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BOOLEAN SUBTRACTIVE ALGEBRAS

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1 Introduction In a recent paper [2], R. Güting has investigated structures $\langle K, -\rangle$, called colonies, which possess a distinguished element 1 and satisfy the following axioms:

K1 $(a - b) - c = (a - c) - b$
K2 $1 - (1 - a) = a$
K3 $a - a = 1 - 1$
K4 $a - (a - b) = a - (1 - b)$.

Güting shows that the study of such structures is equivalent to the study of Boolean algebras in the sense that every colony $\langle K, -\rangle$ gives rise to a Boolean algebra $\langle K, \vee, \wedge, '\rangle$ via the definitions $a' = 1 - a$, $a \wedge b = a - b'$, and $a \vee b = (a' - b)'$, and every Boolean algebra $\langle K, \vee, \wedge, '\rangle$ gives rise to a colony via the definition $a - b = a \wedge b'$.

In the present paper, we consider structures $\langle S, -\rangle$ which satisfy

S1 $(a - b) - c = (a - c) - b$
S2 $a - (b - a) = a$
S3 $\forall a, b \in S, \exists x \in S$ *such that* $x - (a - b) = b$ *and* $x - (b - a) = a$,

and prove that the study of such structures is equivalent to the study of generalized Boolean algebras. We call such structures Boolean subtractive algebras since they are subtractive algebras in the sense of Crapo and Rota ([1], 3.7). Alternatively, such structures might be called generalized colonies since, as we later prove, every colony is a Boolean subtractive algebra.

As an example of a Boolean subtractive algebra which is not a colony, we mention the set of all finite subsets of an infinite set, with set difference as composition. This (infinite) model shows the consistency of our axioms. There are also many finite models of S1, S2, and S3, all of which are colonies.

The independence of these axioms is also easily demonstrated. Let S be any two-element set and let $x - y = x$ for all $x, y \in S$. This composition shows the independence of S3. If, on the other hand, one sets $x - y = y$ for

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all $x, y \in S$, the resulting composition shows the independence of S1. Finally, let $S = \{a, b, c\}$ and define - by $a - a = a - b = b - b = c - a = c - b = c - c = c$, $b - a = a - c = a$, and $b - c = b$. This structure satisfies S1 and S3, but not S2, since $a - (b - a) = a - a = c \neq a$.

2 Preliminaries Following Stone ([3], p. 721), we call a structure $\langle S, \vee, \wedge \rangle$ a generalized Boolean algebra if it satisfies the following axioms:

(i) $a \vee b = b \vee a$
(ii) $a \vee a = a$
(iii) $\exists 0 \in S$ *such that* $\forall a \in S$, $a \vee 0 = a$
(iv) $a \wedge b = b \wedge a$
(v) $a \wedge a = a$
(vi) $a \wedge (b \wedge c) = (a \wedge b) \wedge c$
(vii) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$
(viii) $\forall a, b \in S$ *such that* $a \wedge b = a$, $\exists x \in S$ *such that* $x \vee a = b$ *and* $x \wedge a = 0$.

It may be proved from (i)-(viii) that $\vee$ is also associative and distributes over $\wedge$, and that the absorption identities $a \wedge (a \vee b) = a \vee (a \wedge b) = a$ hold ([3], p. 725). Thus a generalized Boolean algebra is a distributive lattice and therefore has the weak cancellation property: $(x \wedge a = y \wedge a$ *and* $x \vee a = y \vee a) \Longrightarrow x = y$. It follows from this property that the simultaneous equations of (viii) have a unique solution. We may now prove the following theorem:

Theorem 1. *Let $\langle S, \vee, \wedge \rangle$ be a generalized Boolean algebra. $\forall a, b \in S$ denote by $b - a$ the unique solution in S of the simultaneous equations $x \vee a = a \vee b$ and $x \wedge a = 0$. Then $\langle S, - \rangle$ is a Boolean subtractive algebra.*

