, period. There is no such thing as a truth table for $\Leftrightarrow$, because only connectives have truth tables, and $\Leftrightarrow$ is not a connective. There is, however, a connection between $\Leftrightarrow$ and $\leftrightarrow$. Suppose that P and Q are wffs of propositional logic (also known as *propositional forms*). In this context, "P and Q are true in exactly the same circumstances" simply means that P *and* Q *have identical truth tables*. But by the truth table for $\leftrightarrow$, P and Q have identical truth tables precisely when the proposition $P \leftrightarrow Q$ is a tautology. So here is the connection between $\Leftrightarrow$ and $\leftrightarrow$:

If P *and* Q *are wffs of propositional logic,* $P \Leftrightarrow Q$ *precisely when* $P \leftrightarrow Q$ *is a tautology.*

The author of our text, aware of the fact that people often confuse $\Leftrightarrow$ and $\leftrightarrow$, has chosen never to use the symbol $\Leftrightarrow$ at all, always using the phrase "is equivalent to" instead. But the symbol $\Leftrightarrow$ is ubiquitous in mathematical writing, and one really needs to learn to use it correctly, so I will make use of it all the time in my lectures and notes. To help keep the two distinct, let us agree always to read $P \Leftrightarrow Q$ as "P is equivalent to Q," and $P \leftrightarrow Q$ as "P if and only if Q."

If P and Q are propositions, $P \Rightarrow Q$, which is read "P implies Q," asserts that in any circumstance in which P is true, Q is also true. (Although the author of our text mentions "P implies Q" as one way of expressing $P \rightarrow Q$ in words, let us agree to use "implies" only in connection with $\Rightarrow$, always reading $P \rightarrow Q$ as "if P, then Q.") The assertion $P \Rightarrow Q$ is either true or false, period. There is no such thing as a truth table for $P \Rightarrow Q$, because only connectives have truth tables, and $\Rightarrow$ is not a connective. There is, however, a connection between $\Rightarrow$ and $\rightarrow$. Suppose that P and Q are wffs of propositional logic (also known as *propositional forms*). In this context, "in any circumstance in which P is true, Q is also true" simply means that the truth tables of P and Q have the property that whenever a line in the truth table of P takes the value T, that same line in the truth table of Q also takes the value T. But by the truth table for $\rightarrow$, this will be the case precisely when the proposition $P \rightarrow Q$ is a tautology. So here is the connection between $\Rightarrow$ and $\rightarrow$:

If P *and* Q *are wffs of propositional logic,* $P \Rightarrow Q$ *precisely when* $P \rightarrow Q$ *is a tautology.*

Remark. The relations $\Leftrightarrow$ and $\Rightarrow$ occur not just in propositional logic, but in any context in which there are unquantified variables. These symbols can be eliminated by universal quantification over those variables. For example,

1. (n is prime $\wedge$ n > 2) $\Rightarrow$ n is odd *may be equivalently expressed as:*

$$\forall \text{ n } [(\text{n is prime} \wedge \text{ n} > 2) \rightarrow \text{n is odd}].$$

2. f is differentiable $\Rightarrow$ f is continuous *may be equivalently expressed as:*

$$\forall \text{ f } (\text{f is differentiable } \rightarrow \text{ f is continuous}).$$

3. $\neg \forall x\, P(x) \Leftrightarrow \exists x\, \neg P(x)$ *may be equivalently expressed as:*

$$\forall P\, [\neg \forall x\, P(x) \leftrightarrow \exists x\, \neg P(x)].$$