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CONSENSUS FOR BELIEF FUNCTIONS AND RELATED UNCERTAINTY MEASURES*

ABSTRACT. We extend previous work of Lehrer and Wagner, and of McConway, on the consensus of probabilities, showing under axioms similar to theirs that (1) a belief function consensus of belief functions on a set with at least three members and (2) a belief function consensus of Bayesian belief functions on a set with at least four members must take the form of a weighted arithmetic mean. We observe that these results are unchanged when consensual uncertainty measures are allowed to take the form of Choquet capacities of low order monotonicity.

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1. INTRODUCTION

A *belief function* on a set $\Theta = \{\theta_1 \ldots \theta_k\}$ is a mapping $b: 2^\Theta \to [0, 1]$ such that $b(\emptyset) = 0$, $b(\Theta) = 1$, and for all positive integers r and every collection $A_1, \ldots, A_r$ of subsets of Θ,

$$(1.1) \quad b(A_1 \cup \cdots \cup A_r) \geq \sum_{\substack{I \subseteq \{1, \ldots, r\} \\ I \neq \emptyset}} (-1)^{|I|-1} b\Big(\bigcap_{i \in I} A_i\Big).$$

The theory of belief functions was introduced by Shafer (1976) in *A Mathematical Theory of Evidence* and provides, among other things, an abstract formulation of a certain class of lower probabilities, studied earlier by Dempster (1967). Every probability measure on the algebra 2^Θ is clearly a belief function, and we follow Shafer in calling such probability measures *Bayesian belief functions*.

Closely related to belief functions are mappings $m: 2^\Theta \to [0, 1]$, called *basic probability assignments* (BPAs), defined by the properties

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$m(\emptyset) = 0$ and $\Sigma_{A \subseteq \Theta} m(A) = 1$. Every BPA m on Θ induces a belief function $b^{(m)}$ on Θ by

$$(1.2) \qquad b^{(m)}(A) = \sum_{H \subseteq A} m(H), \qquad \forall A \subseteq \Theta,$$

and every belief function b on Θ induces a BPA $m^{(b)}$ on Θ by

$$(1.3) \qquad m^{(b)}(A) = \sum_{H \subseteq A} (-1)^{|A-H|} b(H), \qquad \forall A \subseteq \Theta,$$

with $m^{(b^{(m)})} = m$ and $b^{(m^{(b)})} = b$ (Shafer, 1976, pp. 38–40). To show that a mapping $b: 2^{\Theta} \to [0, 1]$ is a belief function one may thus avoid checking (1.1), either by exhibiting a BPA m such that $b^{(m)} = b$, or by checking that $b(\emptyset) = 0$, $b(\Theta) = 1$ and the quantities $m^{(b)}(A)$ defined by (1.3) are nonnegative for all $A \subseteq \Theta$. Bayesian belief functions are precisely those belief functions whose associated BPAs are positive only on singleton subsets of Θ (Shafer 1976, p. 45).

Denote by $\mathscr{B}(\Theta)$, $\mathscr{P}(\Theta)$, and $\mathscr{M}(\Theta)$, respectively, the set of all belief functions, Bayesian belief functions, and BPAs on Θ. We shall refer to elements of $\mathscr{B}(\Theta)$, $\mathscr{P}(\Theta)$, and $\mathscr{M}(\Theta)$ generically as *uncertainty measures*. For $n \geq 2$, n-tuples $B = (b_1, \ldots, b_n) \in \mathscr{B}^n(\Theta)$, $P = (p_1, \ldots, p_n) \in \mathscr{P}^n(\Theta)$, and $M = (m_1, \ldots, m_n) \in \mathscr{M}^n(\Theta)$ are called *n-profiles* and may be regarded as registering the individual opinions of n experts as to 'where the truth lies' in Θ, cast in terms of the relevant uncertainty measure. In this note we consider the problem of aggregating such opinions into a single consensual measure, subject to two simple axiomatic restrictions. With the exception of a few cases where Θ has small cardinality, these axioms are shown to imply aggregation by weighted arithmetic averaging, thus extending previous results of Lehrer and Wagner (1981) and McConway (1981) on the consensus of probabilities.