Proof: The existence of a solution to $x \vee a = a \vee b$ and $x \wedge a = 0$ follows from (viii) and the absorption identity $a \wedge (a \vee b) = a$. In order to prove that $(a - b) - c = (a - c) - b$, it suffices to show that $(a - b) - c = a - (b \vee c)$ and then use $b \vee c = c \vee b$. Let $x = a - b$, so that $x \vee b = a \vee b$ and $x \wedge b = 0$. Let $x = a - b$, so that $x \vee b = a \vee b$ and $x \wedge b = 0$. Let $y = x - c$, so that $y \vee c = x \vee c$ and $y \wedge c = 0$. Let $z = a - (b \vee c)$, so that $z \vee (b \vee c) = a \vee (b \vee c)$ and $z \wedge (b \vee c) = 0$. Then $y \vee (b \vee c) = (y \vee c) \vee b = (x \vee c) \vee b = (x \vee b) \vee c = (a \vee b) \vee c = a \vee (b \vee c) = z \vee (b \vee c)$. Also, $y \wedge (b \vee c) = (y \wedge (y \vee c)) \wedge (b \vee c) = (y \wedge (x \vee c)) \wedge (b \vee c) = (y \wedge x \wedge b) \vee (y \wedge x \wedge c) \vee (y \wedge c \wedge b) \vee (y \wedge c) = 0 = z \wedge (b \vee c)$. Hence $y = z$ by weak cancellation. To prove that $a - (b - a) = a$, it suffices to observe that $a \vee (b - a) = (b - a) \vee a$ and $a \wedge (b - a) = (b - a) \wedge a = 0$.

Finally, we remark that $x = a \vee b$ is a solution of the simultaneous equations $x - (a - b) = b$ and $x - (b - a) = a$. For $b \vee (a - b) = (a - b) \vee b = b \vee a = (b \vee a) \vee a = (b \vee (a - b)) \vee a = (a - b) \vee (a \vee b)$, and $b \wedge (a - b) = (a - b) \wedge b = 0$. Hence $(a \vee b) - (a - b) = b$. Similarly, $(a \vee b) - (b - a) = a$.

We call $\langle S, - \rangle$, as defined in Theorem 1, the Boolean subtractive algebra associated to the generalized Boolean algebra $\langle S, \vee, \wedge \rangle$.

3 A Sequence of Lemmas We now prove a sequence of lemmas which lead to a theorem complementary to Theorem 1. Throughout this section $\langle S, - \rangle$ is a Boolean subtractive algebra, a, b, c, d, x, and y are elements of S, and these variables are understood to be universally quantified unless otherwise indicated. We define a composition $\wedge$ in S by $a \wedge b = a - (a - b)$.

Lemma 1. $a \wedge b = b \wedge a$.

Proof: Let x be any solution of the simultaneous equations $x - (a - b) = b$ and $x - (b - a) = a$ of S3. Then

$$\begin{aligned} a - (a - b) &= (x - (b - a)) - (a - b) \\ &= (x - (a - b)) - (b - a) = b - (b - a). && \text{(S1)} \end{aligned}$$

Lemma 2. $(a - b) - (b - a) = a - b$.

Proof: $(a - b) - (b - a) = (a - (b - a)) - b = a - b$. (S1); (S2)

Lemma 3. $a - a = b - b$.

Proof:
$$\begin{aligned} a - a &= (a - a) - (((a - a) - (b - b)) - (a - a)) && \text{(S2)} \\ &= (a - a) - (((a - a) - (a - a)) - (b - b)) && \text{(S1)} \\ &= (a - a) - ((a - a) - (b - b)). && \text{(Lemma 2)} \end{aligned}$$

Similarly, $b - b = (b - b) - ((b - b) - (a - a))$. Hence $a - a = b - b$ by Lemma 1.

In view of Lemma 3, there is a distinguished element of S (which we denote by 0) with the property $a - a = 0$ for all $a \in S$.

Lemma 4. *$a - 0 = a$ and $0 - a = 0$.*

Proof: $a - 0 = a - (a - a) = a$ by S2. By S2 and the preceding line, $0 - a = 0 - (a - 0) = 0$.

