ELSEVIER

Discrete Mathematics 222 (2000) 291–298

DISCRETE MATHEMATICS

www.elsevier.com/locate/disc

Note

The Carlitz lattice path polynomials

Carl G. Wagner *

Department of Mathematics, University of Tennessee, Knoxville, 121 Ayres Hall, TN 37996-1300, USA

Received 11 May 1999; revised 20 December 1999; accepted 22 January 2000

In Memoriam, Leonard Carlitz, 1907–1999

Abstract

We study some polynomials of Carlitz as generating functions for some natural statistics on lattice paths with diagonals. © 2000 Elsevier Science B.V. All rights reserved.

Keywords: Lattice path statistics; Recurrence relation; Generating function

1. Introduction

For $m,n\in\mathbb{N}$, let $\mathscr{L}(=\mathscr{L}_{m,n})$ be the set of all minimal lattice paths from $(0,0)$ to (m,n), allowing only vertical and horizontal moves, and let $\Lambda(=\Lambda_{m,n})$ be the set of such paths, with diagonal moves allowed as well.

It is a basic result of elementary combinatorics that $|\mathscr{L}| = \binom{m+n}{n}$. Moreover, the q-binomial coefficient

$$\begin{bmatrix} m+n \\ n \end{bmatrix} := \frac{(q^{m+n}-1)(q^{m+n}-q)\dots(q^{m+n}-q^{n-1})}{(q^n-1)(q^n-q)\dots(q^n-q^{n-1})} \tag{1}$$

is a generating function for $\mathscr{L}$, in the sense that

$$\begin{bmatrix} m+n \\ n \end{bmatrix} = \sum_{s=0}^{mn} p(m,n,s)q^s, \tag{2}$$

where $p(m,n,s)$ is the number of lattice paths in $\mathscr{L}$ subtending area s. Since such paths may be viewed as Ferrers diagrams of partitions of the integer s, $p(m,n,s)$ is also the number of partitions of s with n or fewer parts and no part greater than m [5, p. 29].

The numbers $L(m,n):=|\Lambda|$ not only enumerate lattice paths with diagonals, but specify the volume of a sphere of radius m in n dimensions for the Lee metric [5], and

* Tel.: 865-974-2461; fax: 865-974-6576.

E-mail address: cwagner@utk.edu (C.G. Wagner).

PII: S0012-365X(00)00058-3

also the number of n-element subsets S of the set $\{a_1,\dots,a_{m+n-1},b_1,\dots,b_{m+n}\}$ with $|i-j|\geqslant 2$ for all $a_i,b_j\in S$ [3].

The penultimate lattice point on a path in Λ may be either $(m,n-1)$, $(m-1,n)$, or $(m-1,n-1)$. This leads in the obvious way to a partitioning of Λ into three classes, which we shall simply call 'the usual partitioning' on the numerous occasions below where it is invoked. A basic consequence of this partitioning is the recurrence [2,4]

$$L(m,n)=L(m,n-1)+L(m-1,n)+L(m-1,n-1),\quad m,n>0, \tag{3}$$

with $L(m,0)=L(0,n)=1, m,n\geqslant 0$.

Each $\lambda\in\Lambda$ may be represented as a word in the alphabet $\{x,y,d\}$, the three letters representing, respectively, horizontal, vertical, and diagonal segments of λ. This observation leads to the formulas [1,4,5]

$$L(m,n)=\sum_{s=0}^{m}\binom{n+s}{n+s-m,s,m-s}=\sum_{s=0}^{m}\binom{n}{m-s}\binom{n+s}{n}, \tag{4}$$

and

$$L(m,n)=\sum_{s=0}^{\min(m,n)}\binom{m+n-s}{n-s,m-s,s}=\sum_{s=0}^{\min(m,n)}\binom{m+n-2s}{m-s}\binom{m+n-s}{s}, \tag{5}$$

where paths in Λ are enumerated in (4) and (5), respectively, according to their numbers, s, of horizontal and diagonal segments. From (3), (4), or (5), it may easily be proved [1] that

$$\sum_{m,n\geqslant 0}L(m,n)x^m y^n=\frac{1}{1-x-y-xy}. \tag{6}$$

In [1] Carlitz introduces a polynomial generalization $A(m,n;p,q)$ of the numbers $L(m,n)$, defined by the initial conditions $A(m,0;p,q)=A(0,n;p,q)=1$, $m,n\geqslant 0$, and the recurrence

$$\begin{aligned}A(m,n;p,q)=&\,p^m A(m,n-1;p,q)+q^n A(m-1,n;p,q)\\&+A(m-1,n-1;p,q),\quad m,n>0.\end{aligned} \tag{7}$$

We shall call these polynomials the *Carlitz lattice path polynomials*, for reasons that will soon be clear. Note that $A(m,n;1,1)=L(m,n)$.

