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Finitary bases and formal generating functions

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Abstract

Let k be a nonzero, commutative ring with 1, and let R be a k-algebra with a countably-infinite ordered free k-basis $B=\{p_n\colon n\geqslant 0\}$. We characterize and analyze those bases from which one can construct a k-algebra of 'formal B-series' of the form $f=\sum c_n p_n$, with $c_n\in k$, showing *inter alia* that many classical polynomial bases fail to have this property.

1. Introduction

Let k be a nonzero, commutative ring with 1, and let R be a k-algebra with a countably-infinite ordered free k-basis $B=\{p_n\colon n\geqslant 0\}$. Define a family of coefficients $\{{n \atop i,j}\}_B$ by

$$p_i p_j = \sum_{n\geqslant 0} \left\{ {n \atop i,j} \right\}_B p_n. \tag{1}$$

It is clear that, for all $i,j\geqslant 0$,

$$\#\left\{ n\colon \left\{ {n \atop i,j} \right\}_B \neq 0 \right\}$$

is finite. If it is also the case that, for all $n\geqslant 0$,

$$\#\left\{ (i,j)\colon \left\{ {n \atop i,j} \right\}_B \neq 0 \right\}$$

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is finite, we say that B is *finitary*. Every finitary k-basis B of R gives rise to a k-algebra of 'formal B-series'

$$f = \sum_{n\geqslant 0} c_n p_n, \quad c_n \in k, \tag{2}$$

the product of two such series being given by the formula

$$\left(\sum_{i\geqslant 0} a_i p_i\right)\left(\sum_{j\geqslant 0} b_j p_j\right) = \sum_{n\geqslant 0}\left(\sum_{i,j\geqslant 0}\left\{ {n \atop i,j} \right\}_B a_i b_j\right) p_n. \tag{3}$$

We regard f, as given by (2), as a kind of abstract generating function of the k-sequence $\{c_n\}$ relative to the k-basis $\{p_n\}$.

This paper is organized as follows. In Section 2 we establish necessary and sufficient conditions for a given basis to be finitary (Theorem 1). It follows that if k is a field and R is the polynomial ring $k[x_1,\dots,x_n]$, then any k-basis consisting of homogeneous polynomials is finitary. In particular, the bases consisting of "standard monomials" ([1,5]) fall into this category. Theorem 1 also furnishes a recipe for constructing a number of finitary k-bases of the polynomial ring $R=k[x]$. On the other hand, it enables us to show that many classical families of polynomials, including Abel polynomials, exponential polynomials, and all families of orthogonal polynomials, fail to be finitary. It is thus understandable that none of these polynomial families has ever been employed in combinatorics as a basis for generating functions (cf. [8, p. 370]). We also show (Theorem 2) that the only finitary Appell families $\{A_n\}$ of polynomials over a field k of characteristic zero are given by $A_n = c(x-u)^n$, where $c, u \in k$.

In Section 3 we give a rigorous account of the k-algebra of formal B-series defined by (2) and (3) by showing that this k-algebra is the completion of R with respect to a certain linear topology naturally associated to the basis B (Theorem 3). Any basis $B=\{p_n\colon n\geqslant 0\}$ of R, finitary or not, induces a sequence $(V_j)_{j\geqslant 0}$ of free k-submodules of R, where $V_j := \sum_{r\geqslant j} kp_r$, with $R = V_0 > V_1 > \cdots > V_j > V_{j+1} > \cdots$ and $\bigcap_{j\geqslant 0} V_j = \{0\}$. Taken as a neighborhood basis of zero, $(V_j)_{j\geqslant 0}$ induces a topology on R. The multiplication of R is uniformly continuous with respect to this topology precisely when B is finitary. When k is a field and R is the polynomial ring $k[x]$, it is in fact the case that every Hausdorff linear topology on R comes from some finitary basis. In such a case we show (Corollary 3) that there is a fairly restricted family $\mathscr{F}$ of finitary k-bases of $k[x]$ having the property that for every finitary k-basis $\{p_n\}$, there exists at least one basis $\{f_n\}$ in $\mathscr{F}$ such that the associated $\{p_n\}$-series and $\{f_n\}$-series are isomorphic. In particular, if k is algebraically closed, $\{f_n\}$ may be chosen so that $\deg f_n = n$ and $f_n | f_{n+1}$ for all n. We conclude Section 3 with a brief account of formal Dirichlet series.

In Section 4 we develop tools for a detailed analysis of the structure of finitary bases of a polynomial ring $R=k[x]$. A nondecreasing function $\sigma:\mathbb{N}\to\mathbb{N}$ is called a *bound* of the basis B if, for all quadruples (n,m,i,j) with $0\leqslant m\leqslant n$ and $\max\{i,j\}>\sigma(n)$, we have $\pi_m(p_i p_j)=0$. Such a bound clearly exists if and only if B is finitary. For finitary B the *order* λ of B is defined by $\lambda(n)=\min\{\sigma(n)$: σ is a bound of $B\}$. If $p_n|p_{n+1}$ for all n, then $\sigma(n)=n$. In Theorem 5 we give an explicit equational criterion for a function σ to be a bound of B, showing in particular that $\sigma(n)\geqslant n$. This enables us to prove that if k is an integral domain, then the function $\sigma(n)=n+1$ cannot be the order of any finitary basis of $k[x]$. We leave unresolved the larger question of whether any linear function other than the identity function can serve as the order of a finitary basis, as well as the interesting, if less well defined, problem of furnishing the order function with a salient combinatorial interpretation.

