GENERATINGFUNCTIONOLOGY

How to Solve Counting Problems by Multiplying Power Series

Carl Wagner

Mathematics Department Junior Colloquium – September 21, 2017

Abstract. We'll see how multiplying various power series leads to the solution of many interesting problems in enumerative combinatorics. In particular, we will derive what is arguably the most beautiful formula in mathematics, the so-called "exponential generating function" of the sequence $(B_n)_{n\geq 0}$ of Bell numbers, which enumerate the equivalence relations on an n-element set.

Notation and Terminology

a. Notation: $\mathbb{N} = \{0,1,2,\ldots\}$, $\mathbb{P} = \{1,2,\ldots\}$, $\mathbb{R}$ = the set of real numbers,
$[0] = \varnothing$, and $[n] = \{1,\ldots,n\}$ if $n \in \mathbb{P}$. We use the set $[n]$ as our generic $n-$ element set.
If A is a finite set, $|A|$ denotes the cardinality of A.

b. For every $x \in \mathrm{R}$, $\binom{x}{0} := 1$ and $\binom{x}{k} := \dfrac{x(x-1)\cdots(x-k+1)}{k!}$ if $k \in \mathbb{P}$.

If $n,k \in \mathbb{N}$, $[n]$ has $\binom{n}{k}$ subsets of cardinality k.

If $k > n$, then $\binom{n}{k} = 0$. If $0 \leq k \leq n$, then $\binom{n}{k} = \dfrac{n!}{k!(n-k)!}$.

c. The set $[n]$ has 2^n subsets

d. If $(a_n)_{n\geq 0}$ is a sequence in $\mathbb{R}$, $\sum_{n\geq 0} a_n x^n$ is the *ordinary generating function* and $\sum_{n\geq 0} a_n \dfrac{x^n}{n!}$ the *exponential generating function* of $(a_n)_{n\geq 0}$.

PART I

1. Definition: A *composition* of the positive integer n is a way of writing n as an *ordered sum* of one or more positive integers, called the *parts* of the composition. Let $comp(n) :=$ the number of compositions of n, and $comp(n,k) :=$ the number of compositions of n with exactly k parts.

Theorem 1. For all $n > 0$, $comp(n) = 2^{n-1}$. If $1 \le k \le n$, $comp(n,k) = \binom{n-1}{k-1}$.

2. Now let $n, k \ge 0$ and suppose that I is a nonempty set of nonnegative integers.

Let $C(n,k,I) :=$ the number of ways to write n as an ordered sum of k members of I

Conventions: (i) $C(n,0,I) = \delta_{n,0}$. (ii) $C(0,k,I) = \delta_{0,k}$ if $0 \notin I$.

Note that the *definition* of $C(n,k,I)$ implies that $C(0,k,I) = 1$ if $0 \in I$. Also, if $0 \notin I$ and $k > n$, then $C(n,k,I) = 0$.

Theorem 2. For all $k \ge 0$, $(\sum_{i \in I} x^i)^k = \sum_{n \ge 0} C(n,k,I) x^n$.

Proof. Since $C(n,0,I) = \delta_{n,0}$, the result holds for $k = 0$. Since $C(n,1,I) = 1$ if $n \in I$ and $C(n,1,I) = 0$ otherwise, the result holds for $k = 1$. If $k \ge 2$, then

$$(\sum_{i \in I} x^i)^k = (\sum_{n_1 \in I} x^{n_1}) \cdots (\sum_{n_k \in I} x^{n_k}) = \sum_{n_1,\ldots,n_k \in I} x^{n_1 + \cdots + n_k} = (collecting\ \ like\ \ powers)$$

$$\sum_{n \ge 0} |\{(n_1,\ldots,n_k) : \text{each } n_j \in I \text{ and } n_1 + \cdots + n_k = n\}|\ x^n = \sum_{n \ge 0} C(n,k,I) x^n. \qquad \square$$

Applications. Note that by the definition of $\binom{x}{n}$, $\binom{-k}{n} = (-1)^n \binom{n+k-1}{n}$ if $k \ge 0$, so $\binom{-1}{k} = (-1)^k$.