2. CONSENSUS FUNCTIONS

Informally, a consensus function is simply a method of deriving from each profile of uncertainty measures of some fixed type a consensual uncertainty measure of some fixed type. We shall be interested in four types of consensus functions, corresponding to the (profile type, consensus type) pairs $(\mathscr{P}^n(\Theta), \mathscr{P}(\Theta))$, $(\mathscr{M}^n(\Theta), \mathscr{M}(\Theta))$, $(\mathscr{B}^n(\Theta), \mathscr{B}(\Theta))$, and $(\mathscr{P}^n(\Theta)), \mathscr{B}(\Theta))$. For economy of exposition, the following discussion employs generic n-profiles $U = (u_1, \ldots, u_n) \in \mathscr{U}^n(\Theta)$, where $(\mathscr{U}, U, u) \in \{(\mathscr{P}, P, p), (\mathscr{M}, M, m), (\mathscr{B}, B, b)\}$ and generic (profile type, consensus type) pairs $(\mathscr{U}^n(\Theta), \mathscr{U}^*(\Theta))$, where $(\mathscr{U}, \mathscr{U}^*) \in \Delta = \{(\mathscr{P}, \mathscr{P}), (\mathscr{M}, \mathscr{M}), (\mathscr{B}, \mathscr{B}), (\mathscr{P}, \mathscr{B})\}$.

DEFINITION. A *consensus function* is a mapping $C: \mathscr{U}^n(\Theta) \to \mathscr{U}^*(\Theta)$, for fixed $n \geqslant 2$ and fixed $(\mathscr{U}, \mathscr{U}^*) \in \Delta$.

Since, for every n-profile U, $C(U)$ is either a belief function, a Bayesian belief function, or a BPA on Θ, and since all these uncertainty measures assign $\emptyset$ the measure zero, it is a consequence of the above definition that $C(U)(\emptyset) = 0$. Similarly, when $\mathscr{U}^* \in \{\mathscr{B}, \mathscr{P}\}$, $C(U)(\Theta) = 1$. We wish to study consensus functions which, for each subset $A \subseteq \Theta$ whose measure is not thus predetermined, assigns to A a consensual uncertainty measure which depends only on the measures assigned to A by the n experts. This restriction on aggregation is common in consensus studies and has been variously termed independence, invariance, irrelevance of alternatives, and weak setwise functionality. (In the case of consensus for probabilities, such a restriction was shown by McConway (1981) to be equivalent to requiring the consensus function to commute with marginalization.) For our purposes the relevant axiomatic restriction is formalized as follows:

(I) For all $A \in 2^\Theta - \{\emptyset, \Theta\}$, and if $\mathscr{U} = \mathscr{U}^* = \mathscr{M}$, for $A = \Theta$ as well, there exists a function $F_A : [0, 1]^n \to [0, 1]$ such that for all $U \in \mathscr{U}^n(\Theta)$, $C(U)(A) = F_A(u_1(A), \ldots, u_n(A))$.

In addition, we shall be interested in the consequences of adopting one or more of an infinite number of possible 'unanimity preservation' axioms, (II(c)), where $c \in [0, 1]$, given by

(II(c)) For all $A \subseteq \Theta$ and for all $U \in \mathscr{U}^n(\Theta)$, if $U(A) = (c, \ldots, c)$, then $C(U)(A) = c$.

Our first theorem recapitulates results implicit in Lehrer and Wagner (1981) and McConway (1981).

THEOREM 2.1. *If* $\mathcal{U} = \mathcal{U}^* = \mathcal{P}$ *and* $|\Theta| \geq 3$, *or if* $\mathcal{U} = \mathcal{U}^* = \mathcal{M}$ *and* $|\Theta| \geq 2$, *a consensus function* $C: U^n(\Theta) \to U^*(\Theta)$ *satisfies axioms* (I) *and* $(II(1))$ *iff there exists a sequence of weights* $w_1, \ldots, w_n$, *nonnegative and summing to one, such that for all* $A \subseteq \Theta$ *and for all* $U \in \mathcal{U}^n(\Theta)$, $C(U)(A) = w_1 u_1(A) + \cdots + w_n u_n(A)$.