It follows from Lemma 4 that a structure $\langle S, -\rangle$ satisfying S1, S2, and S3 is a subtractive algebra in the sense of Crapo and Rota (i.e., a structure $\langle S, -\rangle$ with distinguished element 0 satisfying I. $a - a = 0$ and II. $a - 0 = 0 \Rightarrow a = 0$), *cf.* [1], 3.7.

Lemma 5. $a \wedge a = a$.

Proof: $a \wedge a = a - (a - a) = a - 0 = a$. (Lemma 4)

Lemma 6. *If $a - b = b - a$, then $a = b$.*

Proof:
$$\begin{aligned} a &= a - (b - a) = a - (a - b) = b - (b - a) && \text{(S2); (Lemma 1)} \\ &= b - (a - b) = b. && \text{(S2)} \end{aligned}$$

Lemma 7. *If $x - c = y - c$ and $c - x = c - y$, then $x = y$.*

Proof:
$$\begin{aligned} x - y &= (x - y) - ((y - c) - (x - y)) && \text{(S2)} \\ &= (x - y) - ((y - (x - y)) - c) && \text{(S1)} \\ &= (x - y) - (y - c) = (x - y) - (x - c) && \text{(S2)} \\ &= (x - (x - c)) - y && \text{(S1)} \\ &= (c - (c - x)) - y && \text{(Lemma 1)} \\ &= (c - y) - (c - x) = 0. && \text{(S1)} \end{aligned}$$

Similarly, $y - x = 0$ and so $x = y$ by Lemma 6.

Lemma 8. *The simultaneous equations* $x - (a - b) = b$ *and* $x - (b - a) = a$ *of* (S3) *have a unique solution in* S.

Proof: Suppose that $x - (a - b) = y - (a - b) = b$ and $x - (b - a) = y - (b - a) = a$. Then

$$\begin{aligned}(a - b) - x &= ((x - (b - a)) - b) - x \\ &= ((x - b) - (b - a)) - x && \text{(S1)} \\ &= ((x - b) - x) - (b - a) && \text{(S1)} \\ &= ((x - x) - b) - (b - a) = 0. && \text{(S1); (Lemma 4)}\end{aligned}$$

Similarly, $(a - b) - y = 0$, and so $x = y$ by Lemma 7.

In view of Lemma 8 we may introduce a composition $\vee$ in S by letting $a \vee b$ be the unique solution of the simultaneous equations $x - (a - b) = b$ and $x - (b - a) = a$.

Lemma 9. *The following hold:*

(i) $a \vee b = b \vee a$

(ii) $a \vee a = a$

(iii) $a \vee 0 = a$.

Proof: The first two assertions are clear. The third follows from $a - (a - 0) = a - a = 0$ and $a - (0 - a) = a - 0 = a$.

Lemma 10. *The statements* $a - b = 0$, $a \wedge b = a$, *and* $a \vee b = b$ *are all equivalent.*

Proof: If $a - b = 0$, then $a \wedge b = a - (a - b) = a - 0 = a$. If $a \wedge b = a$, then by Lemma 1 $b \wedge a = a$, i.e., $b - (b - a) = a$. By (S2) $b - (a - b) = b$. Hence $a \vee b = b$. Finally, if $a \vee b = b$, then $b - (b - a) = a$ and, by Lemma 1, $a - (a - b) = a$. Hence $a - b = (a - (a - b)) - b = (a - b) - (a - b) = 0$, by (S1) and Lemma 3.

Lemma 11. $(a - b) - b = a - b$.

Proof:
$$\begin{aligned}(a - b) - b &= (a - b) - (b - (a - b)) && \text{(S2)} \\ &= a - b. && \text{(S2)}\end{aligned}$$

Lemma 12. $a - (a - (a - b)) = a - b$.

Proof: Let $x = a \vee (a - b)$. Then $x - ((a - b) - a) = a$ and by (S1) and Lemmas 3 and 4, $x - ((a - a) - b) = x - (0 - b) = x = a$. Also, $x - (a - (a - b)) = a - b$ and so $a - (a - (a - b)) = a - b$.

Lemma 13. $(a \wedge b) \wedge c = a \wedge (b \wedge c)$.

