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Parity Theorems for Statistics on Lattice Paths and Laguerre Configurations

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Abstract

We examine the parity of some statistics on lattice paths and Laguerre configurations, giving both algebraic and combinatorial treatments. For the former, we evaluate q-generating functions at $q = -1$; for the latter, we define appropriate parity-changing involutions on the associated structures. In addition, we furnish combinatorial proofs for a couple of related recurrences.

1 Introduction

To establish the familiar result that a finite nonempty set has equally many subsets of odd and of even cardinality it suffices either to set $q = -1$ in the generating function

$$\sum_{S\subseteq[n]} q^{|S|} = \sum_{k=0}^{n} \binom{n}{k} q^k = (1+q)^n, \tag{1.1}$$

where $[n] := \{1, \dots, n\}$, or to observe that the map

$$S \mapsto \begin{cases} S \cup \{1\}, & \text{if } 1 \notin S; \\ S - \{1\}, & \text{if } 1 \in S, \end{cases} \tag{1.2}$$

is a parity changing involution of $2^{[n]}$.

With this simple example as a model, we analyze the parity of a well known statistic on lattice paths, as well as two statistics on what Garsia and Remmel [3] call *Laguerre configurations*, i.e., distributions of labeled balls to unlabeled, contents-ordered boxes. These statistics have in common the fact that their generating functions all involve q-binomial coefficients.

In §2 we evaluate such coefficients and their sums, known as *Galois numbers*, when $q = -1$, giving both algebraic and bijective proofs. We also give a bijective proof of a recurrence for Galois numbers, furnishing an elementary alternative to Goldman and Rota's proof by the method of linear functionals [4]. In §3 we carry out a similar evaluation of the two types of q-Lah numbers that arise as generating functions for the aforementioned Laguerre configuration statistics. In addition, we supply a combinatorial proof of a recurrence for sums of Lah numbers.

The notational conventions of this paper are as follows: $\mathbb{N} := \{0, 1, 2, \dots\}$, $\mathbb{P} := \{1, 2, \dots\}$, $[0] := \varnothing$, and $[n] := \{1, \dots, n\}$ for $n \in \mathbb{P}$. If q is an indeterminate, then $0_q := 0$, $n_q := 1 + q + \cdots + q^{n-1}$ if $n \in \mathbb{P}$, $0_q^! := 1$, $n_q^! := 1_q 2_q \cdots n_q$ if $n \in \mathbb{P}$, and

$$\binom{n}{k}_q := \begin{cases} \frac{n_q^!}{k_q^!(n-k)_q^!}, & \text{if } 0 \le k \le n; \\ 0, & \text{if } k < 0 \text{ or } 0 \le n < k. \end{cases} \tag{1.3}$$

Our notation in (1.3) for the q-binomial coefficient, which agrees with Knuth's [5], has the advantage over the traditional notation $\left[{n \atop k}\right]$ that it can be used to reflect particular values of the parameter q.

2 A Statistic on Lattice Paths

Let $\Lambda(n, k)$ denote the set of (minimal) lattice paths from $(0, 0)$ to $(k, n-k)$, where $0 \leqslant k \leqslant n$. Each $\lambda \in \Lambda(n, k)$ corresponds to a sequential arrangement $t_1 \cdots t_n$ of the multiset $\{1^k, 2^{n-k}\}$, with 1 representing a horizontal and 2 a vertical step. Hence, $|\Lambda(n, k)| = \binom{n}{k}$. Moreover, since the area $\alpha(\lambda)$ subtended by λ is equal to the number of inversions in the corresponding word (i.e., the number of ordered pairs (i, j) with $1 \leqslant i < j \leqslant n$ such that $t_i > t_j$), and since the q-binomial coefficient is the generating function for the statistic that records the number of inversions in such words [10, Prop. 1.3.17], it follows that

$$\sum_{\lambda \in \Lambda(n,k)} q^{\alpha(\lambda)} = \binom{n}{k}_q, \tag{2.1}$$

a result that Berman and Fryer [1, p. 218] attribute to Polya. With

$$\Lambda(n) := \bigcup_{0 \leqslant k \leqslant n} \Lambda(n, k), \tag{2.2}$$

it follows that

$$\sum_{\lambda \in \Lambda(n)} q^{\alpha(\lambda)} = G_q(n) := \sum_{k=0}^{n} \binom{n}{k}_q. \tag{2.3}$$

The polynomials $G_q(n)$ have been termed *Galois numbers* by Goldman and Rota [4].