2. Finitary k-bases

In what follows $B=\{p_n\}$ is a free k-basis of R, with $\{{n \atop i,j}\}_B$ defined by (1). For $g\in R$, let $\pi_n(g)\in k$ be the p_n-component of g. In particular,

$$\pi_n(p_i p_j)=\left\{{n \atop i,j}\right\}_B.$$

Lemma 1. *For all $n,m\geqslant 0$, the following are equivalent*:
(i) $\{{n \atop i,j}\}_B=0$ *for all* $i\geqslant 0$ *and all* $j\geqslant m$.
(ii) $\pi_n(fp_j)=0$ *for all* $f\in R$ *and all* $j\geqslant m$.

Proof. Setting $f=p_i$ in (ii) yields (i). Given (i), let $f=\sum_i c_i p_i$, with $c_i\in k$. Then

$$\pi_n(fp_j)=\pi_n\left(\sum_i c_i p_i p_j\right)=\sum_i c_i\pi_n(p_i p_j)=0$$

for all $j\geqslant m$. $\square$

Lemma 2. *For all $n\geqslant 0$, the following are equivalent*:
(i) $\#\left\{(i,j)\text{: } \{{n \atop i,j}\}_B\neq 0\right\}$ *is finite.*
(ii) *There exists an* $m(=m(n))$ *such that* $\left\{{n \atop i,j}\right\}_B=0$ *for all* $i\geqslant 0$ *and all* $j\geqslant m$.

Proof. That (i) implies (ii) is obvious. That (ii) implies (i) follows from the symmetry property

$$\left\{{n \atop i,j}\right\}_B=\left\{{n \atop i,j}\right\}_B. \qquad \square$$

Our first theorem states two useful characterizations of finitary k-bases. We note first that any basis $B=\{p_n\}$ induces a sequence $(V_j)_{j\geqslant 0}$ of free k-submodules of R and a sequence $(I_j)_{j\geqslant 0}$ of ideals of R, where

$$V_j:=\sum_{r\geqslant j} kp_r \tag{4}$$

and

$$I_j := \sum_{r \geqslant j} Rp_r. \tag{5}$$

Clearly,

$$R = V_0 > V_1 > \cdots > V_j > V_{j+1} > \cdots, \tag{6}$$

with

$$\bigcap_{j \geqslant 0} V_j = \{0\}, \tag{7}$$

and

$$R = I_0 \geqslant I_1 \geqslant \cdots \geqslant I_j \geqslant I_{j+1} \geqslant \cdots, \tag{8}$$

with I_j being the ideal generated by V_j.

Theorem 1. *The following are equivalent:*
(i) *B is finitary.*
(ii) *There exists a function $\sigma : \mathbb{N} \to \mathbb{N}$ such that $I_{\sigma(t)} \subseteq V_t$ for all $t \in \mathbb{N}$.*
(iii) *There exists a descending chain $J_0 \geqslant J_1 \geqslant \cdots \geqslant J_r \geqslant J_{r+1} \geqslant \cdots$ of ideals of R, and functions $\alpha, \beta : \mathbb{N} \to \mathbb{N}$, such that $J_{\alpha(t)} \subseteq V_t$ and $V_{\beta(t)} \subseteq J_t$ for all $t \in \mathbb{N}$.*

Proof. (i)⇒(ii). Since $\{p_n\}$ is finitary, for each $n \in \mathbb{N}$ there exists a j_n such that $\{{n \atop i,j}\}_B = 0$ for all $i \geqslant 0$ and all $j \geqslant j_n$. For each $t \in \mathbb{N}$, set $\sigma(t) = \max\{j_n : 0 \leqslant n < t\}$. Then, for all $n < t$, $\{{n \atop i,j}\}_B = 0$ for all $i \geqslant 0$ and all $j \geqslant \sigma(t)$. Thus, for all $n < t$, it follows from Lemma 1 that $\pi_n(fp_j) = 0$ for all $f \in R$ and all $j \geqslant \sigma(t)$. Hence, $fp_j \in V_t$ for all $f \in R$ and all $j \geqslant \sigma(t)$, i.e., $I_{\sigma(t)} \subseteq V_t$.

(ii)⇒(iii). Set $J_r = I_r$ for all $r \in \mathbb{N}$, and let $\alpha = \sigma = \beta$. Since $V_{\sigma(t)} \subseteq I_{\sigma(t)} \subseteq V_t \subseteq I_t$, we have $J_{\alpha(t)} \subseteq V_t$ and $V_{\beta(t)} \subseteq J_t$ for all $t \in \mathbb{N}$.

(iii)⇒(i). From (iii) we shall show that, for each $n \in \mathbb{N}$, $\{{n \atop i,j}\}_B = 0$ for all $i \geqslant 0$ and all $j \geqslant \beta(\alpha(n+1))$, from which it follows by Lemma 2 that B is finitary.

We have from (iii) that $V_{\beta(\alpha(n+1))} \subseteq J_{\alpha(n+1)} \subseteq V_{n+1}$ for all $n \in \mathbb{N}$. Since $I_{\beta(\alpha(n+1))}$ is the ideal generated by the set $V_{\beta(\alpha(n+1))}$, we have $I_{\beta(\alpha(n+1))} \subseteq J_{\alpha(n+1)}$ and hence $I_{\beta(\alpha(n+1))} \subseteq V_{n+1}$ for all $n \in \mathbb{N}$. Thus, $fp_j \in V_{n+1}$ for all $f \in R$ and all $j \geqslant \beta(\alpha(n+1))$, and so $\pi_n(fp_j) = 0$ for all $f \in R$ and all $j \geqslant \beta(\alpha(n+1))$. By Lemma 1, $\{{n \atop i,j}\}_B = 0$ for all $i \geqslant 0$ and all $j \geqslant \beta(\alpha(n+1))$, as claimed. □

The following corollary of Theorem 1 enables us to construct a number of finitary k-bases of the polynomial ring $R = k[x]$.