We omit the minor details required to modify the aforementioned results to yield this theorem, except to note that the lower threshold $|\Theta| = 2$ for BPAs obtains because a consensus function in this case is an 'allocation aggregation method' (Lehrer and Wagner, 1981, Theorem 6.4) for the *three* 'decision variables' $m(\{\theta_1\})$, $m(\{\theta_2\})$, and $m(\{\theta_1, \theta_2\})$. The case $\mathcal{U} = \mathcal{U}^* = \mathcal{P}$ and $|\Theta| = 2$ is essentially characterized by Theorem 6.5 of Lehrer and Wagner (1981).

3. CONSENSUS IN THE FORM OF A BELIEF FUNCTION

We now examine consensus functions $C: \mathcal{U}^n(\Theta) \to \mathcal{U}^*(\Theta)$ constrained by axioms (I), $(II(1))$, and $(II(1/2))$, where $(\mathcal{U}, \mathcal{U}^*) \in \{(\mathcal{B}, \mathcal{B}), (\mathcal{P}, \mathcal{B})\}$. In what follows $X = (x_1, \ldots, x_n)$ and $Y = (y_1, \ldots, y_n)$ denote elements of $[0, 1]^n$, $\mathbf{c}$ denotes the n-dimensional vector $(c, \ldots, c)$, and all inequalities between vectors are to be understood coordinatewise. We observe first that if $|\Theta| \geq 3$, (I) and $(II(1))$ imply that the functions F_A posited by (I) must be identical.

THEOREM 3.1. *If* $(\mathcal{U}, \mathcal{U}^*) \in \{(\mathcal{B}, \mathcal{B}), (\mathcal{P}, \mathcal{B})\}$, $|\Theta| \geq 3$, *and* $C: \mathcal{U}^n(\Theta) \to \mathcal{U}^*(\Theta)$ *satisfies axioms* (I) *and* $(II(1))$, *then for all* H *and* $K \in 2^\Theta - \{\emptyset, \Theta\}$, $F_H = F_K$, *and* C *satisfies* $(II(0))$.

Proof. Suppose first that H is a proper subset of K. For every $X \in [0, 1]^n$, there is obviously a profile $P = (p_1, \ldots, p_n) \in \mathcal{P}^n(\Theta) \subseteq \mathcal{B}^n(\Theta)$ such that $P(H) = (p_1(H), \ldots, p_n(H) = X$, $P(K - H) = \mathbf{0}$, and $P(\bar{K}) = \mathbf{1} - X$. Let $A_1 = H$ and $A_2 = K - H$. Since $C(P) \in \mathcal{B}(\Theta)$, (1.1) and axiom (I) yield

$$(3.1) \quad C(P)(A_1 \cup A_2) = F_K(X) \geq F_H(X) + F_{K-H}(\mathbf{0}) - C(P)(\emptyset)$$
$$= F_H(X) + F_{K-H}(\mathbf{0}) \geq F_H(X)\,.$$

Next let $A_1 = H \cup \bar{K}$ and $A_2 = K$. In this case (1.1) and axiom (I) yields

$$C(P)(A_1 \cup A_2) = C(P)(\Theta) = 1$$
$$\geq F_{H \cup \bar{K}}(\mathbf{1}) + F_K(X) - F_H(X)\,,$$

which, with axiom ($II(1)$), yields

$$(3.2) \quad F_H(X) \geq F_K(X)\,.$$

It follows from (3.1) and (3.2) that $F_H = F_K$ whenever $H \subseteq K$.