Let $\Lambda_r(n) := \{\lambda \in \Lambda(n) : \ \alpha(\lambda) \equiv r \pmod 2\}$, and let $\Lambda_r(n,k) := \Lambda(n,k) \cap \Lambda_r(n)$. Clearly,

$$\binom{n}{k}_{-1} = |\Lambda_0(n,k)| - |\Lambda_1(n,k)|, \tag{2.4}$$

and

$$G_{-1}(n) = |\Lambda_0(n)| - |\Lambda_1(n)|. \tag{2.5}$$

In evaluating (2.4) and (2.5) we shall employ several alternative characterizations of $\binom{n}{k}_q$, namely, the recurrence

$$\binom{n}{k}_q = \binom{n-1}{k-1}_q + q^k \binom{n-1}{k}_q, \qquad \forall n, k \in \mathbb{P}, \tag{2.6}$$

with $\binom{n}{0}_q = \delta_{n,0}$ and $\binom{0}{k}_q = \delta_{k,0}$, $\forall\ n, k \in \mathbb{N}$, the generating function

$$\sum_{n \geqslant 0} \binom{n}{k}_q x^n = \frac{x^k}{(1-x)(1-qx)\cdots(1-q^k x)}, \qquad \forall k \in \mathbb{N}, \tag{2.7}$$

and the summation formula

$$\binom{n}{k}_q = \sum_{\substack{d_0+d_1+\cdots+d_k=n-k \\ d_i \in \mathbb{N}}} q^{d_1+2d_2+\cdots+kd_k}. \tag{2.8}$$

See [11, pp. 201–202] for further details.

Setting $q=-1$ in (2.7) and treating separately the even and odd cases for k yields

Theorem 2.1. *If $0 \leqslant k \leqslant n$, then*

$$\binom{n}{k}_{-1} = \begin{cases} 0, & \text{if } n \text{ is even and } k \text{ is odd;} \\ \binom{\lfloor n/2 \rfloor}{\lfloor k/2 \rfloor}, & \text{otherwise.} \end{cases} \tag{2.9}$$

A straightforward application of (2.9) yields

Corollary 2.1.1. *For all $n \in \mathbb{N}$,*

$$G_{-1}(n) = 2^{\lceil n/2 \rceil}. \tag{2.10}$$

The above results are well known and apparently very old. But the following bijective proofs of (2.9) and (2.10), which convey a more visceral understanding of these formulas, are, so far as we know, new.

Bijective proofs of Theorem 2.1 and Corollary 2.1.1.

As above, we represent a lattice path $\lambda \in \Lambda(n)$ by a word $t_1 t_2 \cdots t_n$ in the alphabet $\{1, 2\}$, recalling that $\alpha(\lambda)$ is equal to the number of inversions in this word, which we also denote by $\alpha(\lambda)$. By (2.5), formula (2.10) asserts that

$$|\Lambda_0(n)| - |\Lambda_1(n)| = 2^{\lceil n/2 \rceil}. \tag{2.11}$$

Our strategy for proving (2.11) is to identify a subset $\Lambda_0^+(n)$ of $\Lambda_0(n)$ having cardinality $2^{\lceil n/2 \rceil}$, along with an α-parity changing involution of $\Lambda(n) - \Lambda_0^+(n)$. Let $\Lambda_0^+(n)$ comprise those words $\lambda = t_1 t_2 \cdots t_n$ such that for $i = 1, 2, \ldots, \lfloor n/2 \rfloor$,

$$t_{2i-1} t_{2i} = 11 \text{ or } 22. \tag{2.12}$$

Clearly, $\Lambda_0^+(n) \subseteq \Lambda_0(n)$ and $|\Lambda_0^+(n)| = 2^{\lceil n/2 \rceil}$. If $\lambda \in \Lambda(n) - \Lambda_0^+(n)$, let i_0 be the smallest index for which (2.12) fails to hold and let λ' be the result of switching t_{2i_0-1} and t_{2i_0} in λ. The map $\lambda \mapsto \lambda'$ is clearly an α-parity changing involution of $\Lambda(n) - \Lambda_0^+(n)$, which proves (2.11) and hence (2.10).