Recall *Newton's generalization of the binomial theorem:* For all $\alpha \in \mathbb{R}$,

$$(1+x)^\alpha = \sum_{n \ge 0} \binom{\alpha}{n} x^n \quad \text{for all } x \in \mathbb{R} \text{ such that } |x| < 1.$$

Application 2.1: $\sum_{n \ge 0} C(n,k,\mathbb{N})\ x^n =$

$(1 + x + x^2 + \cdots)^k = (1-x)^{-k} = \sum_{n \ge 0} (-1)^n \binom{-k}{n} x^n = \sum_{n \ge 0} \binom{n+k-1}{n} x^n$. So $C(n,k,\mathbb{N})$, the number of *weak compositions of n with k parts*, is equal to $\binom{n+k-1}{n}$

Application 2.2 (*exercise*). Noting that $comp(n,k) = C(n,k,\mathbb{P})$, use Theorem 2 to show that

$comp(0,0) = 1$, $comp(n,k) = \binom{n-1}{k-1}$ if $1 \le k \le n$, and $comp(n,k) = 0$ otherwise.

3. Now suppose that $I \subseteq \mathbb{P}$, so that $C(n,k,I) = 0$ if $k > n$ and we may define $C(n,I)$ by

$$C(n,I) = \sum_{k\ge 0} C(n,k,I)$$

Theorem 3. If $I \subseteq \mathbb{P}$, then $\displaystyle\sum_{n\ge 0} C(n,I)x^n = \frac{1}{1-\sum_{i\in I} x^i}$.

Proof. $$\sum_{n\ge 0} C(n,I)x^n = \sum_{n\ge 0}\sum_{k\ge 0} C(n,k,I)x^n = \sum_{k\ge 0}\sum_{n\ge 0} C(n,k,I)x^n = \sum_{k\ge 0}(\sum_{i\in I} x^i)^k = \frac{1}{1-\sum_{i\in I} x^i} .$$

Application 3.1 $\displaystyle\sum_{n\ge 0} C(n,\{1,2\})x^n = \frac{1}{1-x-x^2}$. Multiplying both sides by $1-x-x^2$ and equating the coefficients of like powers of x yields the following *recursive formula* for $C(n,\{1,2\})$:

$C(0,\{1,2\}) = C(1,\{1,2\}) = 1$, and $C(n,\{1,2\}) = C(n-1,\{1,2\}) + C(n-2,\{1,2\})$ for all $n \ge 2$.
So $C(n,\{1,2\}) = F_n$, the n^{th} Fibonacci number.

Application 3.2 (exercise)

(i) Determine $\displaystyle\sum_{n\ge 0} C(n,\ 2\mathbb{N}+1)\,x^n$, where $2\mathbb{N}+1$ denotes the set of odd positive integers, and use the result to find a recursive formula for $C(n,\ 2\mathbb{N}+1)$.

(ii) Determine $\displaystyle\sum_{n\ge 0} C(n,\ 2\mathbb{P})\,x^n$, where $2\mathbb{P}$ denotes the set of even positive integers, and use the result to find a recursive formula for $C(n,\ 2\mathbb{P})$.

(iii) Let c_n denote the number of compositions of n with all parts greater than 1. Determine $\displaystyle\sum_{n\ge 0} c_n x^n$, and find a recursive formula for c_n.

(iv) Let c_n denote the number of compositions of n with all parts congruent to 2 modulo 3.

Determine $\sum_{n\geq 0} c_n x^n$, and find a recursive formula for c_n.

PART II

1. Functions and Ordered Partitions

If A is any set, a sequence $(A_1,\ldots,A_k)$ of pairwise disjoint subsets of A such that $A_1 \cup \cdots \cup A_k = A$ is called a *weak ordered partition of A*. If each $A_i \neq \varnothing$, then $(A_1,\ldots,A_k)$ is called an *ordered partition of A*. The sets A_i are called *blocks*.

Remark. There is a $1-\text{to}-1$ correspondence between the set of all functions $f : A \to [k]$

and the set of all weak ordered partitions $(A_1,\ldots,A_k)$ of A, given by the mapping $f \mapsto (f^{-1}(\{1\}),\ldots, f^{-1}(\{k\}))$

The restriction of the above map to the set of surjective functions is a $1-\text{to}-1$ correspondence between the set of such functions and the set of all ordered partitions $(A_1,\ldots,A_k)$ of A.