Suppose now that H and K are arbitrary nonempty proper subsets of Θ. If $H \cap K \neq \emptyset$, then by the preceding argument $F_H = F_{H \cap K} = F_K$. If $H \cap K = \emptyset$ and $H \cup K$ is a proper subset of Θ, then $F_H = F_{H \cup K} = F_K$. If $H \cap K = \emptyset$ and $H \cup K = \Theta$ then since $|\Theta| \geq 3$, $|H| \geq 2$ or $|K| \geq 2$. Supposing, with no loss of generality, that $|H| \geq 2$, and that $\theta_i \in H$, it follows that $F_H = F_{\{\theta_i\}} = F_{K \cup \{\theta_i\}} = F_K$. Thus $F_H = F_K$ for all $H, K \in 2^\Theta - \{\emptyset, \Theta\}$, and aggregation is carried out by a single function $F: [0,1)^n \to [0,1]$. Dropping subscripts and setting $X = \mathbf{0}$ in (3.1) then yields $F(\mathbf{0}) \geq 2F(\mathbf{0})$. Hence, $F(\mathbf{0}) = 0$ and C satisfies ($II(0)$).

The preceding theorem fails to hold when $|\Theta| = 2$. For example, the function C, defined for all $B \in \mathcal{B}^n(\{\theta_1, \theta_2\})$ by $C(B)(\emptyset) = 0$, $C(B)(\Theta) = 1$, $C(B)(\{\theta_1\}) = \min\{b_1(\{\theta_1\}), \ldots, b_n(\{\theta_1\})\}$ and $C(B)(\{\theta_2\}) = \max\{b_1(\{\theta_2\}), \ldots, b_n(\{\theta_2\})\}$ yields a belief function on $\{\theta_1, \theta_2\}$ for every profile B, and satisfies axioms (I) and ($II(1)$), while $F_{\{\theta_1\}} \neq F_{\{\theta_2\}}$.

THEOREM 3.2. *If* $|\Theta| \geq 3$, *a consensus function* $C: \mathcal{B}^n(\Theta) \to \mathcal{B}(\Theta)$ *satisfies axioms* (I), ($II(1)$), *and* ($II(\frac{1}{2})$) *iff there exists a sequence of weights* $w_1, \ldots, w_n$, *nonnegative and summing to one, such that for all* $A \subseteq \Theta$ *and all* $B \in \mathcal{B}^n(\Theta)$, $C(B)(A) = w_1 b_1(A) + \cdots + w_n b_n(A)$.[1]

Proof. Sufficiency: straightforward. *Necessity*: By Theorem 3.1 there exists a function $F: [0, 1]^n \to [0, 1]$ such that for all $B \in \mathscr{B}^n(\Theta)$ and for all $A \subseteq \Theta$, $C(B)(A) = F(b_1(A), \ldots, b_n(A))$. We show that for all X, Y such that $\mathbf{0} \leq X, Y, X + Y \leq \mathbf{1}$, $F(X + Y) = F(X) + F(Y)$, which implies, by a standard result of functional equations along with $F(\mathbf{1}) = 1$ (see Lehrer and Wagner, 1981, p. 122) that F is a weighted arithmetic mean. Suppose first that $\mathbf{0} \leq X, Y \leq \mathbf{1/2}$. Let M be the BPA profile defined by $M(\{\theta_1\}) = X$, $M(\{\theta_2\}) = Y$, $M(\{\theta_1, \theta_3\}) = \mathbf{1/2} - X$, $M(\{\theta_2, \theta_3\}) = \mathbf{1/2} - Y$, and $M(A) = \mathbf{0}$ for all other $A \subseteq \Theta$. Let B be the belief function profile induced by M, as described in Section 1. Among other things, $B(\{\theta_1\}) = X$, $B(\{\theta_2\}) = Y$, $B(\{\theta_3\}) = \mathbf{0}$, $B(\{\theta_1, \theta_2\}) = X + Y$, $B(\{\theta_1, \theta_3\}) = B(\{\theta_2, \theta_3\}) = \mathbf{1/2}$, and $B(\{\theta_1, \theta_2, \theta_3\}) = \mathbf{1}$. Letting $C(B) = b$, it follows, using axioms $(II(1))$ and $(II(1/2))$ where appropriate, that $b(\{\theta_1\}) = F(X)$, $b(\{\theta_2\}) = F(Y)$, $b(\{\theta_3\}) = 0$, $b(\{\theta_1, \theta_2\}) = F(X + Y)$, $b(\{\theta_1, \theta_3\}) = b(\{\theta_2, \theta_3\}) = 1/2$, and $b(\{\theta_1, \theta_2, \theta_3\}) = 1$. Since $b \in \mathscr{B}(\Theta)$ by hypothesis, it satisfies (1.1). Instantiating (1.1) for $A_1 = \{\theta_1\}$ and $A_2 = \{\theta_2\}$ yields $F(X + Y) \geq F(X) + F(Y)$; when $A_1 = \{\theta_1, \theta_2\}$, $A_2 = \{\theta_1, \theta_3\}$, and $A_3 = \{\theta_2, \theta_3\}$, (1.1) yields $1 \geq F(X + Y) + \frac{1}{2} + \frac{1}{2} - F(X) - F(Y)$, i.e., $F(X + Y) \leq F(X) + F(Y)$. Hence $F(X + Y) = F(X) + F(Y)$ whenever $\mathbf{0} \leq X, Y \leq \mathbf{1/2}$, and so if $\mathbf{0} \leq X, Y, X + Y \leq \mathbf{1}$, $F(X + Y) = 2F(\frac{1}{2}(X + Y)) = 2F(\frac{1}{2}X + \frac{1}{2}Y) = 2F(\frac{1}{2}X) + 2F(\frac{1}{2}Y) = F(X) + F(Y)$, as desired.