By (2.4), formula (2.9) asserts that

$$|\Lambda_0(n,k)| - |\Lambda_1(n,k)| = \begin{cases} 0, & \text{if } n \text{ is even and } k \text{ is odd;} \\ \binom{\lfloor n/2 \rfloor}{\lfloor k/2 \rfloor}, & \text{otherwise.} \end{cases} \tag{2.13}$$

To show (2.13), let $\Lambda_0^+(n,k) = \Lambda_0^+(n) \cap \Lambda(n,k)$. The cardinality of $\Lambda_0^+(n,k)$ is given by the right-hand side of (2.13), and the restriction of the above map to $\Lambda(n,k) - \Lambda_0^+(n,k)$ is again an involution and inherits the parity changing property. This proves (2.13), and hence (2.9). □

In tabulating the numbers $\binom{n}{k}_{-1}$ it is of course more efficient to use the recurrence

$$\binom{n}{k}_{-1} = \binom{n-1}{k-1}_{-1} + (-1)^k \binom{n-1}{k}_{-1}, \tag{2.14}$$

representing the case $q = -1$ of (2.6).

Comparison of (2.9) with an evaluation of $\binom{n}{k}_{-1}$ based on (2.8) yields a pair of interesting identities.

Corollary 2.1.2. *If* $1 \leqslant m \leqslant \lfloor n/2 \rfloor$, *then*

$$\sum_{j=0}^{n-2m} (-1)^j \binom{m+j-1}{m-1} \binom{n-m-j}{m} = \binom{\lfloor n/2 \rfloor}{m}, \tag{2.15}$$

and if $0 \leqslant m \leqslant \lfloor (n-1)/2 \rfloor$, *then*

$$\sum_{j=0}^{n-2m-1} (-1)^j \binom{m+j}{m} \binom{n-m-j-1}{m} = \begin{cases} 0, & \textit{if } n \textit{ is even;} \\ \binom{\lfloor n/2 \rfloor}{m}, & \textit{if } n \textit{ is odd.} \end{cases} \tag{2.16}$$

Proof. Setting $q = -1$ and $k = 2m$ in (2.8) yields

$$\begin{aligned}\binom{n}{2m}_{-1} &= \sum_{d_0+d_1+\cdots+d_{2m}=n-2m} (-1)^{d_1+d_3+\cdots+d_{2m-1}} \\ &\underset{(j=d_1+d_3+\cdots+d_{2m-1})}{=} \sum_{j=0}^{n-2m} (-1)^j \binom{m+j-1}{m-1}\binom{n-m-j}{m},\end{aligned}$$

which implies (2.15) by (2.9), upon independently choosing the d_i's of even index, which sum to $n-2m-j$. Setting $k = 2m+1$ yields

$$\begin{aligned}\binom{n}{2m+1}_{-1} &= \sum_{d_0+d_1+\cdots+d_{2m+1}=n-2m-1} (-1)^{d_1+d_3+\cdots+d_{2m+1}} \\ &\underset{(j=d_1+d_3+\cdots+d_{2m+1})}{=} \sum_{j=0}^{n-2m-1} (-1)^j \binom{m+j}{m}\binom{n-m-j-1}{m},\end{aligned}$$

which implies (2.16) by (2.9). □

Corollary 2.1.1 above can also be proved by induction from the case $q = -1$ of the following recurrence for $G_q(n)$:

Theorem 2.2. *For all $n \in \mathbb{P}$,*

$$G_q(n+1) = 2G_q(n) + (q^n - 1)G_q(n-1), \tag{2.17}$$

where $G_q(0) = 1$ and $G_q(1) = 2$.

Proof. Let $a(n,i) := |\{\lambda \in \Lambda(n) : \alpha(\lambda) = i\}|$, where $n \in \mathbb{N}$ and $a(n,i) := 0$ if $i < 0$. Showing (2.17) is equivalent to showing that

$$\begin{aligned}a(n+1,i) &= 2a(n,i) + a(n-1,i-n) - a(n-1,i) \\ &= a(n,i) + (a(n,i) - a(n-1,i)) + a(n-1,i-n)\end{aligned} \tag{2.18}$$

for all $i \in \mathbb{N}$. As above, we represent a lattice path $\lambda \in \Lambda(n+1)$ by a word $t_1t_2\cdots t_{n+1}$ in the alphabet $\{1,2\}$, recalling that $\alpha(\lambda)$ is equal to the number of inversions in this word.

