CONVERSATION ON CONSENSUS

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What does haunt late works are the author’s previous works: he is burdensomely aware that he has been cast, unlike his ingénue self, as an author who writes in a certain way, with the inexorable consistency of his own handwriting.

John Updike, in the essay “Late Works,” in the book, Due Considerations

My ingénue enthusiasms:

1. Axiomatic characterizations of methods of aggregating individual opinion, with *universal domain conditions.*

2. Impossibility (i.e., Arrow-type) theorems.

Example: ALLOCATION AGGREGATION

Suppose that each of n individuals is asked to assess the most appropriate values of some set of numerical decision variables $x_1,\ldots,x_m$. Values are constrained to be nonnegative, and to sum to some fixed positive real number s. How should their possibly differing individual assessments be aggregated into a single group assessment?

● Record their individual assessments in an $n \times m$ matrix $A = (a_{ij})$ where a_{ij} denotes the value assigned by individual i to variable x_j.
Any such matrix is called an s-*allocation matrix.* If $n = 1$, it is called an s-*allocation row vector.*

Reformulation of the allocation aggregation problem: Given an s-allocation matrix $A = (a_{ij})$, produce an s-allocation row vector $a = (a_1,\ldots,a_m)$ that incorporates the assessments recorded in A in some reasonable way.

Key question: What are the allowable values of the decision variables? Denote this set by V and call it the *valuation domain.* Clearly, V should be a subset of [0,s] satisfying at least

(1) $0 \varepsilon V$,

(2) $x \varepsilon V \Rightarrow s - x \ \varepsilon V$, and

(3) $x, y \varepsilon V$ and $x + y \leq s \Rightarrow x + y \ \varepsilon V$.

NOTATION and TERMINOLOGY

- $\mathcal{A}(n,m;s,V)$ = the set of all n x m s-allocation matrices, with entries belonging to V (henceforth, "in V."

- $\mathcal{A}$(m;s,V) = the set of all m-dimensional s-allocation row vectors in V.

- Any function F: $\mathcal{A}$(n,m;s,V) $\rightarrow$ $\mathcal{A}$(m;s,V) is called an *allocation aggregation method* (AAM). Each AAM F furnishes a method, applicable to every s-allocation matrix A in V , of reconciling the possibly different opinions recorded in A in the form of the group assignment F(A) = a = $(a_1,\ldots,a_m)$ in V.

- A_j denotes the j^{th} column of matrix A.

- a_j denotes the j^{th} entry of row vector a.

- <u>**c**</u> denotes the n x 1 column vector with all entries equal to c.

AGGREGATION AXIOMS

- *Irrelevance of Alternatives* (IA): For each $j = 1,\ldots,m$, and for all A, B in $\mathscr{A}(n,m;s,V)$,
$A_j = B_j \Rightarrow F(A)_j = F(B)_j$.

Remark. IA is clearly equivalent to the existence of functions $f_j : V^n \to V$,
$j = 1,\ldots,m$, such that for all A in $\mathscr{A}(n,m;s,V)$,

$$F(A)_j = f_j(A_j) \quad \text{and} \quad \sum_{1 \le j \le m} f_j(A_j) = s.$$

- *Zero Preservation* (ZP): For each $j = 1,\ldots,m$, and for all A in $\mathscr{A}(n,m;s,V)$, $A_j = \underline{\mathbf{0}} \Rightarrow F(A)_j = 0$,
i.e., $f_j(\underline{\mathbf{0}}) = 0$ for each $j = 1,\ldots,m$.

THE AAMs CHARACTERIZED BY IA AND ZP DEPEND CRUCIALLY ON THE CHOICE OF V!

CASE 1: $V = [0,s]$

Theorem 1. (L & W 1981) If $V = [0,s]$ and $m \geq 3$, an AAM F satisfies IA and ZP if and only if there exists a *single* sequence $w_1,\ldots,w_n$ of weights, nonnegative and summing to 1, such that for all $A = (a_{ij})$ in $\mathcal{A}(n,m;s,V)$ and $j = 1,\ldots,m$,

$$F(A)_j = f_j(A_j) = w_1 a_{1j} + w_2 a_{2j} + \cdots + w_n a_{nj}.$$

• Note that IA and ZP allow for *dictatorial aggregation* (for some fixed d in $\{1,\ldots,n\}$, $w_d = 1$ and $w_i = 0$ for $i \neq d$).

CASE 2: V is a finite subset of [0,s].

• An AAM F is *imposed* if there exists an allocation vector a such that F(A) = a for all A in $\mathcal{A}(n,m;s,V)$.

Theorem 2. (W, recent). If V is finite and satisfies the closure properties (1), (2), and (3), then an AAM F: $\mathcal{A}(n,m;s,V) \rightarrow \mathcal{A}(m;s,V)$ satisfies IA and ZP if and only if it is dictatorial or imposed.

• Remark. If V is a subset of [0,s] satisfying (1), (2), and (3) and V is *discrete* (i.e., for each x in V there is an open interval containing x, but no other element of V), then V is finite.

Moral: Universal domain conditions more or less force one to adopting IA (or IA, followed by normalization), and IA is unduly restrictive. Allowing an AAM to operate on a smaller domain "opens up the canon" of acceptable aggregation methods.

See my papers

1. Universality and its discontents (to appear in the Balkan J. of Philosophy)

2. Peer disagreement and independence preservation, Erkenntnis 74 (2011), 277-288.