We remark that when $\Theta = \{\theta_1, \theta_2\}$, even if $F_{\{\theta_1\}} = F_{\{\theta_2\}} = F$ (as need not be the case, by the remark following the proof of Theorem 3.1), F is not necessarily a weighted arithmetic mean. For, as is easily checked, setting $C(B)(A) = \min\{b_1(A), \ldots, b_n(A)\}$ for all $A \subseteq \{\theta_1, \theta_2\}$ yields a belief function on $\{\theta_1, \theta_2\}$ for all $B \in \mathscr{B}^n(\{\theta_1, \theta_2\})$, and C satisfies $(II(1))$ and $(II(1/2))$.

Moreover, axioms (I) and $(II(1))$ alone are not sufficient to guarantee the conclusion of Theorem 3.2 for, setting $C(B)(A) = \lfloor b_1(A) \rfloor$, the greatest integer less than or equal to $b_1(A)$, defines a mapping $C: \mathscr{B}^n(\Theta) \to \mathscr{B}(\Theta)$ satisfying (I) and $(II(1))$, and C is not a weighted arithmetic mean.

THEOREM 3.3. *If* $|\Theta| \geq 4$, *a consensus function* $C: \mathscr{P}^n(\theta) \to \mathscr{B}(\Theta)$ *satisfies axioms* (I), $(II(1))$, *and* $(II(1/2))$ *if there exists a sequence of* *weights* $w_1, \ldots, w_n$, *nonnegative and summing to one, such that for all* $P \in \mathcal{P}^n(\Theta)$ *and for all* $A \subseteq \Theta$, $C(P)(A) = w_1 p_1(A) + \cdots + w_n p_n(A)$.

Proof. Sufficiency: straightforward. *Necessity*: By Theorem 3.1 there exists a function $F: [0,1]^n \to [0,1]$ such that for all $\mathcal{P} \in \mathcal{P}^n(\Theta)$ and for all $A \subseteq \Theta$, $C(P)(A) = F(p_1(A), \ldots, p_n(A))$. As in the proof of the preceding theorem, to establish that F is a weighted arithmetic mean we need only show that $F(X+Y) = F(X) + F(Y)$ for all X and Y such that $\mathbf{0} \leqslant X, Y, X+Y \leqslant \mathbf{1}$.