The term $a(n+1,i)$ thus counts all words of length $n+1$ with i inversions. The term $a(n,i)$ counts the subclass of such words for which $t_{n+1} = 2$. The term $a(n,i) - a(n-1,i)$ counts the subclass of such words for which $t_1 = t_{n+1} = 1$. For deletion of t_1 is a bijection from this subclass to the class of words $u_1u_2\cdots u_n$ with i inversions and $u_n = 1$, and there are clearly $a(n,i) - a(n-1,i)$ words of the latter type. Finally, the term $a(n-1,i-n)$ counts the subclass of words for which $t_1 = 2$ and $t_{n+1} = 1$. For deletion of t_1 and t_{n+1} is a bijection from this subclass to the class of words $v_1v_2\cdots v_{n-1}$ with $i-n$ inversions (both classes being empty if $i < n$). □

The above proof provides an elementary alternative to Goldman and Rota's proof of (2.17) using the method of linear functionals [4].

3 Two Statistics on Laguerre Configurations

Let $\mathcal{L}(n,k)$ denote the set of distributions of n balls, labeled $1,2,\ldots,n$, among k unlabeled, *contents-ordered* boxes, with no box left empty. Garsia and Remmel [3] term such distributions *Laguerre configurations*. If $L(n,k) := |\mathcal{L}(n,k)|$, then $L(n,0) = \delta_{n,0}$, $\forall\ n \in \mathbb{N}$, $L(n,k) = 0$ if $0 \leqslant n < k$, and

$$L(n,k) = \frac{n!}{k!}\binom{n-1}{k-1}, \qquad 1 \leqslant k \leqslant n. \tag{3.1}$$

The numbers $L(n,k)$ are called *Lah numbers*, after Ivo Lah [6], who introduced them as the connection constants in the polynomial identities

$$x(x+1)\cdots(x+n-1) = \sum_{k=0}^{n} L(n,k)x(x-1)\cdots(x-k+1), \qquad \forall n \in \mathbb{N}. \tag{3.2}$$

From (3.1) it follows that

$$\sum_{n\geqslant k} L(n,k)\frac{x^n}{n!} = \frac{1}{k!}\left(\frac{x}{1-x}\right)^k, \qquad \forall k \in \mathbb{N}. \tag{3.3}$$

The Lah numbers also satisfy the recurrence relations

$$L(n,k) = L(n-1,k-1) + (n+k-1)L(n-1,k), \qquad \forall n,k \in \mathbb{P}, \tag{3.4}$$

and

$$L(n,k) = \frac{n}{k}L(n-1,k-1) + nL(n-1,k), \qquad \forall n,k \in \mathbb{P}. \tag{3.5}$$

The set $\mathcal{L}(n) := \bigcup_k \mathcal{L}(n,k)$ comprises all distributions of n balls, labeled $1,2,\ldots,n$, among n unlabeled, contents-ordered boxes. If $L(n) := |\mathcal{L}(n)|$, it follows from (3.3) that

$$\sum_{n\geqslant 0} L(n)\frac{x^n}{n!} = e^{x/(1-x)}, \tag{3.6}$$

and differentiating (3.6) yields [7, p. 171], [9, A000262]

Theorem 3.1. *For all* $n \in \mathbb{P}$,

$$L(n+1) = (2n+1)L(n) - (n^2-n)L(n-1), \tag{3.7}$$

where $L(0) = L(1) = 1$.

Combinatorial proof of Theorem 3.1.

We'll argue that the cardinality of $\mathcal{L}(n+1)$ is given by the right-hand side of (3.7) when $n \geqslant 1$. Let us represent members of $\mathcal{L}(m)$ by partitions of $[m]$ in which the elements of each block are ordered. As there are clearly $L(n)$ members of $\mathcal{L}(n+1)$ in which the singleton

$\{n+1\}$ occurs, we need only show that the members of $\mathcal{L}(n+1)$ in which the singleton $\{n+1\}$ doesn't occur number $2nL(n) - n(n-1)L(n-1)$.