Corollary 1.1. *Assume $R = k[x]$, where x is an indeterminate over k. Assume p_n is monic of degree n for all $n \geqslant 0$. Assume there exists a function $\mu : \mathbb{N} \to \mathbb{N}$ such that*

$$\mu(t) \geqslant t \quad \text{for all } t \geqslant 0, \tag{9}$$

$$p_{\mu(t)} | p_n \quad \text{for all } t \geqslant 0 \text{ and all } n \geqslant \mu(t). \tag{10}$$

Then the basis B is finitary.

Proof. By Theorem 1, part (ii), it suffices to show that $I_{\mu(t)} \subseteq V_t$ for all $t \geqslant 0$. By (10), we have $I_{\mu(t)} = k[x]p_{\mu(t)}$, and so for each $i \geqslant 0$ there exists an $f_i \in k[x]$ such that $p_{\mu(t)+i} = f_i p_{\mu(t)}$. Clearly, f_i is monic of degree i for each $i \geqslant 0$. Hence $\{f_i\}_{i \geqslant 0}$ is a k-basis of $k[x]$. Given any $fp_{\mu(t)} \in I_{\mu(t)}$, with $f \in k[x]$, write $f = \sum_{i \geqslant 0} c_i f_i$, with $c_i \in k$. Then $fp_{\mu(t)} = \sum_{i \geqslant 0} c_i f_i p_{\mu(t)} = \sum_{i \geqslant 0} c_i p_{\mu(t)+i}$, whence $I_{\mu(t)} \subseteq V_{\mu(t)} \subseteq V_t$, since $\mu(t) \geqslant t$. □

In particular, any family $\{p_n\}$ such that $\deg(p_n) = n$, the coefficient of x^n in p_n is a unit of k, and $p_n | p_{n+1}$ for all $n \geqslant 0$ is a finitary k-basis of $k[x]$, the simplest case of this being the obviously finitary k-basis $\{p_n = x^n\}$. If k contains the field of rational numbers, then $\{p_n = x^{\overline{n}} := x(x+1)\cdots(x+n-1)\}$, $\{p_n = x^{\underline{n}} := x(x-1)\cdots(x-n+1)\}$, and $\{p_n = \binom{x}{n} := x^{\underline{n}}/n!\}$ are finitary k-bases of $k[x]$. Generating functions of the form $\sum c_n \binom{x}{n}$, and q-generalizations thereof, have in fact been employed in the combinatorial analysis of covers [4]. Another noteworthy consequence of the above corollary is that any finite set of polynomials having mutually distinct degrees and whose leading coefficients are units of k, can be extended to a finitary k-basis of $k[x]$.

There exist finitary bases not satisfying the property mentioned in the above corollary. For example, the sequence $\{p_n\}$ where $p_0 = 1$, $p_{2m+1} = x(x^2+x+1)^{2m}$ for all $m \geqslant 0$ and $p_{2m} = (x^2+1)(x^2+x+1)^{2m-2}$ for all $m \geqslant 1$ is clearly a free k-basis of the polynomial ring $k[x]$. It is easy to verify that $I_{n+3} \subseteq V_n$ for all $n \geqslant 0$. Hence $\{p_n\}$ is finitary. Here, in fact, for each $n \geqslant 1$, p_{n+1} fails to be divisible by p_n.

A second corollary of Theorem 1 states a simple necessary condition for $\{p_n\} \subset R = k[x]$ to be finitary.

Corollary 1.2. *Assume $R = k[x]$, where x is an indeterminate over k. Let $\{p_n\}$ be a free k-basis of $k[x]$, with $(I_t)_{t \geqslant 0}$ the associated chain of ideals defined by* (5). *Let d_t be the minimum of the degrees of nonzero elements of I_t. If $\{p_n\}$ is finitary, then* $\lim_{t \to \infty} d_t = \infty$.

Proof. Let δ_t be the minimum of the degrees of nonzero elements of V_t. Clearly $\delta_t \leqslant \delta_{t+1}$ for all $t \in \mathbb{N}$. For $\{p_n\}$ finitary there exists by Theorem 1 a function $\sigma : \mathbb{N} \to \mathbb{N}$ such that $I_{\sigma(t)} \subseteq V_t$ for all $t \in \mathbb{N}$. Hence $d_{\sigma(t)} \geqslant \delta_t$. It suffices to show that the sequence $\{\delta_t\}$ is not bounded above.

For each t let $W_t = \sum_{0 \leqslant i \leqslant t} kp_i$ and $R_t = \sum_{0 \leqslant i \leqslant t} kx^i$. Given a positive integer m there exists a positive integer $s(m)$ such that $R_m \subseteq W_{s(m)-1}$. Since $\{p_n\}$ is a free k-basis, $W_{s(m)-1} \cap V_{s(m)} = 0$. Therefore, $\delta_{s(m)} > m$. $\square$

If k is a field, it follows from Corollary 1.2 that no family of polynomials satisfying $\deg p_i = i$ for all $i \geqslant 0$ and

$$p_{n+1} = (a_n + xb_n)p_n - c_n p_{n-1} \quad \text{for all } n \geqslant 1, \tag{11}$$

with a_n, b_n, c_n in k, is finitary since $\gcd\{p_t, p_{t+1}\} = 1$, and so $d_t = 0$ for all $t \in \mathbb{N}$. In particular, no family of orthogonal polynomials [2, p. 773] can be finitary.

The Abel polynomials $\{q_n\}$, defined by $q_0 = 1$ and $q_n = x(x+n)^{n-1}$ for $n \geqslant 1$, with $d_t = 1$ for all $t \geqslant 1$, also fail to be finitary.