Proof: It suffices to prove that $((a \wedge b) \wedge c) - (a \wedge (b \wedge c)) = 0$, for this implies by Lemma 1 that $(c \wedge (b \wedge a)) - ((c \wedge b) \wedge a) = 0$ and, hence, that $(a \wedge (b \wedge c)) - ((a \wedge b) \wedge c) = 0$. The desired result then follows by Lemma 6. Now

$$\begin{aligned}&((a \wedge b) \wedge c) - (a \wedge (b \wedge c)) = ((a \wedge b) - ((a \wedge b) - c)) - (a \wedge (b \wedge c)) \\ &= ((a \wedge b) - (a \wedge (b \wedge c))) - ((a \wedge b) - c) && \text{(S1)} \\ &= ((a - (a - b)) - (a - (a - (b \wedge c)))) - ((a \wedge b) - c) \\ &= ((a - (a - (a - (b \wedge c)))) - (a - b)) - ((a \wedge b) - c) && \text{(S1)}\end{aligned}$$

$$\begin{aligned}
&= ((a - (b \wedge c)) - (a - b)) - ((a \wedge b) - c) && \text{(Lemma 12)}\\
&= ((a - (a - b)) - (b \wedge c)) - ((a \wedge b) - c) && \text{(S1)}\\
&= ((b - (b - a)) - (b - (b - c))) - ((a \wedge b) - c) && \text{(Lemma 1)}\\
&= ((b - (b - (b - c))) - (b - a)) - ((a \wedge b) - c) && \text{(S1)}\\
&= ((b - c) - (b - a)) - ((a \wedge b) - c) && \text{(Lemma 12)}\\
&= ((b - (b - a)) - c) - ((a \wedge b) - c) && \text{(S1)}\\
&= ((b \wedge a) - c) - ((a \wedge b) - c) = 0. && \text{(Lemma 1)}
\end{aligned}$$

Lemma 14. *If* $a - d = 0$, *then* $a - (a - b) = a - (d - b)$ *for all* b.

Proof: By (S2), $(a - b) - (a - (a - b)) = a - b$. Also,

$$\begin{aligned}
a - b &= (a - (a - d)) - b && \text{(Lemma 4)}\\
&= (d - (d - a)) - b = (d - b) - (d - a) && \text{(Lemma 1); (S1)}
\end{aligned}$$

and so

$$\begin{aligned}
(a - b) - (a - (d - b)) &= ((d - b) - (d - a)) - (a - (d - b))\\
&= ((d - b) - (a - (d - b))) - (d - a) && \text{(S1)}\\
&= (d - b) - (d - a) && \text{(S2)}\\
&= (d - (d - a)) - b && \text{(S1)}\\
&= (a - (a - d)) - b && \text{(Lemma 1)}\\
&= (a - 0) - b = a - b. && \text{(Lemma 4)}
\end{aligned}$$

Furthermore, $(a - (a - b)) - (a - b) = a - (a - b)$ by Lemma 11, and

$$\begin{aligned}
(a - (d - b)) - (a - b) &= (a - (a - b)) - (d - b) && \text{(S1)}\\
&= (b - (b - a)) - (d - b) && \text{(Lemma 1)}\\
&= (b - (d - b)) - (b - a) && \text{(S1)}\\
&= b - (b - a) && \text{(S2)}\\
&= a - (a - b). && \text{(Lemma 1)}
\end{aligned}$$

Hence, $a - (a - b) = a - (d - b)$ by Lemma 7.

Lemma 15. $(d - c) \wedge b = (d \wedge b) - (c \wedge b)$.

Proof: Since $(d - c) - d = (d - d) - c = 0 - c = 0$, it follows from Lemma 14 that

$$\begin{aligned}
(d - c) \wedge b &= (d - c) - ((d - c) - b) = (d - c) - (d - b)\\
&= (d - (d - b)) - c = (b - (b - d)) - c && \text{(S1); (Lemma 1)}\\
&= (b - c) - (b - d) && \text{(S1)}\\
&= (b - (b - (b - c))) - (b - d) && \text{(Lemma 12)}\\
&= (b - (b - d)) - (b - (b - c)) && \text{(S1)}\\
&= (b \wedge d) - (b \wedge c) = (d \wedge b) - (c \wedge b). && \text{(Lemma 1)}
\end{aligned}$$

Lemma 16. $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$.