Carlitz makes a detailed study of these polynomials, including a number of special cases. His analysis is, however, purely algebraic. In the next section we study these polynomials from a combinatorial perspective, based on the observation that they are generating functions for some obvious and natural statistics on Λ. In Section 3 we extend some of our results to lattice paths with diagonals in $\mathbb{R}^3$.

2. Some statistics on lattice paths

Given a lattice path $\lambda\in\Lambda$, let $\alpha(\lambda)$ be the area (of that part of the $m\times n$ rectangle with vertices $(0,0)$, $(m,0)$, $((m,n)$, and $(0,n))$ lying to the left of the vertical segments of λ, let $\beta(\lambda)$ be the area lying below the horizontal segments of λ, and let $\delta(\lambda)$ be the area lying to the left of and below the diagonal segments of λ. Clearly, $\alpha(\lambda)+\beta(\lambda)+\delta(\lambda)=mn$. Let

$$L_{p,q,r}(m,n):=\sum_{\lambda\in\Lambda} p^{\alpha(\lambda)}q^{\beta(\lambda)}r^{\delta(\lambda)}. \tag{8}$$

By the usual partitioning of Λ, it follows that $L_{p,q,r}(m,n)$ satisfies the recurrence

$$\begin{aligned}L_{p,q,r}(m,n)&=p^mL_{p,q,r}(m,n-1)+q^nL_{p,q,r}(m-1,n)\\&\quad+r^{m+n-1}L_{p,q,r}(m-1,n-1),\quad m,n>0,\end{aligned} \tag{9}$$

with $L_{p,q,r}(m,0)=L_{p,q,r}(0,n)=1$, $m,n\geqslant 0$. Comparing (9) with (7) we see that the polynomial $A(m,n;\,p,q)$ is a special case of $L_{p,q,r}(m,n)$, namely,

$$A(m,n;\,p,q)=L_{p,q,1}(m,n)=\sum_{\lambda\in\Lambda} p^{\alpha(\lambda)}q^{\beta(\lambda)}, \tag{10}$$

which endows the coefficients of $A(m,n;\,p,q)$ with a salient combinatorial interpretation.

Of course by (2) we have

$$\begin{aligned}L_{1,q,0}(m,n)&=\sum_{\lambda\in\Lambda} q^{\beta(\lambda)}0^{\delta(\lambda)}=\sum_{\lambda\in\mathscr{L}} q^{\beta(\lambda)}\\&=\begin{bmatrix}m+n\\n\end{bmatrix},\end{aligned} \tag{11}$$

so both the q-binomial coefficients and the polynomials $A(m,n;\,p,q)$ are special cases of $L_{p,q,r}(m,n)$. It should be noted, however, that $L_{p,q,r}(m,n)$ is simply the homogenization of $A(m,n;\,p,q)$, i.e., $L_{p,q,r}(m,n)=r^{mn}A(m,n;\,p/r,q/r)$. We shall find it more convenient to work with $L_{p,q,r}(m,n)$ than with $A(m,n;\,p,q)$, but the two polynomials contain exactly the same information.

We now consider two special cases of the above, namely, generating functions for the statistics β and δ. First, let

$$A_1(m,n):=L_{1,q,1}(m,n)=\sum_{\lambda\in\Lambda} q^{\beta(\lambda)}=\sum_{t=0}^{mn} a_1(m,n,t)q^t, \tag{12}$$

where the coefficients $a_1(m,n,t)$ have the obvious combinatorial interpretation. By the usual partitioning of Λ, we get the recurrence

$$A_1(m,n)=A_1(m,n-1)+q^nA_1(m-1,n)+A_1(m-1,n-1),\quad m,n>0. \tag{13}$$

To determine the coefficients $a_1(m,n,t)$, suppose that the lattice path λ contains exactly s horizontal segments, so that λ is represented by a word $w_1w_2\dots w_{n+s}$ comprising $s\,x$'s, $m-s\,d$'s and $n+s-m\,y$'s. If $w_{i_1}=w_{i_2}=\cdots=w_{i_s}=x$, where $1\leqslant i_1<i_2<\cdots<i_s\leqslant n+s$, then clearly