As for the exponential polynomials $\{e_n\}$, (here k is the field $\mathbb{C}$ of complex numbers) defined by

$$e_n = e_n(x) = \sum_{k=0}^{n} S(n,k)x^k, \tag{12}$$

with $S(n,k)$ the Stirling number of the second kind [7, p. 42], it may be shown that

$$\sum_{n \geqslant 0} e_n(x)\frac{z^n}{n!} = \mathrm{e}^{x(\mathrm{e}^z - 1)}. \tag{13}$$

Let g_t denote the monic generator of I_t. Suppose that r is a root in $\mathbb{C}$ of g_t for some $t \geqslant 1$. Setting $x = r$ in (13), we have $e_n(r) = 0$ for all $n \geqslant t$, whence $\mathrm{e}^{r(\mathrm{e}^z - 1)}$ is a polynomial in z. But $\mathrm{e}^{r(\mathrm{e}^z - 1)} = 1$ for the infinitely many $z = 2\pi m_i$, $m \in \mathbb{Z}$. Thus (13) is the constant polynomial 1, and so in fact $e_n(r) = 0$ for all $n \geqslant 1$. Since $e_1(x) = x$, we have $r = 0$. Hence g_t is a power of x for all $t \geqslant 1$. That $g_t = x$ for all $t \geqslant 1$ follows from the fact that $S(n,0) = 0$ and $S(n,1) = 1$ for all $n \geqslant 1$. So the family of exponential polynomials also fails to be finitary.

For k of characteristic zero, a family $\{A_n\}$ in $k[x]$ is called an *Appell family* [7, p. 59] if A_0 is a nonzero constant and $D_x A_n = nA_{n-1}$ for all $n \geqslant 1$. It is easy to show that $\{A_n\}$ is Appell if and only if there exists a sequence $(a_n)_{n \geqslant 0}$ in k with $a_0 \neq 0$ and

$$A_n = \sum_{k=0}^{n} \binom{n}{k} a_{n-k} x^k \quad \text{for all } n \geqslant 0. \tag{14}$$

The following theorem characterizes finitary k-bases within the class of Appell families.

Theorem 2. *Assume k is a field of characteristic 0. An Appell family $\{A_n\} \subset k[x]$ is finitary if and only if there exist $u, c \in k$ with $c \neq 0$ such that $A_n = c(x-u)^n$ for all $n \geqslant 0$.*

Proof. Assume A_n is a finitary Appell family. Let I_t be the associated chain of ideals and let g_t denote the monic generator of I_t. From Corollary 1.2 it follows that for some positive integer m we have $\deg g_m \geqslant 1$. Fix such an integer m. Let f be an irreducible factor of g_m. Write $A_j = f^{s(j)} B_j$ with B_j relatively prime to f. Note that $s(j) \geqslant 1$ for all $j \geqslant m$. Since $\operatorname{char} k = 0$ the equation

$$\begin{aligned} s(j+1) f^{s(j+1)-1}(D_x f) B_{j+1} + f^{s(j+1)}(D_x B_{j+1}) = D_x A_{j+1} &= (j+1) A_j \\ &= (j+1) f^{s(j)} B_j \end{aligned}$$

forces $s(j+1) = s(j) + 1$ for all $j \geqslant m$.

Let $g_m = f_1^{e(1)} \cdots f_r^{e(r)}$ with $e(i) \geqslant 1$ and f_i monic, irreducible for $1 \leqslant i \leqslant r$. Then the above argument allows us to write

$$A_j = f_1^{e(1)+j-m} \cdots f_r^{e(r)+j-m} H_j$$

with H_j relatively prime to g_m, for all $j \geqslant m$. Letting d_i be the degree of f_i for $1 \leqslant i \leqslant r$, we get

$$0 \leqslant \deg H_j = \deg H_m - (j-m)(d_1 + \cdots + d_r - 1)$$

for all $j \geqslant m$. This equation can be valid for all $j \geqslant m$ only if $d_1 + \cdots + d_r = 1$, i.e., $r = 1 = d_1$. Hence we have $A_j = (x-u)^{e+j-m} H_j$, where $e = e(1) > 0$, for all $j \geqslant m$.

Let μ denote a positive integer such that $I_\mu \subseteq V_m$. Clearly $\mu \geqslant m$. Let n denote the greatest integer such that $I_\mu \subseteq V_n$. Then, we must have $m \leqslant n \leqslant \mu$. If $\mu > n$ then, there exists an $h \in I_\mu \backslash V_{n+1}$. Clearly $h = aA_n + \psi$ for some $0 \neq a \in k$ and $\psi \in V_{n+1}$. Observe that both h, ψ are divisible by $(x-u)^{n+1-m+e}$. This is absurd since A_n is not divisible by $(x-u)^{n+1-m+e}$. Consequently, $n = \mu$, i.e., $I_\mu = V_\mu$. Now we can find $a_1, a_2 \in k$ such that $(x-u)A_\mu = a_1 A_\mu + a_2 A_{\mu+1}$. Dividing by the common factor $(x-u)^{e+\mu-m}$ we obtain

$$(x-u) H_\mu = a_1 H_\mu + a_2 (x-u) H_{\mu+1}.$$

Since $(x-u)$ and H_μ are coprime, $a_1 = 0$, i.e., $H_\mu = a_2 H_{\mu+1}$. The Appell condition $D_x A_{\mu+1} = (\mu+1) A_\mu$ leads to the equation

$$(e + \mu + 1 - m) H_{\mu+1} + (x-u)(D_x H_{\mu+1}) = (\mu+1) H_\mu = a_2 (\mu+1) H_{\mu+1}.$$