Proof: We have

$$\begin{aligned}
(a \wedge (b \vee c)) - ((a \wedge b) - (a \wedge c)) &= ((b \vee c) \wedge a) - ((b \wedge a) - (c \wedge a)) && \text{(Lemma 1)}\\
&= ((b \vee c) \wedge a) - ((b - c) \wedge a) && \text{(Lemma 15)}\\
&= ((b \vee c) - (b - c)) \wedge a && \text{(Lemma 15)}\\
&= a \wedge ((b \vee c) - (b - c)) = a \wedge c. && \text{(Lemma 1)}
\end{aligned}$$

Similarly, $(a \wedge (b \vee c)) - ((a \wedge c) - (a \wedge b)) = a \wedge b$, and so $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ by the definition of $\vee$.

Lemma 17. *If* $a \wedge b = a$, *then the equations* $x \vee a = b$ *and* $x \wedge a = 0$ *admit* $x = b - a$ *as a simultaneous solution.*

Proof: By Lemmas 11 and 1, we have $b - ((b - a) - a) = b - (b - a) = b \wedge a = a \wedge b = a$; by S2, $b - (a - (b - a)) = b - a$. Hence $b = (b - a) \vee b$ by the definition of $\vee$. Also, $(b - a) \wedge a = (b \wedge a) - (a \wedge a) = (a \wedge b) - (a \wedge a) = a - a = 0$, by Lemmas 15, 1, 5, and 3.

Lemmas 1, 5, 9, 13, 16, and 17 imply the following theorem.

Theorem 2. *Let* $\langle S, -\rangle$ *be a Boolean subtractive algebra. For all* $a, b \in S$, *let* $a \wedge b = a - (a - b)$ *and let* $a \vee b$ *be the unique solution of the simultaneous equations* $x - (a - b) = b$ *and* $x - (b - a) = a$. *Then* $\langle S, \vee, \wedge\rangle$ *is a generalized Boolean algebra.*

We call $\langle S, \vee, \wedge\rangle$, as defined in the preceding theorem, the generalized Boolean algebra associated to $\langle S, -\rangle$. In the next two theorems, we demonstrate a one-to-one correspondence between generalized Boolean algebras and Boolean subtractive algebras.

Theorem 3. *Let* $\langle S, \sim\rangle$ *be the Boolean subtractive algebra associated to the generalized Boolean algebra* $\langle S, \vee, \wedge\rangle$ *associated to the Boolean subtractive algebra* $\langle S, -\rangle$. *Then* $a \sim b = a - b$ *for all* $a, b \in S$.

Proof: By the definition of $\sim$, $a \sim b$ is the unique solution in S of the simultaneous equations $x \vee b = b \vee a$ and $x \wedge b = 0$. It therefore suffices to show that $a - b$ is also a solution of these equations. By the definition of $\wedge$ and Lemma 11, $(a - b) \wedge b = (a - b) - ((a - b) - b) = (a - b) - (a - b) = 0$.

We prove that $(a - b) \vee b = a \vee b = b \vee a$ by showing that (I) $(a \vee b) - ((a - b) - b) = b$ and (II) $(a \vee b) - (b - (a - b)) = a - b$: (I). By Lemma 11 and the definition of $\vee$, $(a \vee b) - ((a - b) - b) = (a \vee b) - (a - b) = b$. (II) By S2 it suffices to show that $(a \vee b) - b = a - b$. Now $((a \vee b) - b) - (a - b) = ((a \vee b) - (a - b)) - b = b - b = 0$ by S1 and the definition of $\vee$. Also, since $a = (a \vee b) - (b - a)$, it follows that $a - b = ((a \vee b) - (b - a)) - b = ((a \vee b) - b) - (b - a)$. Hence, $(a - b) - ((a \vee b) - b) = ((a \vee b) - b) - (b - a) - ((a \vee b) - b) = (((a \vee b) - b) - ((a \vee b) - b)) - (b - a) = 0 - (b - a) = 0$. Thus $(a \vee b) - b = a - b$ by Lemma 6.

