Public Lecture

University of the Cumberlands

April 16, 2012

WHEN "COMMON SENSE" FAILS: SOME PARADOXES OF PROBABILITY,STATISTICS, AND MAJORITY RULE

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PARADOX : AN APPARENT CONTRADICTION, i.e., A STATE OF AFFAIRS THAT SEEMS IMPOSSIBLE, BUT CAN ACTUALLY OCCUR.

PARADOXES ARE SAID TO BE:

“COUNTER-INTUITIVE,”

“NON-INTUITIVE,”

“CONTRARY TO COMMON SENSE”

1. THE BERKELEY ADMISSIONS CASE

- Acceptance rate for female applicants to graduate school at the University of California at Berkeley =

  # of female acceptees /

  # of female applicants = $P(A|F)$

- # of male acceptees /

  # of male applicants = $P(A|M)$

- In the early 70's, the Berkeley administration noticed that

$$P(A|F) < P(A|M),$$

and they commissioned a study to answer the following question:

"In which departments is the acceptance rate for female applicants less than the acceptance rate for male applicants?"

ANSWER: NONE

Admissions data for entire university:

	A	R	
F	210	390	600
M	275	325	600
	485	715	1200

$P(A) =$ acc. rate $= 485/1200 \approx 0.40$

$P(A|F) = 210/600 = 0.35$

$P(A|M) = 275/600 \approx 0.46$

$$P(A|F) < P(A) < P(A|M)$$

Imagine two departments, humanities (H) and sciences (S)

	A	R	
F	150	350	500
M	25	75	100
	175	425	600

(humanities data)

	A	R	
F	60	40	100
M	250	250	500
	310	290	600

(sciences data)

1. $P_H(A|F) = 0.30 > 0.25 = P_H(A|M)$
2. $P_S(A|F) = 0.60 > 0.50 = P_S(A|M)$

Explanation:

$P_H(A) = 175/600 \approx 0.27$

$P_S(A) = 310/600 \approx 0.62$

5/6 of females apply to humanities;

5/6 males apply to sciences !

The Berkeley admissions case is an example of SIMPSON'S PARADOX.

Here is another example of Simpson's Paradox, the BATTING AVERAGE PARADOX:

	Player x			Player y	
	At-bats	Hits	Av	At-bats	Hits	Av
1.	300	75	.250	100	24	.240
2.	120	36	.300	300	87	.290
3.	300	72	.240	100	23	.230
4.	100	31	.310	300	90	.300
5.	200	52	.260	220	55	.250
	1020	266	.261	1020	279	.274

There are many other examples, e.g., testing a drug against a placebo in several hospitals. It is possible that the drug has a higher cure rate in each hospital, but lower overall cure rate when the data from all hospitals are merged. (Assignment for math majors: Show that this situation can be avoided if we give the drug to the same percentage of subjects in each hospital.)

2. THE ACCURACY OF POLLS

Population of TN: 6.4 million

Population of CA: 37.7 million

Question: A newspaper reports that in a random sample of 400 Tennessee voters, x % support a particular presidential candidate, with a "margin of error" of 5 percentage points. How many voters in California would need to be sampled on this issue to produce the same margin of error?

Surprisingly, just 400!

THE POLLING PARADOX

The accuracy of a poll depends only on the size of the sample, not on the fraction of the population sampled.

The square root rule: If n individuals are sampled, the margin of error is equal to 1 / $\sqrt{n}$, converted to a percentage. If this rule of thumb for calculating the margin of error is used many times, the true value of the percentage being estimated will be within the margin of error about 95% of the time.

Question: How many people need to be polled to produce a margin of error of 1 percentage point?

3. CAN MOST STUDENTS BE ABOVE AVERAGE?

In Garrison Keillor's mythical Lake Woebegone, all the women are strong, all the men are good-looking, and all the children are above average.

Could the latter ever be the case?

Suppose the scores of n students on a test are $x_1, \ldots, x_n$. There are two numbers commonly referred to as the "average" of these scores:

(1) the mean $\mu = (x_1 + \ldots + x_n)/n$

(2) the median m = a number for which the number of scores greater than m is equal to the number of scores less than m.

• If "above" means <u>strictly larger than</u>, then, it is impossible for everyone to be above the mean or above the median. But can almost everyone be above average? Not if "average" means "median." But yes, if "average" means "mean." Example:

Scores: 10, 20, 80, 80, 80, 90, 90, 90, 100, 100. Mean score = 74. So 80% of the individuals are above average!

4. WHERE DO I STAND?

Suppose that on a certain test:

1. My score is 90 out of 100.
2. The mean score is 60.

I am pretty pleased with myself, and think that my class standing must be pretty high, i.e., that only a small percentage of the class could have done at least as well as I did. Is this necessarily true?

ANSWER: NO

FACT: As many as two-thirds of the students in the class might have done at least as well on the test as I did !

EXAMPLE: 3 students score 0, 90, and 90. Mean score = 60. My score =90. Only one-third of the students attain a lower score than I do!

MARKOV'S INEQUALITY:

If the mean of a list of numbers $x_1,\ldots,x_n$ is equal to μ and s is any number at least as large as μ, then the fraction of numbers in the list that are greater than or equal to s can be as large as μ/s !

$$\frac{\#\text{ of scores } \geq s}{\text{total }\#\text{ of test-takers}}$$

is always $\leq \mu/s$.

MAJORITY RULE IS REGARDED AS THE ESSENCE OF DEMOCRATIC DECISION-MAKING.

BUT THERE ARE SEVERAL PARADOXES ASSOCIATED WITH THIS APPARENTLY IMPECCABLE METHOD OF RESOLVING DISAGREEMENT

HERE ARE TWO, CONDORCET'S PARADOX AND ANSCOMBE'S PARADOX:

5. WHEN MAJORITY RULE FAILS

- Voters 1, 2, and 3 rank alternatives a, b, and c as follows:

1	2	3
a	b	c
b	c	a
c	a	b

If we use majority rule to get a group ranking, we would

1. rank a over b (a beats b for 1 & 3)
2. rank b over c (b beats c for 1 & 2)
3. rank c over a! (c beats a for 2 & 3)

This is CONDORCET'S PARADOX.

6. WHEN A MAJORITY OF VOTERS ARE DISSATISFIED WITH A MAJORITY OF OUTCOMES DETERMINED BY MAJORITY RULE

Proposals

VOTERS		A	B	C
	* 1	yes	yes	no
	* 2	no	no	no
	* 3	no	yes	yes
	4	yes	no	yes
	5	yes	no	yes
Decision:		yes	no	yes

This is ANSCOMBE'S PARADOX

“EDUCATION IS THE PATH
FROM COCKY
IGNORANCE
TO MISERABLE
UNCERTAINTY”

Mark Twain