Once again the coprimality of $(x-u)$ and $H_{\mu+1}$ allows us to deduce $D_x H_{\mu+1} = 0$. Since $\operatorname{char} k = 0$, $H_{\mu+1}$ must be a constant. Thus $A_{\mu+1} = c(x-u)^{e+\mu+1-m}$ with $c \in k$. Since $A_{\mu+1}$ is a nonzero polynomial of degree $\mu+1$, we have $e = m$ and $c \neq 0$. Using integration and the fact that $(x-u)$ divides A_j for all $j \geqslant \mu+1$ we see that $A_j = c(x-u)^j$ for all $j \geqslant \mu+1$. Differentiation yields $A_j = c(x-u)^j$ for all $0 \leqslant j \leqslant \mu$. □

3. Completions

In this section we identify the formal series ring associated to a finitary basis B as the completion of R with respect to a linear topology associated to B. Our reference for the basics of topological rings and their completions is [3].

In the following we assume $B=\{p_n\}$ to be finitary and fix a descending chain $J:=\{J_r\}$ of ideals of R, with $J_0=R$, satisfying (iii) of Theorem 1, i.e., there are functions $\alpha,\beta:\mathbb{N}\to\mathbb{N}$, such that $J_{\alpha(t)}\subseteq V_t$ and $V_{\beta(t)}\subseteq J_t$ for all $t\in\mathbb{N}$. Since $\bigcap V_t=0$, we have $\bigcap J_t=0$ and hence the linear topology on R corresponding to the filtration J is a Hausdorff topology. Let S denote the completion of R with respect to this topology.

Let $\{f_i\}_{i\in\mathbb{N}}$ be a sequence of elements of R. If there exists a map $v:\mathbb{N}\to\mathbb{N}$ such that $f_{i+1}-f_i\in J_m$ for all $i\geqslant v(m)$ and all $m\in\mathbb{N}$ then, $\{f_i\}$ is a cauchy-sequence. Moreover, if $f_i\in J_m$ for all $i\geqslant n$ then, $\{f_i\}$ is a null-sequence. Two cauchy-sequences $\{f_i\}$ and $\{g_i\}$ are equivalent if $\{f_i-g_i\}$ is a null-sequence. Our k-algebra S can be thought of as the set of equivalence classes of cauchy-sequences with componentwise addition and multiplication.

Let $W_t=\sum_{0\leqslant i\leqslant t}kp_i$ and let $\pi_t(f)$ denote the p_t-component of f. By a *standard-sequence* we mean a sequence $\{g_i\}$ of elements of R satisfying

(i) $g_t\in W_t$ for all $t\geqslant 0$, and

(ii) $g_{t+1}-g_t\in V_{t+1}$ for all $t\geqslant 0$.

It follows from the condition (ii) that $g_j-g_i\in V_{t+1}$ for all $j>i\geqslant t$. In particular, $\{g_i\}$ is a Cauchy-sequence. If $\{g_i\}$ and $\{h_i\}$ are standard sequences then so is their sum $\{g_i+h_i\}$. A sequence $\{g_t\}$ is standard if and only if there exists a sequence $\{c_i\}$ of elements of k such that $g_t=\sum_{0\leqslant i\leqslant t}c_ip_i$ for all $t\geqslant 0$.

Lemma 3. *Every Cauchy sequence $\{f_i\}$ is equivalent to a unique standard sequence.*

Proof. Suppose two standard-sequences $\{g_i\}$ and $\{h_i\}$ are equivalent, i.e., $g_i-h_i\in J_m$ for all $i\geqslant v(m)$ and all $m\geqslant 0$. Then $g_i-h_i\in V_m$ for all $i\geqslant v(\alpha(m))$ and all $m\geqslant 0$. Writing $g_t-h_t=(g_r-h_r)-(g_r-g_t)+(h_r-h_t)$, where $r=\max\{t+1,v(\alpha(t+1))\}$, we observe that the three paranthetical terms on the right are in V_{t+1}. Hence $g_t-h_t\in V_{t+1}$. Since $g_t,h_t\in W_t$ we must have $g_t-h_t\in W_t\cap V_{t+1}=0$, i.e., $g_t=h_t$ for all $t\geqslant 0$. This establishes uniqueness. To prove the existence consider a cauchy sequence $\{f_i\}$. Replacing $\{f_i\}$ by a suitable subsequence, if necessary, we may assume that $f_{i+1}-f_i\in V_{i+1}$ for all $i\geqslant 0$. Then, we have $\pi_i(f_t)=\pi_i(f_i)$ for $0\leqslant i\leqslant t$. Define $\{g_t\}$ inductively by setting $g_0=\pi_0(f_0)p_0$ and $g_{i+1}=g_i+\pi_{i+1}(f_{i+1})p_{i+1}$ for all $i\geqslant 0$. Clearly, $\{g_t\}$ is a standard-sequence equivalent to $\{f_i\}$. □

Theorem 3. *The k-algebra of formal B-series is k-isomorphic to S.*

Proof. Map a formal B-series $f:=\sum a_ip_i$ to the equivalence class of the standard-sequence $\{f_t\}$ where $f_t=\sum_{0\leqslant i\leqslant t}a_ip_i$ for all $t\geqslant 0$. By the above lemma this map is one–one, onto. The map is clearly k-linear. Let $g:=\sum b_ip_i$ be formal B-series and say $fg:=\sum c_ip_i$. Consider the associated standard-sequences $\{f_i\}$, $\{g_i\}$, and $\{(fg)_i\}$. Since B is finitary, for every $m\geqslant 0$ there is an integer $\tau(m)$ such that $\pi_t(f_ig_i)=c_t$ for all $i\geqslant\tau(m)$ and $0\leqslant t\leqslant m-1$, i.e., $(fg)_i-f_ig_i\in V_m$ for all $i\geqslant\tau(m)$. Thus our map preserves the multiplication. □