Theorem 1. If $n_1 + n_2 + \cdots + n_k = n$, where each $n_i \geq 0$, then the number of weak ordered partitions $(A_1,\ldots,A_k)$ of the set $[n]$ with $|A_i| = n_i$ is equal to the *$k-$nomial coefficient*

$$\binom{n}{n_1, n_2, \ldots, n_k} = \frac{n!}{n_1!n_2!\cdots n_k!} \quad .$$

By the **Remark** above, the $k-$nomial coefficient $\binom{n}{n_1, n_2, \ldots, n_k}$ also counts a certain class of functions $f:[n]\to[k]$, namely, those f for which $|f^{-1}(\{i\})| = n_i$, $i = 1,\ldots,k$.

2. Generating Functions for Ordered Partitions

Suppose that $I \subseteq \mathbb{N}$ is nonempty. Let $P(n,k,I) :=$ the number of weak ordered partitions $(A_1,\ldots,A_k)$ of $[n]$, with each $|A_i| \in I$. By the convention that empty unions are equal to the empty set, we have $P(n,0,I)=\delta_{n,0}$. If $k>0$, $P(0,k,I)=1$ if $0 \in I$ and $P(0,k,I)=0$ if $0 \notin I$.

Theorem 2. For all $k \geq 0$, $\sum_{n\geq 0} P(n,k,I)\frac{x^n}{n!} = (\sum_{i\in I}\frac{x^i}{i!})^k$.

Proof. The case $k=0$ follows from the fact that $P(n,0,I)=\delta_{n,0}$. The case $k=1$ follows from the fact that $P(n,1)=1$ if $n \in I$ and $P(n,1)=0$ if $n \notin I$. If $k \geq 2$, then

$$(\sum_{i\in I}\frac{x^i}{i!})^k = (\sum_{i_1\in I}\frac{x^{i_1}}{i_1!})\cdots(\sum_{i_k\in I}\frac{x^{i_k}}{i_k!}) = \sum_{i_1,\ldots,i_k\in I}\frac{x^{i_1+\cdots+i_k}}{i_1!\cdots i_k!} = \sum_{n\geq 0}\Big(\sum_{\substack{i_1+\cdots+i_k=n \\ i_j\in I}}\binom{n}{i_1,\ldots,i_k}\Big)\frac{x^n}{n!} = \sum_{n\geq 0}P(n,k,I)\frac{x^n}{n!}. \quad \square$$

Application 2 .1 (exercise) Show that $P(n,k,\mathbb{N}) = k^n$. This result is predictable since the weak ordered partitions of $[n]$ with k blocks are in $1-\text{to}-1$ correspondence with the set of functions $f:[n]\to[k]$. $\square$

Application 2.2 $\sum_{n\geq 0} P(n,k,\mathbb{P})\frac{x^n}{n!} = (e^x-1)^k$.

The numbers $P(n,k,\mathbb{P})$, often denoted by $\sigma(n,k)$, count the number of ordered partitions of $[n]$ with k blocks, and hence the number of surjective functions $f:[n]\to[k]$.

Corollary 2.2.a $\quad \sigma(n,k) = \sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j} j^n$.

Proof. $\sum_{n\geq 0}\sigma(n,k)\frac{x^n}{n!} = (e^x-1)^k = \sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j} e^{jx}$.

So $\sigma(n,k) = D^n\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j}e^{jx}|_{x=0} = \sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j} j^n$. $\square$

Remark 2.2.1 $\sigma(n,0)=\delta_{n,0}$, $\sigma(0,k)=\delta_{0,k}$ if $n,k\geq 0,$ and

$$\sigma(n,k)=k\sigma(n-1,k-1)+k\sigma(n-1,k) \text{ if } n,k>0.$$

Remark 2.2.2 $x^n=\sum_{k=0}\sigma(n,k)\binom{x}{k}$.

Now suppose that $I\subseteq\mathbb{P}$, so that $P(n,k,I)=0$ if $k>n$, and we may define $P(n,I)$ by

$$P(n,I):=\sum_{k\geq 0}P(n,k,I).$$

Theorem 3. If $I\subseteq\mathbb{P}$, then $\sum_{n\geq 0}P(n,I)\frac{x^n}{n!}=\frac{1}{1-\sum_{i\in I}\frac{x^i}{i!}}$.