Suppose $\lambda \in \mathcal{L}(n)$ and consider the $2n$ members of $\mathcal{L}(n+1)$ gotten from λ by inserting $n+1$ either directly before or directly after an element of $[n]$ within λ. Then $2nL(n)$ double counts members of $\mathcal{L}(n+1)$ for which $n+1$ is neither first nor last in its block and counts once all other members of $\mathcal{L}(n+1)$ for which $n+1$ goes in a block with at least one element of $[n]$. But there are $n(n-1)L(n-1)$ configurations of the former type as seen upon choosing an element j of $[n]$ to directly follow $n+1$ and then inserting $n+1, j$ directly after an element of $[n]-\{j\}$ in a Laguerre configuration of the set $[n]-\{j\}$. □

In what follows, we consider two statistics on Laguerre configurations.

3.1 The Statistic i

Given a distribution $\delta \in \mathcal{L}(n,k)$, let us represent the ordered contents of each box by a word in $[n]$, and then arrange these words in a sequence $W_1, \ldots, W_k$ in decreasing order of their least elements. Replacing the commas in this sequence by zeros and counting inversions in the resulting single word yields the value $i(\delta)$, i.e.,

$$i(\delta) = \text{ the number of inversions in } W_1 0 W_2 0 \cdots 0 W_{k-1} 0 W_k. \tag{3.8}$$

As an illustration, for the distribution $\delta \in \mathcal{L}(9,4)$ given by

$$\lfloor 3,4,9 \rfloor \quad \lfloor 8,1 \rfloor \quad \lfloor 2,6 \rfloor \quad \lfloor 7,5 \rfloor, \tag{3.9}$$

we have $i(\delta) = 35$, the number of inversions in the word 750349026081.

The statistic i is due to Garsia and Remmel [3], who show that the generating function

$$L_q(n,k) := \sum_{\delta \in \mathcal{L}(n,k)} q^{i(\delta)} = q^{k(k-1)} \frac{n_q^!}{k_q^!} \binom{n-1}{k-1}_q, \quad 1 \leqslant k \leqslant n. \tag{3.10}$$

Generalizing (3.4), the q-Lah number $L_q(n,k)$ satisfies the recurrence

$$L_q(n,k) = q^{n+k-2} L_q(n-1,k-1) + (n+k-1)_q L_q(n-1,k), \forall n,k \in \mathbb{P}. \tag{3.11}$$

Garsia and Remmel also show that

$$x_q (x+1)_q \cdots (x+n-1)_q = \sum_{k=1}^{n} L_q(n,k) x_q (x-1)_q \cdots (x-k+1)_q, \tag{3.12}$$

where $x_q := (q^x - 1)/(q-1)$. It seems not to have been noted that (3.12) is equivalent to

$$x(qx + 1_q) \cdots (q^{n-1}x + (n-1)_q) = \sum_{k=1}^{n} L_q(n,k) x \left(\frac{x - 1_q}{q}\right) \cdots \left(\frac{x-(k-1)_q}{q^{k-1}}\right), \tag{3.13}$$

which generalizes (3.2).

Theorem 3.2. *If $1 \le k \le n$, then*

$$L_{-1}(n,k) = \delta_{n,k}. \tag{3.14}$$

Proof. Formula (3.14) is an immediate consequence of (3.10) and (2.9), upon considering even and odd cases for n, as $j_{-1} = 0$ if j is even (cf. [8]). For a bijective proof of (3.14), first note that $L_{-1}(n,k) = |\mathcal{L}_0(n,k)| - |\mathcal{L}_1(n,k)|$, where $\mathcal{L}_r(n,k) := \{\delta \in \mathcal{L}(n,k) : \ i(\delta) \equiv r \pmod 2\}$. Now $\mathcal{L}(n,n)$ consists of a single distribution δ, with $i(\delta) = n(n-1) =$ the number of inversions in $n0(n-1)0\cdots0201$, whence $|\mathcal{L}_0(n,n)| = 1$ and $|\mathcal{L}_1(n,n)| = 0$. If $1 \leqslant k < n$ and $\delta \in \mathcal{L}(n,k)$ gives rise to the sequence $W_1,\ldots,W_k$, then locate the leftmost word W_i containing at least two letters and interchange its first two letters. The resulting map is a parity changing involution of $\mathcal{L}(n,k)$, whence $|\mathcal{L}_0(n,k)| - |\mathcal{L}_1(n,k)| = 0$. □

Remark. Note that $\mathcal{L}(n,1) = \mathcal{S}_n$, the set of permutations of $[n]$, and so (3.10) is a generalization of the well known result that

$$\sum_{\pi \in \mathcal{S}_n} q^{i(\pi)} = n_q^!, \tag{3.15}$$

and (3.14) a generalization of the fact that among the permutations of $[n]$, if $n \geqslant 2$, there are as many with an odd number of inversions as there are with an even number of inversions.