$$\beta(\lambda)=\sum_{j=1}^{s}(i_j-j). \tag{14}$$

Thus to construct those $\lambda \in \Lambda$ containing exactly s horizontal segments, and such that $\beta(\lambda)=t$, it suffices (with $i'_j:=i_j-j$) to choose $0 \leqslant i'_1 \leqslant i'_2 \leqslant \cdots \leqslant i'_s \leqslant n$ with $i'_1+\cdots+i'_s=t$, which can be done in $p(n,s,t)$ ways, and then to distribute the $m-s$ d's and $n+s-m$ y's among the remaining positions in $w_1w_2\dots w_{n+s}$. Hence

$$a_1(m,n,t)=\sum_{s=0}^{m}\binom{n}{m-s}p(n,s,t). \tag{15}$$

It follows from (12), (15), and (2) that

$$A_1(m,n)=\sum_{s=0}^{m}\binom{n}{m-s}\begin{bmatrix} n+s \\ n \end{bmatrix}. \tag{16}$$

Note that, as one might expect, (16) reduces when $q=1$ to formula (4) for $L(m,n)$.

Finally, with $(z)_s:=(1-z)(1-qz)\cdots(1-q^{s-1}z)$, it follows from (2) that

$$\sum_{n\geqslant 0}\begin{bmatrix} n+s \\ n \end{bmatrix} z^n=\frac{1}{(z)_{s+1}}. \tag{17}$$

From (16) and (17), it is straightforward to show that

$$\sum_{m,n\geqslant 0} A_1(m,n)x^m y^n=\sum_{s\geqslant 0}\frac{x^s}{(xy+y)_{s+1}}, \tag{18}$$

which of course reduces to (6) when $q=1$.

Next, let

$$A^*(m,n):=L_{1,1,q}(m,n)=\sum_{\lambda\in\Lambda} q^{\delta(\lambda)}=\sum_{t=0}^{mn} a^*(m,n,t)q^t, \tag{19}$$

where the coefficients $a^*(m,n,t)$ have the obvious combinatorial interpretation. From the usual partitioning of Λ we get the recurrence

$$A^*(m,n)=A^*(m,n-1)+A^*(m-1,n)+q^{m+n-1}A^*(m-1,n-1),$$
$$m,n>0. \tag{20}$$

To determine the coefficients $a^*(m,n,t)$, suppose that the lattice path λ contains exactly s diagonal moves, so that λ is represented by a word $w_1w_2\dots w_{m+n-s}$ comprising s d's, $n-s$ y's, and $m-s$ x's. If $w_{i_1}=w_{i_2}=\cdots=w_{i_s}=d$, where $1\leqslant i_1<i_2<\cdots<i_s\leqslant m+n-s$, then it is easy to see that

$$\delta(\lambda)=\sum_{j=1}^{s}(i_j+j-1). \tag{21}$$

Thus, to construct those $\lambda\in\Lambda$ containing exactly s diagonal segments and such that $\delta(\lambda)=t$, we must first choose $1\leqslant i_1<i_2<\cdots<i_s\leqslant m+n-s$ with $i_1+i_2+\cdots+i_s=t-\binom{s}{2}$. With $i'_j:=i_j-j$, this is equivalent to choosing $0\leqslant i'_1\leqslant i'_2\leqslant\cdots\leqslant i'_s\leqslant m+n-2s$ such that $i'_1+\cdots+i'_s=t-s^2$, which can be done in $p(m+n-2s,s,t-s^2)$ ways.

Next we must distribute the $n-s$ y's and $m-s$ x's among the remaining positions in $w_1\dots w_{m+n-s}$. Summing over all possible values of s then yields the formula

$$a^*(m,n,t)=\sum_{s=0}^{\min(m,n)}\binom{m+n-2s}{m-s}p(m+n-2s,s,t-s^2). \tag{22}$$

By (19), (22), and (2), it follows that

$$A^*(m,n)=\sum_{s=0}^{\min(m,n)}\binom{m+n-2s}{m-s}\begin{bmatrix}m+n-s\\ s\end{bmatrix}q^{s^2} \tag{23}$$

and from (23) and (17) that

$$\sum_{m,n\geqslant 0}A^*(m,n)x^m y^n=\sum_{s\geqslant 0}\frac{(xy)^s q^{s^2}}{(x+y)_{s+1}}. \tag{24}$$

Of course, when $q=1$, (23) reduces to (5) and (24) reduces to (6).