Proof. $\sum_{n\geq 0}P(n,I)\frac{x^n}{n!}=\sum_{n\geq 0}\sum_{k\geq 0}P(n,k,I)\frac{x^n}{n!}=\sum_{k\geq 0}(\sum_{n\geq 0}P(n,k,I)\frac{x^n}{n!})=\sum_{k\geq 0}(\sum_{i\in I}\frac{x^i}{i!})^k=\frac{1}{1-\sum_{i\in I}\frac{x^i}{i!}}$. □

$P(n,\mathbb{P})$, usually denoted simply by P_n, counts the total number of ordered partitions of $[n]$. The numbers P_n are sometimes called *horse- race numbers.* P_n also counts the number of *weak orders* of $[n]$, i.e., reflexive, transitive, complete binary relations on $[n]$.

Application 3.1 $\sum_{n\geq 0}P_n\frac{x^n}{n!}=\frac{1}{2-e^x}.$

Corollary 3.1.a $P_0=1$ and $P_n=\sum_{k=1}^{n}\binom{n}{k}P_{n-k}$ if $n\geq 1.$

Corollary 3.1.b $P_n\sim\frac{n!}{2(\ln 2)^{n+1}}$.

Corollary 3.1.c $P_n=\sum_{k\geq 0}\frac{k^n}{2^{k+1}}.$

3. Generating Functions for Partitions.

If A is any set, a *partition of A* is a set of nonempty, pairwise disjoint subsets of A, with union equal to A. The sets comprising a partition are called *blocks.* There is a 1 – to – 1 correspondence between the set of partitions of A and the set of equivalence relations on A, where the blocks are called *equivalence classes.*

Let $S(n,k) :=$ the number of partitions of $[n]$ with k blocks (= the number of distributions of balls labeled $1,\ldots,n$ among k unlabeled boxes, with no box left empty)

Clearly, $S(n,k) = \frac{1}{k!}\sigma(n,k)$. So,

(1) $S(n,0) = \delta_{n,0}$ and $S(0,k) = \delta_{0,k}$ if $n,k \geq 0$, with $S(n,k) = S(n-1,k-1) + kS(n-1,k)$ if $n,k > 0$.

(2) $S(n,k) = \frac{1}{k!}\sum_{j=0}^{k}(-1)^{k-j}\binom{k}{j} j^n$.

(3) $\sum_{n\geq 0} S(n,k)\frac{x^n}{n!} = \frac{1}{k!}(e^x-1)^k$.

Let $B_n := \sum_{k\geq 0} S(n,k)$ = the total number of partitions of / equivalence relations on $[n]$.

Theorem 4. $\sum_{n\geq 0} B_n \frac{x^n}{n!} = e^{e^x-1}$.

Corollary 4.1 $B_0 = 1$ and $B_n = \sum_{k=1}^{n}\binom{n-1}{k-1} B_{n-k}$.

Corollary 4.2 (Dobinski's formula). For all $n \geq 0$, $B_n = \frac{1}{e}\sum_{k=0}^{\infty}\frac{k^n}{k!}$.

Proof. $B_n = D^n e^{e^x-1}|_{x=0} = \frac{1}{e}D^n e^{e^x}|_{x=0} = \frac{1}{e}D^n\sum_{k=0}^{\infty}\frac{e^{kx}}{k!}|_{x=0} = \frac{1}{e}\sum_{k=0}^{\infty}\frac{k^n}{k!}$. □

Remark 4.1 It follows from Dobinski's formula that any infinite series of the form $\sum_{k=0}^{\infty} \frac{p(k)}{k!}$, where $p(k)$ is a polynomial in k, may easily be summed. For if $p(k) = a_0 + a_1 k + \cdots + a_n k^n$, then

$$\sum_{k=0}^{\infty} \frac{p(k)}{k!} = \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{j=0}^{n} a_j k^j = \sum_{j=0}^{n} a_j \sum_{k=0}^{\infty} \frac{k^j}{k!} = (\sum_{j=0}^{n} a_j B_j) e \ .$$

Remark 4.2 As in the case of ordered partitions, one can enumerate partitions subject to various limitations on their block cardinalities, and also where the blocks of a partition are equipped with various binary relations.