For X and Y as above, consider the Bayesian belief function profile P for which $P(\{\theta_1\}) = X$, $P(\{\theta_2\}) = Y$, $P(\{\theta_3\}) = \mathbf{1} - X - Y$, and $P(\{\theta_i\}) = 0$ for all $i \geqslant 4$. Letting $C(P) = b$, it follows, using axiom (*II*) where appropriate, that $b(\{\theta_1\}) = F(X)$, $b(\{\theta_2\}) = F(Y)$, $b(\{\theta_1, \theta_2\}) = F(X+Y)$, $b(\{\theta_2, \theta_3\}) = F(\mathbf{1} - X)$, and $b(\{\theta_1, \theta_2, \theta_3\}) = 1$. For $A_1 = \{\theta_1\}$ and $A_2 = \{\theta_2\}$, (1.1) implies that

$$(3.3) \qquad F(X+Y) \geqslant F(X) + F(Y), \qquad \mathbf{0} \leqslant X, Y, X+Y \leqslant \mathbf{1}.$$

For $A_1 = \{\theta_1, \theta_2\}$ and $A_2 = \{\theta_1, \theta_3\}$, (1.1) implies that $1 \geqslant F(X+Y) + F(\mathbf{1} - X) - F(Y)$, which is equivalent to

$$(3.4) \qquad F(X) + F(Y) \geqslant F(X+Y) + [F(X) + F(\mathbf{1} - X) - 1], \quad \mathbf{0} \leqslant X, Y, X+Y \leqslant \mathbf{1}.$$

Now suppose that $X \geqslant \mathbf{1/2}$ and let P be the Bayesian belief function profile for which $P(\{\theta_1\}) = P(\{\theta_3\}) = X - \mathbf{1/2}$, $P(\{\theta_2\}) = P(\{\theta_4\}) = \mathbf{1} - X$, and $P(\{\theta_i\}) = \mathbf{0}$ for all $i \geqslant 5$. Letting $C(P) = b$, it follows, using axiom (*II*(1/2)) where appropriate, that $b(\{\theta_2\}) = F(\mathbf{1} - X)$, $B(\{\theta_1, \theta_2\}) = b(\{\theta_2, \theta_3\}) = 1/2$, and $b(\{\theta_1, \theta_2, \theta_3\}) = F(X)$. For $A_1 = \{\theta_1, \theta_2\}$ and $A_2 = \{\theta_2, \theta_3\}$, (1.1) implies that $F(X) \geqslant 1/2 + 1/2 - F(\mathbf{1} - X)$, i.e., that

$$(3.5) \qquad F(X) + F(\mathbf{1} - X) \geqslant 1, \qquad \mathbf{1/2} \leqslant X \leqslant \mathbf{1},$$

which, with (3.3) for $Y = \mathbf{1} - X$, yields

(3.6) $F(X) + F(\mathbf{1} - X) = 1\,, \qquad \mathbf{1/2} \leqslant X \leqslant \mathbf{1}\,,$

and hence, of course,

(3.7) $F(X) + F(\mathbf{1} - X) = 1\,, \qquad \mathbf{0} \leqslant X \leqslant \mathbf{1/2}\,.$

Combining (3.3), (3.4), and (3.7), we see that

(3.8) $F(X + Y) = F(X) + F(Y)\,,$

$\mathbf{0} \leqslant X, Y, X + Y \leqslant \mathbf{1}\,; \quad X \leqslant \mathbf{1/2}\,.$

It follows from (3.8) that for all X, Y such that $\mathbf{0} \leqslant X, Y, X + Y \leqslant \mathbf{1}$, $F(X + Y) = F(\frac{1}{2}X + (\frac{1}{2}X + Y)) = F(\frac{1}{2}X) + F(\frac{1}{2}X + Y) = F(\frac{1}{2}X) + F(\frac{1}{2}X) + F(Y) = F(X) + F(Y)$, which completes the proof.

The condition $|\Theta| \geqslant 4$ in the preceding theorem is essential. When $|\Theta| = 3$, for example, setting $C(p_1, \ldots, p_n)(A) = \min\{p_1(A), p_2(A)\}$ for all $A \subseteq \Theta$ defines a mapping $C: \mathcal{P}^n(\Theta) \to \mathcal{B}(\Theta)$ for which axioms (I), ($II(1)$), and ($II(1/2)$) hold, and C is not a weighted arithmetic mean.