Corollary 3. *Assume $R = k[x]$ where k is a field and x is an indeterminate over k. Given a finitary k-basis $\{q_n\}$ of R there exists a finitary k-basis $\{f_n\}$ of R and a subsequence $\{f_{\mu(n)}\}$ satisfying the following three conditions:*
(i) *f_n is monic of degree n for all $n \geqslant 0$.*
(ii) *$\mu(0) = 0$, $\mu(n) \geqslant n$, $f_{\mu(n)} | f_{\mu(n+1)}$, and $f_{\mu(n+1)}/f_{\mu(n)}$ is irreducible (in R) for all $n \geqslant 0$.*
(iii) *The algebra of formal $\{q_n\}$-series is k-isomorphic to the algebra of formal $\{f_n\}$-series.*

Proof. Let $\{I_n\}$ be the sequence of ideals associated with the basis $\{q_n\}$. Let $J := \{J_n\}$ be a (strictly) descending chain of ideals of R such that $J_0 = R$, J_i/J_{i+1} is a nonzero simple R-module for all $i \geqslant 0$, and for each $n \geqslant 0$ we have $I_n = J_r$ for some $r \geqslant 0$. Since $\{q_n\}$ is assumed to be finitary J satisfies the requirement (iii) of Theorem 1. Let g_t denote the monic generator of J_t for all $t \geqslant 0$. Then, $g_0 = 1$, $t \leqslant \deg g_t \leqslant \deg g_{t+1}$ and $g_t | g_{t+1}$ for all $t \geqslant 0$. Also, the quotient g_{t+1}/g_t is irreducible for all $t \geqslant 0$. Define $\mu(t) := \deg g_t$ and let $f_n := x^{n-\mu(t)} g_t$ where t is uniquely determined by $\mu(t) \geqslant n < \mu(t+1)$. Clearly $\{f_n\}$ satisfies the conditions (i) and (ii) above. Consequently, Corollary 1.1 implies that $\{f_n\}$ is finitary. To establish (iii), apply Theorem 3 to the pairs $\{f_n\}, J$. and $\{q_n\}, J$. □

When $R = k[x]$, with x an indeterminate over k, and $\{p_n = x^n\}$, Theorem 3 yields a standard construction of the algebra $k[\![x]\!]$ of formal power series over k. (See [6] for an elegant elementary alternative to this construction.) This theorem also yields the following construction of formal Dirichlet series. Let S be any set of real numbers that is not bounded above, and let A be the $\mathbb{R}$-algebra $\mathbb{R}^S$, with all operations taken pointwise. For all $n \in \mathbb{P}$ and all $s \in S$, let $f_n(s) := n^{-s}$. Since $f_i f_j = f_{ij}$, the subalgebra R of A generated by $B := \{f_n : n \geqslant 1\}$ consists of all finite linear combinations of elements of B. It is straight-forward to show that B is linearly independent, and hence a free $\mathbb{R}$-basis of R. Also, B is finitary since, clearly,

$$\left\{ \begin{matrix} n \\ i,j \end{matrix} \right\} = \begin{cases} 1 & \text{if } ij = n, \\ 0 & \text{otherwise.} \end{cases}$$

By Theorem 3, series of the form $\sum c_n f_n$ are well-defined, and for all sequences (a_i) and (b_j) in $\mathbb{R}$,

$$\left(\sum_{i \geqslant 1} a_i f_i \right) \left(\sum_{j \geqslant 1} b_j f_j \right) = \sum_{n \geqslant 1} \left(\sum_{ij=n} a_i b_j \right) f_n.$$

4. Bounds and equations

Throughout the remaining we assume $R = k[x]$ where x is an indeterminate over the ring k. Let $B := \{p_n\}$ be a free k-basis of R. A nondecreasing function $\sigma : \mathbb{N} \to \mathbb{N}$ is called a *bound* of B if for all quadruples (n,m,i,j) with $0 \leqslant m \leqslant n$ and $\max\{i,j\} > \sigma(n)$, we have $\pi_m(p_i p_j) = 0$. Using the notation of Theorem 1, σ is a bound if and only if $I_{\sigma(n)+1} \subseteq V_{n+1}$ for all n. Obviously, such a bound exists if and only if B is finitary. For a finitary B we define the *order* of B to be the least bound λ of B, i.e. $\lambda(n) := \min\{\sigma(n)\}$ where the minimum is taken over all bounds σ of B. If σ is a bound of B, then necessarily $\sigma(n) \geqslant n$ for all n (this is a corollary of the following lemma).

Lemma 4. *Let $\{p_n\}$ be a free k-basis of R and let $T : \mathbb{N} \to \mathbb{N}$ be a function. Then the following are equivalent.*

(i) $\pi_n(p_i p_j) = 0$ *for all (i,j) such that* $\max\{i,j\} > T(n)$.

(ii) $\pi_n(x^d p_i) = 0$ *for all (d,i) such that $d \geqslant 0$ and $i > T(n)$.*

Proof. Obvious. □

It follows from the above lemma that whether a k-free basis B of R is finitary or not is completely determined by the matrix of the (k-linear) 'multiplication by x' map with respect to the basis B. Below we make this notion more explicit and obtain an equational criterion for specific bounds. We begin with some observations about matrices.