3.2 The Statistic $\tilde{w}$

As above, given $\delta \in \mathcal{L}(n,k)$, we represent the ordered contents of each box by a word in $[n]$. Now, however, we arrange these words in a sequence $W_1,\ldots,W_k$ in increasing order of their initial elements, defining $\tilde{w}(\delta)$ by the formula

$$\tilde{w}(\delta) = \sum_{i=1}^{k} (i-1)(|W_i| - 1), \tag{3.16}$$

where $|W_i|$ denotes the length of the word W_i. As an illustration, for the distribution $\delta \in \mathcal{L}(9,4)$ given above by (3.9), we have W_1, W_2, W_3, $W_4 = 26$, 349, 75, 81 and $\tilde{w}(\delta) = 7$. The statistic $\tilde{w}$ is an analogue of a now well known partition statistic first introduced by Carlitz [2] (see also [11]).

Theorem 3.3. *The generating function*

$$\tilde{L}_q(n,k) := \sum_{\delta \in \mathcal{L}(n,k)} q^{\tilde{w}(\delta)} = \frac{n!}{k!} \binom{n-1}{k-1}_q, \qquad 1 \leqslant k \leqslant n. \tag{3.17}$$

Proof. In running through $\delta \in \mathcal{L}(n,k)$, we are running through all sequences of words $W_1,\ldots,W_k$ whose initial elements form an increasing sequence, and such that $|W_i| = n_i$, with $\sum n_i = n$. For fixed such $n_1,\ldots,n_k$, there are $\binom{n}{k}(n-k)!$ such sequences, $\binom{n}{k}$ being the number of ways to choose and place the initial elements, and $(n-k)!$ the number of ways to place the remaining elements. By (3.16) and (2.8), it follows that

$$\sum_{\delta\in\mathcal{L}(n,k)} q^{\tilde{w}(\delta)} = \binom{n}{k}(n-k)! \sum_{\substack{n_1+\cdots+n_k=n\\ n_i\in\mathbb{P}}} q^{0(n_1-1)+1(n_2-1)+\cdots+(k-1)(n_k-1)}$$
$$= \frac{n!}{k!}\binom{n-1}{k-1}_q.$$

□

From (3.17) and (2.7), it follows that

$$\sum_{n\geqslant k} \tilde{L}_q(n,k)\frac{x^n}{n!} = \frac{1}{k!}\frac{x^k}{\prod\limits_{0\leqslant j\leqslant k-1}(1-q^j x)}, \quad \forall k\in\mathbb{N}, \tag{3.18}$$

which generalizes (3.3). The q-Lah number $\tilde{L}_q(n,k)$ also satisfies the recurrence

$$\tilde{L}_q(n,k) = \frac{n}{k}\tilde{L}_q(n-1,k-1) + nq^{k-1}\tilde{L}_q(n-1,k), \tag{3.19}$$

which generalizes (3.5).

Theorem 3.4. *If $1\le k\le n$, then*

$$\tilde{L}_{-1}(n,k) = \begin{cases} 0, & \textit{if } n \textit{ is odd and } k \textit{ is even;}\\ \frac{n!}{k!}\binom{\lfloor (n-1)/2\rfloor}{\lfloor (k-1)/2\rfloor}, & \textit{otherwise.}\end{cases} \tag{3.20}$$

Proof. This follows immediately from (3.17) and (2.9), but the following bijective proof yields a deeper insight into this result: with $\mathcal{L}_r(n,k) := \{\delta\in\mathcal{L}(n,k) :\ \tilde{w}(\delta)\equiv r \pmod 2\}$, we have $\tilde{L}_{-1}(n,k) = |\mathcal{L}_0(n,k)| - |\mathcal{L}_1(n,k)|$. To prove (3.20) it thus suffices to identify a subset $\mathcal{L}_0^+(n,k)$ of $\mathcal{L}_0(n,k)$ such that

$$|\mathcal{L}_0^+(n,k)| = \begin{cases} 0, & \text{if } n \text{ is odd and } k \text{ is even;}\\ \frac{n!}{k!}\binom{\lfloor (n-1)/2\rfloor}{\lfloor (k-1)/2\rfloor}, & \text{otherwise,}\end{cases} \tag{3.21}$$

along with a parity changing involution of $\mathcal{L}(n,k) - \mathcal{L}_0^+(n,k)$.