Readers may wish to compare Carlitz's algebraic treatment [1] of the polynomials $A_1(m,n)$ and $A^*(m,n)$, which proceeds from recurrence to generating function to closed form. His paper also includes a treatment of the polynomial $A(m,n):=L_{q,q,1}(m,n)$. Since $A(m,n)$ is the reciprocal polynomial of $A^*(m,n)$, its properties are easily derived from (22)–(24).

3. Lattice paths in $\mathbb{R}^3$

For $m,n,r\in\mathbb{N}$, let $\mathscr{L}_{m,n,r}$ be the set of all minimal lattice paths from $(0,0,0)$ to (m,n,r), allowing only moves parallel to the x-, y-, and z-axes. Clearly, $|\mathscr{L}_{m,n,r}|=\binom{m+n+r}{m,n,r}$. Let $\Lambda_{m,n,r}$ be the set of such paths, with diagonal moves from a lattice point (a,b,c) to $(a+1,b+1,c+1)$ allowed as well, and let $L(m,n,r):=|\Lambda_{m,n,r}|$.

Partitioning $\Lambda_{m,n,r}$ according to the four possible penultimate lattice points on a path $\lambda\in\Lambda_{m,n,r}$ leads to the recurrence

$$\begin{aligned}L(m,n,r)&=L(m,n,r-1)+L(m,n-1,r)\\&\quad+L(m-1,n,r)+L(m-1,n-1,r-1),\quad m,n,r>0\end{aligned} \tag{25}$$

with $L(m,n,0)=\binom{m+n}{n}$, $L(m,0,r)=\binom{m+r}{r}$, and $L(0,n,r)=\binom{n+r}{n}$, $m,n,r\geqslant 0$. Enumerating paths by their number, s, of diagonal segments yields the formula

$$L(m,n,r)=\sum_{s=0}^{\min(m,n,r)}\binom{m+n+r-2s}{m-s,n-s,r-s,s}. \tag{26}$$

Among a number of interesting statistics on $\Lambda_{m,n,r}$, we shall investigate just one, a three-dimensional analogue of the statistic δ (see Section 2 above), which we will denote by the same symbol in what follows. In calculating $\delta(\lambda)$ for $\lambda\in\Lambda_{m,n}$ we summed the areas of all the L-shaped pieces extending from the squares traversed by the diagonal segments of λ, as illustrated above in Fig. 1 for the square traversed by a diagonal segment from $(a-1,b-1)$ to (a,b).

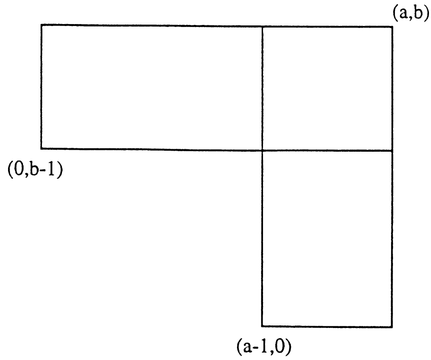

Fig. 1.

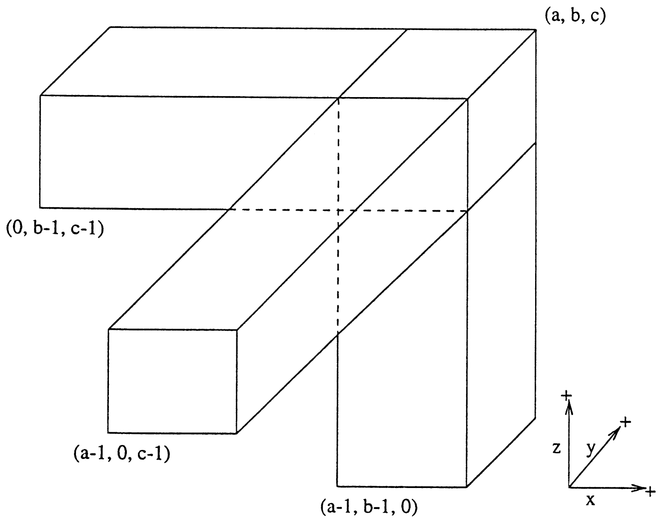

Fig. 2.

In calculating $\delta(\lambda)$ for $\lambda \in \Lambda_{m,n,r}$ we sum the volumes of all the 'tripods' extending from the cubes traversed by the diagonal segments of λ, as illustrated above in Fig. 2 for the cube traversed by a diagonal segment from $(a-1, b-1, c-1)$ to (a, b, c).