Let $\bar{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ and $(r,s) \in \bar{\mathbb{N}} \times \bar{\mathbb{N}}$. Let $\mathbb{M}(r,s;k)$ denote the set of $r \times s$ matrices with entries in k such that each row contains only finitely many non-zero entries. By a *k-matrix* we mean an element of $\mathbb{M}(r,s;k)$ for some $(r,s) \in \bar{\mathbb{N}} \times \bar{\mathbb{N}}$. By a square k-matrix we mean an element of $\mathbb{M}(r,r;k)$ for some $r \in \bar{\mathbb{N}}$. Note that if $A \in \mathbb{M}(r,s;k)$ and $D \in \mathbb{M}(s,t;k)$ then the product AD is well-defined and is in $\mathbb{M}(r,t;k)$. Clearly, a submatrix of a k-matrix is itself a k-matrix.

Consider a 3×3 block-matrix

$$A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix},$$

where each A_{ij} is a k-matrix and the diagonal blocks A_{ii} are square k-matrices. Then A itself is a square k-matrix. For a non-negative integer d we write A^d as a 3×3 block-matrix with blocks $A_{ij}^{(d)}$ where the size of each $A_{ij}^{(d)}$ is the same as that of the A_{ij}.

Lemma 5. *Assume d to be a positive integer and $A_{31}^{(r)} = 0$ for $0 \leqslant r \leqslant d$. Then we have*

$$A_{31}^{(d+1)} = \sum_{1 \leqslant i \leqslant d} (A_{32} A_{22}^{i-1} A_{21}) A_{11}^{(d-i)}.$$

Proof. Under the assumption $A_{31}^{(r)} = 0$ for $0 \leqslant r \leqslant d-1$ it is easy to establish (by induction)

$$A_{21}^{(r)} = \sum_{1 \leqslant i \leqslant r} A_{22}^{i-1} A_{21} A_{11}^{(r-i)},$$

for $0 \leqslant r \leqslant d$. Using $A_{31} = 0 = A_{31}^{(d)}$ we get $A_{31}^{(d+1)} = A_{32}A_{21}^{(d)}$. Now our assertion follows by substitution. □

Theorem 4. *For $m \in \mathbb{N}$ the following statements $E(m)$ and $F(m)$ are equivalent.*
E(m) : $A_{31}^{(d)} = 0$ *for* $0 \leqslant d \leqslant m+1$.
F(m) : $A_{31} = 0$ *and* $A_{32}A_{22}^{d}A_{21} = 0$ *for* $0 \leqslant d \leqslant m-1$.

Proof. The proof is by induction on m. Clearly $E(0)$ is equivalent to $F(0)$. Henceforth assume $1 \leqslant r$ and $E(i)$ is equivalent to $F(i)$ for $0 \leqslant i \leqslant r-1$. Note that if $E(r)$ holds then, a priori, $E(r-1)$ holds and hence by the induction hypothesis $F(r-1)$ also holds. Likewise, $F(r)$ implies both $F(r-1)$, $E(r-1)$. Now $E(r-1)$ and $F(r-1)$ together, in view of the Lemma 5, yield $A_{31}^{(r+1)} = A_{32}A_{22}^{r-1}A_{21}$. Thus $E(r)$, $F(r)$ are equivalent. □

Corollary 4. *Suppose $A_{22} \in \mathbb{M}(s+1, s+1; k)$ for some $s \in \mathbb{N}$. Then the following are equivalent*:
(i) $A_{31}^{(d)} = 0$ *for* $d \geqslant 0$.
(ii) $A_{31} = 0$ *and* $A_{32}A_{22}^{d}A_{21} = 0$ *for* $0 \leqslant d \leqslant s$.

Proof. If $A_{32}A_{22}^{d}A_{21} = 0$ for $0 \leqslant d \leqslant s$ then, the Cayley–Hamilton theorem implies $A_{32}A_{22}^{d}A_{21} = 0$ for all $d \geqslant 0$. Now the asserted equivalence follows from the above theorem. □

If $A := [a_{ij}] \in \mathbb{M}(r, r; k)$ and if $m < r$, $n < r$ are non-negative integers then, we define $A\langle m, n\rangle$ to be the submatrix $[a_{ij}]$ where $m < i$ and $j \leqslant n$.

Let $B = \{p_n\}$ be an ordered free k-basis of R as before. We will think of B as a column, p_i being its ith row. For $(i, j) \in \mathbb{N} \times \mathbb{N}$ let $u_{ij} := \pi_j(xp_i)$. By $M(x, B)$ we denote the square k-matrix $[u_{ij}]$. Then $x^d B = M(x, B)^d B$ for $d \geqslant 0$.

Lemma 6. *Let $M := M(x, B)$ and let $\sigma : \mathbb{N} \to \mathbb{N}$ be a non-decreasing function. Then the following are equivalent*:
(i) *σ is a bound of B.*
(ii) $M^d\langle\sigma(n), n\rangle = 0$ *for* $(n, d) \in \mathbb{N} \times \mathbb{N}$.
(iii) $\sigma(n) \geqslant n$ *for* $n \in \mathbb{N}$, $M\langle\sigma(n), n\rangle = 0$ *for* $n \in \mathbb{N}$ *and* $M^d\langle\sigma(n), n\rangle = 0$ *for all* $(n, d) \in \mathbb{N} \times \mathbb{N}$ *with* $\sigma(n) \geqslant n+1$.

Proof. In view of Lemma 4 (i) holds if and only if $\pi_j(x^d p_i) = 0$ for all (d, n, i, j) with $0 \leqslant d, 0 \leqslant n, 0 \leqslant j \leqslant n$ and $\sigma(n) < i$. The equivalence of (i) and (ii) is now a matter of unwinding the definitions.