Theorem 4. *Let* $\langle S, \cup, \cap\rangle$ *be the generalized Boolean algebra associated to the Boolean subtractive algebra* $\langle S, -\rangle$ *associated to the generalized Boolean algebra* $\langle S, \vee, \wedge\rangle$. *Then* $a \wedge b = a \cap b$ *and* $a \vee b = a \cup b$ *for all* $a, b \in S$.

Proof: By definition $a \cap b = a - (a - b)$. We prove that $a \wedge b = a - (a - b)$ by showing that $a \wedge b$ is a solution of the equations $x \wedge (a - b) = 0$ and $x \vee (a - b) = (a - b) \vee a$. First, $(a \wedge b) \wedge (a - b) = a \wedge (b \wedge (a - b)) = a \wedge 0 = 0$. Next, note that $(a - b) \vee a = a$. For $((a - b) \vee a) \vee b = ((a - b) \vee b) \vee a = (a \vee b) \vee a = a \vee b$, and $((a - b) \vee a) \wedge b = ((a - b) \wedge b) \vee (a \wedge b) = 0 \vee (a \wedge b) = a \wedge b$. Hence $(a - b) \vee a = a$ by weak cancellation in $\langle S, \vee, \wedge\rangle$. Finally, we have $(a \wedge b) \vee (a - b) = (a \vee (a - b)) \wedge (b \vee (a - b)) = a \wedge (a \vee b) = a = (a - b) \vee a$, as desired.

Now $a \cup b$ is the unique solution of the equations $x - (a - b) = b$ and $x - (b - a) = a$. To show that $a \vee b$ also satisfies these equations, note that $a \vee b = (a - b) \vee b$ and hence $(a \vee b) - (a - b) = ((a - b) \vee b) - (a - b)$. But $((a - b) \vee b) - (a - b) = b$, for $b \vee (a - b) = (a - b) \vee ((a - b) \vee b)$ and $b \wedge (a - b) = 0$. Similarly, $(a \vee b) - (b - a) = a$.

We conclude by remarking that the notion of an ideal I in the generalized Boolean algebra $\langle S, \vee, \wedge \rangle$ may be reformulated in subtractive terminology by

(i) $a \in \mathrm{I}$ and $b \in S \Rightarrow a - b \in \mathrm{I}$

and

(ii) $a \in \mathrm{I}$ and $x - a \in \mathrm{I} \Rightarrow x \in \mathrm{I}$.

4 Colonies, Boolean Subtractive Algebras, and Difference Domains We remarked in the Introduction that every colony $\langle K, - \rangle$ is a Boolean subtractive algebra. This may be proved easily by passing to the Boolean algebra associated to $\langle K, - \rangle$, for S1 is K1; and $a - (b - a) = a \wedge (b \wedge a')' = a \wedge (b' \vee a) = a$, so S2 holds. Finally $a \vee b$ is easily shown to be a solution of the simultaneous equations $x - (a - b) = b$ and $x - (b - a) = a$, so that S3 holds. The following theorem gives a simple criterion for a Boolean subtractive algebra to be a colony.

Theorem 5. *A Boolean subtractive algebra $\langle S, - \rangle$ is a colony if and only if it contains a distinguished element* 1 *such that $a - 1 = 0$ for all $a \in S$.*

Proof: Necessity. ([2], p. 214, Lemma 4). Sufficiency. K1 is S1. To prove K2, note that since $a - 1 = 0, a - (a - 1) = a - 0 = a$. But by Lemma 1, $a - (a - 1) = 1 - (1 - a) = a$. Also $a - a = 1 - 1 = 0$, so K3 holds. Finally, K4 is a special case of Lemma 14.

In conclusion we mention that Güting [2] has also studied a class of structures called difference domains (defined by axioms K1, K2, and S2), and has related their study to that of the class of lattices with Boolean involution ([2], p. 219). Difference domains represent a generalization of of colonies in an essentially different direction from that of Boolean subtractive algebras, for it may easily be shown that the intersection of the class of difference domains with the class of Boolean subtractive algebras is precisely the class of colonies.

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