4. DISCUSSION

At the outset of this investigation we had hoped that enlarging the co-domain of a consensus function $C: \mathcal{P}^n(\Theta) \to \mathcal{P}(\Theta)$ to $\mathcal{B}(\Theta)$ might allow for some interesting ways of resolving disagreement in a Bayesian profile by means of a non-Bayesian consensus. Since, however, any weighted arithmetic mean of Bayesian belief functions is again Bayesian, Theorem (3.2) demonstrates that Axioms (I), ($II(1)$) and ($II(1/2)$) vitiate this hope.

Might we evade this limitation by allowing a yet broader class of consensual uncertainty measures? While substantially generalizing the class of probability measures, belief functions are, after all, still highly structured uncertainty measures. Indeed $\mathcal{B}(\Theta) = \bigcap_{r \geqslant 2} \mathcal{C}_r(\Theta)$, where $\mathcal{C}_r(\Theta)$ denotes the set of 'r-monotone Choquet capacities' on Θ, i.e., mappings $b: 2^{\Theta} \to [0, 1]$ such that $b(\phi) = 0$, $b(\Theta) = 1$, and (1.1) holds for the *fixed* integer r. An examination of the proofs of Theorems 3.1, 3.2, and 3.3 reveals that (1.1) was invoked only for small r. In particular, with no change in their proofs, Theorems 3.1 and 3.3 hold with $\mathscr{B}(\Theta)$ replaced by $\mathscr{C}_2(\Theta)$ and Theorem 3.2 holds with $\mathscr{B}(\Theta)$ replaced by $\mathscr{C}_3(\Theta)$. Thus even allowing consensus in the form of a Choquet capacity of low order monotonicity does not enlarge the set of consensus functions if one continues to stipulate Axioms (I), ($II(1)$), and ($II(1/2)$).

As shown by the example $C(P)(A) = \lfloor p_1(A) \rfloor$, the greatest integer less than or equal to $p_1(A)$, deleting ($II(1/2)$) as a restriction on consensus formation makes room for at least one (admittedly crude) non-Bayesian consensus. While it would be interesting to delete ($II(1)$) as well, and to try to characterize consensus functions, regulated only by Axiom (I), results of Aczél, Ng, and Wagner (1984) and Genest (1984) on consensual probability without unanimity preservation[2] suggest that it is not unanimity axioms, but rather Axiom (I) that is primarily responsible for circumscribing the class of acceptable consensus functions. We have thus begun a study of consensual uncertainty measures which may, for each $A \subseteq \Theta$, depend on the individual uncertainty measures assigned to A as well as to subsets $H \subseteq \Theta$ in certain classes naturally related to A such as $\{H : A \subseteq H\}$ and $\{H : H \cap A \neq \emptyset\}$.

NOTES

[1] A cursory reading of Theorem 3.2 might tempt one to think that it is a simple corollary of Theorems 2.1 and 3.1. After all, any consensus function $C: \mathscr{B}^n(\Theta) \to \mathscr{B}(\Theta)$ induces a consensus function C': $\mathscr{M}^n(\Theta) \to \mathscr{M}(\Theta)$ by the formula

$$C'(M) = m^{(C(B^{(M)}))},$$

where $B^{(M)} = (b^{(m_1)}, \ldots, b^{(m_n)})$. Since, correspondingly,

$$C(B) = b^{(C'(M^{(B)}))},$$

where $M^{(B)} = (m^{(b_1)}, \ldots, m^{(b_n)})$, if C' is based on weighted arithmetic averaging (as Theorem 2.1 assures us it is, given that it satisfies I and $II(1)$, then by (1.2), C will also be based on weighted arithmetic averaging. But from the fact that C satisfies I and $II(1)$ it follows from Theorem 3.1 only that

$$C'(M)(A) = \sum_{E \subseteq A} (-1)^{|A-E|} F\left(\sum_{H \subseteq E} m_1(H), \ldots, \sum_{H \subseteq E} m_n(H) \right).$$

Until one shows that F is linear (precisely the point of Theorem 3.2), it is not at all clear that C' satisfies I.

[2] Regulated only by Axiom (I), a consensus of n probability measures must take the form of a linear combination of those measures and some fixed, but arbitrary, 'external' probability measure. Interestingly, subject to certain restrictions, some weights may be negative.

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