Let

$$A^*(m,n,r) := \sum_{\lambda \in \Lambda_{m,n,r}} q^{\delta(\lambda)} = \sum_{t \geqslant 0} a^*(m,n,r,t) q^t. \tag{27}$$

The natural partitioning of $\Lambda_{m,n,r}$ leads to the recurrence

$$A^*(m,n,r) = A^*(m,n,r-1) + A^*(m,n-1,r) + A^*(m-1,n,r) + q^{m+n+r-2} A^*(m-1,n-1,r-1), \quad m,n,r > 0 \tag{28}$$

with

$$A^*(m,n,0)=\binom{m+n}{n},\quad A^*(m,0,r)=\binom{m+r}{r}$$

and

$$A^*(0,n,r)=\binom{n+r}{r},\quad m,n,r\geqslant 0.$$

To get an explicit formula for $a^*(m,n,r,t)$, we represent paths $\lambda\in\Lambda_{m,n,r}$ as words in the alphabet $\{x,y,z,d\}$ in the obvious way. Suppose that λ contains exactly s diagonal segments, so that λ is represented by a word $w_1w_2\dots w_{m+n+r-2s}$ comprising $s\,d$'s, $m-s\,x$'s, $n-s\,y$'s, and $r-s\,z$'s. If $w_{i_1}=w_{i_2}=\cdots=w_{i_s}=d$, where $1\leqslant i_1<i_2<\cdots<i_s\leqslant m+n+r-2s$, then

$$\delta(\lambda)=\sum_{j=1}^{s} i_j+2(j-1)=2\binom{s}{2}+\sum_{j=1}^{s} i_j, \tag{29}$$

for the volume of the tripod extending from the cube traversed by the jth diagonal segment of λ is $i_j+2(j-1)$, by the following argument: Among the symbols $w_1,w_2,\dots,w_{i_j-1}$, there are $(j-1)\,d$'s. If there are $u\,x$'s and $v\,y$'s among these symbols, there are $i_j-j-u-v\,z$'s. So this jth diagonal segment connects the lattice point $(u+j-1,v+j-1,i_j-u-v-1)$ to the lattice point $(u+j,v+j,i_j-u-v)$, and so the volume of the tripod in question is $(u+j)+(v+j)+(i_j-u-v)-2=i_j+2(j-1)$.

Thus, to construct those $\lambda\in\Lambda_{m,n,r}$ containing exactly s diagonal segments, and such that $\delta(\lambda)=t$, we must first (with $i'_j:=i_j-j$) choose $0\leqslant i'_1\leqslant i'_2\leqslant\cdots\leqslant i'_s\leqslant m+n+r-3s$ such that $i'_1+\cdots+i'_s=t-2\binom{s}{2}-\binom{s+1}{2}=t-s(3s-1)/2$, which can be done in $p(m+n+r-3s,s,t-s(3s-1)/2)$ ways. We must then distribute the $m-s\,x$'s, $n-s\,y$'s, and $r-s\,z$'s among the remaining $m+n+r-3s$ positions in $w_1\dots w_{m+n+r-3s}$. Hence,

$$a^*(m,n,r,t)=\sum_{s=0}^{\min(m,n,r)}\binom{m+n+r-3s}{m-s,n-s,r-s}p(m+n+r-3s,s,t-s(3s-1)/2). \tag{30}$$

It follows from (27), (30), and (2) that

$$A^*(m,n,r)=\sum_{s=0}^{\min(m,n,r)}\binom{m+n+r-3s}{m-s,n-s,r-s}\begin{bmatrix}m+n+r-2s\\ s\end{bmatrix}q^{s(3s-1)/2}, \tag{31}$$

which it is interesting to compare to formula (23). From (31) and (17), we may derive the generating function formula

$$\sum_{m,n,r}A^*(m,n,r)x^my^nz^r=\sum_{s=0}^{\infty}\frac{(xyz)^sq^{s(3s-1)/2}}{(x+y+z)_{s+1}}, \tag{32}$$

which it is interesting to compare to formula (24).

References

[1] L. Carlitz, Some q-analogues of certain combinatorial numbers, SIAM J. Math. Anal. 4 (1973) 433–446.
[2] S. Kaparthi, H. Rao, Higher dimensional restricted lattice paths with diagonal steps, Discrete Appl. Math. 31 (1991) 279–289.
[3] I. Kaplansky, The asymptotic distribution of runs and consecutive elements, Ann. Math. Statistics 16 (1945) 200–203.
[4] L. Moser, W. Zayachkowski, Lattice paths with diagonal steps, Scripta Mathematica 26 (1963) 223–229.
[5] R. Stanton, D. Cowan, Note on a “square” functional equation, SIAM Rev. 12 (1970) 277–279.