The set $\mathcal{L}_0^+(n,k)$ consists of those distributions whose associated sequences $W_1, W_2, \ldots, W_k$ satisfy

$$|W_{2i-1}| \text{ is odd and } |W_{2i}| = 1,\ 1\leqslant i\leqslant \lfloor k/2\rfloor. \tag{3.22}$$

Clearly, $\mathcal{L}_0^+(n,k)=\varnothing$ if n is odd and k is even. In the remaining cases, the factor $n!/k!$ arises as the product $\binom{n}{k}(n-k)!$, just as it does in the proof of Theorem 3.3, and

$$\binom{\lfloor (n-1)/2\rfloor}{\lfloor (k-1)/2\rfloor} = \Big|\Big\{(n_1,\ldots,n_k) :\ \sum n_i = n,\ n_{2i-1} \text{ is odd,}$$
$$\text{and } n_{2i}=1,\ 1\leqslant i\leqslant\lfloor k/2\rfloor\Big\}\Big|, \tag{3.23}$$

upon halving compositions of an integer whose parts are all even.

Suppose now that $\delta \in \mathcal{L}(n,k) - \mathcal{L}_0^+(n,k)$ is associated with the sequence $W_1, \dots, W_k$ and that i_0 is the smallest index for which (3.22) fails to hold. If $|W_{2i_0-1}|$ is even, take the last member of W_{2i_0-1} and place it at the end of W_{2i_0}. If $|W_{2i_0-1}|$ is odd, whence $|W_{2i_0}| \geqslant 2$, take the last member of W_{2i_0} and place it at the end of W_{2i_0-1}. The resulting map is a parity changing involution of $\mathcal{L}(n,k) - \mathcal{L}_0^+(n,k)$. □

In tabulating the numbers $\tilde{L}_{-1}(n,k)$ it is of course more efficient to use the recurrence

$$\tilde{L}_{-1}(n,k) = \frac{n}{k}\tilde{L}_{-1}(n-1,k-1) + (-1)^{k-1} n \tilde{L}_{-1}(n-1,k), \tag{3.24}$$

representing the case $q = -1$ of (3.19). This yields the following table for $0 \leqslant k \leqslant n \leqslant 8$:

Table 3.1: The numbers $\tilde{L}_{-1}(n,k)$ for $0 \leqslant k \leqslant n \leqslant 8$.

	$k=0$	1	2	3	4	5	6	7	8
$n=0$	1								
1	0	1							
2	0	2	1						
3	0	6	0	1					
4	0	24	12	4	1				
5	0	120	0	40	0	1			
6	0	720	360	240	60	6	1		
7	0	5040	0	2520	0	126	0	1	
8	0	40320	20160	20160	5040	1008	168	8	1

The row sums of Table 3.1 correspond to the quantities $\tilde{L}_{-1}(n)$ [9, A089656], where

$$\tilde{L}_q(n) := \sum_{\delta \in \mathcal{L}(n)} q^{\tilde{w}(\delta)} = \sum_k \tilde{L}_q(n,k). \tag{3.25}$$

We have been unable to find a simple closed form or recurrence for $\tilde{L}_{-1}(n)$. However, using the case $q = -1$ of formula (3.18), it is straightforward to show that

$$\sum_{n \geqslant 0} \tilde{L}_{-1}(n) \frac{x^n}{n!} = \cosh \frac{x}{\sqrt{1-x^2}} + \frac{\sqrt{1-x^2}}{1-x} \sinh \frac{x}{\sqrt{1-x^2}}. \tag{3.26}$$

The values of $\tilde{L}_{-1}(n)$ for $0 \leqslant n \leqslant 10$ are as follows: 1, 1, 3, 7, 41, 161, 1387, 7687, 86865, 623233, 8682131.