Observe that since M^0 is the identity matrix, $M^0\langle\sigma(n), n\rangle = 0$ implies $\sigma(n) \geqslant n$. If $\sigma(n) = n$ then, M can be expressed as a 2×2 block matrix such that the diagonal blocks are square and the bottom-left-block is $M\langle\sigma(n), n\rangle$. Moreover, assuming $M\langle\sigma(n), n\rangle = 0$ it follows that all powers of M are block-upper-triangular, i.e., $M^d\langle\sigma(n), n\rangle = M^d\langle n, n\rangle = 0$ for all $d \geqslant 0$. Thus (ii) and (iii) are equivalent. □

Let $M := M(x,B)$ as above. Fix a a : $\mathbb{N} \to \mathbb{N}$ and an $n \in \mathbb{N}$ such that $\sigma(n) \geqslant n + 1$. Write M as a 3 x 3 block-matrix $[M_{ij}(\sigma, n)]$ in such a way that $M_{31}(\sigma,n) = M\langle\sigma(n),n\rangle$ and the diagonal blocks $M_{ii}(\sigma,n)$ are square matrices. Then, using the notation of Lemma 5, we have $M^d\langle\sigma(n),n\rangle = M_{31}(\sigma,n)^{(d)}$. Note that $M_{22}(\sigma,n)$ is a (a(n) $-n$) x $(a(n)-n)$ matrix. By Corollory 4, $M^d\langle\sigma(n),n\rangle = 0$ for all $d \geqslant 0$ if and only if

$$\begin{aligned} &M\langle\sigma(n),n\rangle = 0,\\ &M_{32}(\sigma,n)M_{22}(\sigma,n)^d M_{21}(\sigma,n) = 0 \quad \text{for } 0 \leqslant d \leqslant \sigma(n) - n - 1. \end{aligned} \tag{$*$}$$

Combining $(*)$ with Lemma 6 we get the following.

Theorem 5. ***Under the assumptions of Lemma 6 the following are equivalent:***
(i) a ***is a bound of B.***
(ii) $\sigma(n) \geqslant n$ ***for*** $n \in \mathbb{N}$, ***and the condition*** $(*)$ ***holds for all n such that*** $\sigma(n) \geqslant n + 1$.

Proof. Straightforward. □

We proceed to extract the set of equations satisfied by the rth row of M. Define

$$S_r(\sigma) := \{i \in \mathbb{N} \mid i + 2 \leqslant \sigma(i) + 1 \leqslant r\}.$$

For an $i \in S_r(\sigma)$ define

$$\rho(r, i, a) := [u_{rj}] \quad \text{where } i + 1 \leqslant j \leqslant \text{a(i)}.$$

Observe that $\rho(r,i,\sigma)$ appears as a row of $M_{32}(\sigma,i)$ for each i in S,(a). So, the equivalence in Theorem 5 can be reformulated as: a is a bound of B if and only if the equations

$$L(a):\quad \begin{cases} u_{ri} = 0 \quad \text{whenever } r > a(i),\\ \rho(r,i,\sigma)M_{22}(\sigma,i)^d M_{21}(\sigma,i) = 0 \ \text{ for all } \ i \in S_r(\sigma)\\ \quad \text{and } 0 \leqslant d \leqslant \sigma(i) - i - 1 \end{cases}$$

are satisfied for all $r \in \mathbb{N}$. For $i \in S_r(\sigma)$ the matrices $M_{22}(\sigma, i)$ and $M_{21}(\sigma, i)$ involve only the rows prior to the rth row. Hence $L,(a)$ is a system of homogeneous linear equations in the entries of the rth row. Further, observe that the entries u_{rj} with $j \geqslant r$ ***are*** not involved in $L_r(\sigma)$.

If $u_{ri} = 0$ for all i with $r >$ a(i) then, $\rho(r, i, a) = 0$ for all i with $r > a(a(i))$. It follows that L,(a) is equivalent to

$$\text{L:(a):}\quad \begin{cases} u_{ri} = 0 \quad \text{whenever } r > \text{a(i)},\\ \rho(r,i,\sigma)M_{22}(\sigma,i)^d M_{21}(\sigma,i) = 0 \ \text{ for all } \ i \in S_r^*(\sigma)\\ \quad \text{and } 0 \leqslant d \leqslant \sigma(i) - i - 1, \end{cases}$$

where $S_r^*(\sigma) := \{i \in \mathbb{N} \mid i + 2 \leqslant \sigma(i) + 1 \leqslant r \leqslant \sigma(\sigma(i))\}$.

Examples. 1. Suppose $\sigma(n)=n$ for all $n\in\mathbb{N}$. In this case $S_r(\sigma)$ is empty for all r and $L_r(\sigma)$ reduces to: $u_{ri}=0$ for all $i<r$. Thus the identity function is a bound (and hence the order) of B if and only if $M(x,B)$ is upper-triangular.

2. Assume $\sigma(n)=n+1$ for all $n\in\mathbb{N}$. It is easy to verify that $L_r^*(\sigma)$ reduces to

$$u_{ri}=0 \quad \text{for all } i\leqslant r-2,$$
$$u_{r(r-1)}u_{(r-1)(r-2)}=0$$

for all $r\geqslant 2$. Assuming σ to be a bound of B we get $u_{20}=u_{21}u_{10}=0$. If k is an integral domain then, we must have either $u_{21}=0$ or $u_{10}=0$. If $u_{21}=0$ then, the function

$$\alpha(n)=\begin{cases}1 & \text{if } n=1,\\ n+1 & \text{otherwise}\end{cases}$$

is a bound of B. If $u_{10}=0$ then,

$$\beta(n)=\begin{cases}0 & \text{if } n=0,\\ n+1 & \text{otherwise}\end{cases}$$

is a bound of B. Thus σ can not be the order of any finitary basis of $k[x]$ in case k is an integral domain.

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