4 Some Concluding Remarks

Reductions from q-binomial coefficients to ordinary binomial coefficients similar to those seen when $q = -1$ occur with higher roots of unity. For example, substituting $q = \rho = \frac{-1+\sqrt{3}i}{2}$, a third root of unity, and $q = i$, a fourth root of unity, into (2.7) and considering cases for k mod 3 and mod 4 yields

Theorem 4.1. *If* $0 \le k \le n$*, then*

$$\binom{n}{k}_\rho = \begin{cases} \binom{\lfloor n/3 \rfloor}{\lfloor k/3 \rfloor}, & \text{if } n \equiv k \pmod 3 \text{ or } k \equiv 0 \pmod 3; \\ -\rho^2 \binom{\lfloor n/3 \rfloor}{\lfloor k/3 \rfloor}, & \text{if } n \equiv 2 \pmod 3 \text{ and } k \equiv 1 \pmod 3; \\ 0, & \text{otherwise.} \end{cases} \tag{4.1}$$

and

Theorem 4.2. *If* $0 \le k \le n$*, then*

$$\binom{n}{k}_i = \begin{cases} \binom{\lfloor n/4 \rfloor}{\lfloor k/4 \rfloor}, & \text{if } n \equiv k \pmod 4 \text{ or } k \equiv 0 \pmod 4; \\ i\binom{\lfloor n/4 \rfloor}{\lfloor k/4 \rfloor}, & \text{if } n \equiv 3 \pmod 4 \text{ and } k \equiv 1,2 \pmod 4; \\ (1+i)\binom{\lfloor n/4 \rfloor}{\lfloor k/4 \rfloor}, & \text{if } n \equiv 2 \pmod 4 \text{ and } k \equiv 1 \pmod 4; \\ 0, & \text{otherwise.} \end{cases} \tag{4.2}$$

Bijective proof of Theorem 4.1.

We modify the combinatorial argument used to establish (2.9). Instead of pairing members of $\Lambda(n,k)$ of opposite α-parity, we partition a portion of $\Lambda(n,k)$ into tripletons each of whose members have different α values mod 3. Each such tripleton contributes 0 towards the sum $\binom{n}{k}_\rho = \sum_{\lambda \in \Lambda(n,k)} \rho^{\alpha(\lambda)}$ since $1 + \rho + \rho^2 = 0$.

As before, we represent lattice paths by words in $\{1,2\}$. Let $\Lambda'(n,k)$ consist of those words $\lambda = t_1 t_2 \cdots t_n$ in $\Lambda(n,k)$ satisfying

$$t_{3i-2} = t_{3i-1} = t_{3i}, \qquad 1 \leqslant i \leqslant \lfloor n/3 \rfloor. \tag{4.3}$$

In all cases, the right-hand side of (4.1) above gives the net contribution of $\Lambda'(n,k)$ towards $\binom{n}{k}_\rho$; note that members of $\Lambda'(n,k)$ may end in either 12 or 21 if $n \equiv 2 \pmod 3$ and $k \equiv 1 \pmod 3$, hence the $1 + \rho = -\rho^2$ factor in this case.

Suppose now that $\lambda = t_1 t_2 \cdots t_n \in \Lambda(n,k) - \Lambda'(n,k)$, with i_0 the smallest i for which (4.3) fails to hold. Group the three members of $\Lambda(n,k) - \Lambda'(n,k)$ gotten by circularly permuting t_{3i_0-2}, t_{3i_0-1}, and t_{3i_0} within $\lambda = t_1 t_2 \cdots t_n$, leaving the rest of λ undisturbed. Note that these three members of $\Lambda(n,k) - \Lambda'(n,k)$ have different α values mod 3, which establishes (4.1). □

A similar proof, which involves partitioning members of $\Lambda(n,k)$ according to their *inv* values mod 4, applies to (4.2), the details of which we leave as an exercise for interested readers.

If $m \in \mathbb{P}$ and $\omega = e^{2\pi i/m}$, a primitive m^{th} root of unity, examining (2.7) when $q = \omega$ reveals that $\binom{n}{k}_\omega$ is of the form $\beta\binom{\lfloor n/m \rfloor}{\lfloor k/m \rfloor}$ for all n and k, where β is some complex number depending on the values of n and k mod m. Even though β can in general be expressed in terms of symmetric functions of certain m^{th} roots of unity, there does not appear to be a simple closed form for $\binom{n}{k}_\omega$ which generalizes (2.9), (4.1), and (4.2). Some particular cases are easily ascertained. For example, when m divides n, we have from (2.7),

$$\binom{n}{k}_\omega = \begin{cases} \binom{n/m}{k/m}, & \text{if } m \text{ divides } k; \\ 0, & \text{otherwise.} \end{cases} \tag{4.4}$$

When m is a prime, the combinatorial argument used for (4.1) readily generalizes to (4.4).

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