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AGGREGATION THEOREMS FOR ALLOCATION PROBLEMS*

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Abstract. Suppose that n individuals assign values to a sequence of m numerical decision variables subject to the constraints that the m values assigned by each individual must be nonnegative and sum to some fixed positive σ. Suppose that we wish to aggregate their individual assignments to produce consensual values of these variables satisfying the aforementioned constraints. Aczél and Wagner have shown that if $m \geqq 3$, then a method of aggregation is based on weighted arithmetic averaging iff (a) the consensual value assigned to each variable depends only on the values assigned by individuals to that variable and (b) the consensual value is zero if all individuals assign that variable the value zero. In the present paper we extend this result in various ways, dropping the unanimity condition (b) and allowing individual and consensual values to be restricted to some subinterval of $[0, \sigma]$.

1. Suppose that a group of n individuals wish to assign values to a sequence of m numerical decision variables. We call such a problem an *allocation problem* if the values assigned must be nonnegative and sum to some fixed positive number σ. Examples of allocation problems abound, including, for example, the assignment of probabilities to a sequence of pairwise disjoint, exhaustive events, and the distribution of a fixed sum of money or other resource σ among m projects.

In general we may expect that individuals will differ in the values that they assign to the variables, and hence be faced with the problem of aggregating their individual assignments to produce consensual values of these variables. Let us denote by the n-dimensional vector z_j the sequence of values assigned by the individuals to the jth variable. (In what follows, lower case Latin letters, other than subscripts and integers describing dimensions, denote vectors, while Greek letters denote real numbers. We abbreviate the vector $(\alpha, \alpha, \cdots, \alpha)$, with equal components, by $\boldsymbol{\alpha}$.) If $m \geqq 3$ a method of aggregation assigns consensual values to decision variables in such a way that (a) the consensual value assigned to the jth variable is $\Phi_j(z_j)$, where $\Phi_j: [0, \sigma]^n \to [0, \sigma]$ and (b) the consensual value is zero if all individuals assign that variable the value zero iff the method is based on weighted arithmetic averaging, with weights invariant over the m decision variables ([7, Thm. 6.4]; see also [2], [3], [4], [9]). In many decisionmaking situations, individual and consensual values of the variables may be constrained to lie in a proper subinterval of $[0, \sigma]$, as, for example, when each of a number of budgetary units must receive not less than a minimal allocation $\mu > 0$, nor more than a maximal allocation $\nu < \sigma$. In such cases aggregation is more appropriately modeled by functions $\Phi_j: I^n \to I$, where $I \subset [0, \sigma]$. Furthermore, the assigned values may be deviations from some preferred value, and thus some may be negative (cf. [2]). So we may wish also to drop the condition $\mu > 0$ (or $\mu \geqq 0$) and let I be any (finite) real interval. In addition, conditions like (b) are only plausible if aggregation is carried out "internally" among the n individuals. If these individuals functioned as advisors to some external decisionmaker and he was responsible for aggregation, he might very well decide to ignore their unanimity. We are thus motivated in the present

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paper to generalize, as described below, the model of aggregation specified by (a) and (b), and to characterize the methods of aggregation which accord with this more general model.

2. With the above observations in mind, we first model aggregation as follows: Let σ be a fixed constant, and I be an interval with end points $\mu < \nu$ compatible with σ in the sense that

$$(1)\quad \begin{aligned} & I=[\mu,\nu] \quad \text{and} \quad (m-1)\mu+\nu \leqq \sigma \leqq \mu+(m-1)\nu, \\ \text{or}\quad & I=[\mu,\nu[\quad \text{and} \quad (m-1)\mu+\nu \leqq \sigma < \mu+(m-1)\nu, \\ \text{or}\quad & I=]\mu,\nu] \quad \text{and} \quad (m-1)\mu+\nu < \sigma \leqq \mu+(m-1)\nu, \\ \text{or}\quad & I=]\mu,\nu[\quad \text{and} \quad (m-1)\mu+\nu \leqq \sigma \leqq \mu+(m-1)\nu. \end{aligned}$$

We suppose that there exist bounded functions

$$(2)\qquad \Phi_j : I^n \to \mathbb{R} \qquad (j=1,2,\cdots,m)$$

(in fact, it will suffice to assume merely that at least one Φ_{j_0} is bounded below on some proper rectangle $[\gamma_1,\delta_1]\times[\gamma_2,\delta_2]\times\cdots\times[\gamma_n,\delta_n]\subseteq I^n$) such that

$$(3)\qquad \left(z_j \in I^n \text{ and } \sum_{j=1}^{m} z_j = \boldsymbol{\sigma}\right) \Rightarrow \sum_{j=1}^{m} \Phi_j(z_j) = \sigma.$$

To motivate the compatibility conditions in (1) for the study of (3), we observe that, in general the set $S=\{z_1\in I^n \mid \exists z_2,\cdots,z_m\in I^n \text{ such that } \sum_{j=1}^m z_j=\boldsymbol{\sigma}\}$ is a subinterval (n-dimensional) of I^n and (3) provides information about Φ_j only on S. Thus it is natural to assume that the domain I^n of Φ_j is equal to S, which is equivalent to (1). For instance, if $I=]\mu,\nu[$, then $z_1=\boldsymbol{\sigma}-\sum_{j=2}^m z_j<\boldsymbol{\sigma}-(m-1)\boldsymbol{\mu}$ (inequality meant componentwise). But z_1 can be anywhere in $]\boldsymbol{\mu},\boldsymbol{\nu}[$, so it is natural to assume $\nu\leqq \sigma-(m-1)\mu$. The argument is similar for $\mu \geqq \sigma-(m-1)\nu$ and for the other cases in (1). Note that (1) implies

$$(4)\qquad \mu < \frac{1}{m}\sigma < \nu.$$

The following theorem characterizes the aggregation methods which satisfy (2) and (3) when there are at least three decision variables ($m \geqq 3$):

THEOREM 1. *For fixed $m\geqq 3$, an aggregation method satisfies* (2) *and* (3) *if, and only if, there exist real numbers $\omega_1,\omega_2,\cdots,\omega_n$ and $\beta_1,\beta_2,\cdots,\beta_m$, with*

$$(5)\qquad \sum_{j=1}^{m}\beta_j=\left(1-\sum_{i=1}^{n}\omega_i\right)\sigma,$$

such that, for all $z=(\zeta_1,\cdots,\zeta_n)\in I^n$,

$$(6)\qquad \Phi_j(z)=\sum_{i=1}^{n}\omega_i\zeta_i+\beta_j \qquad (j=1,\cdots,m).$$

The ω_i $(i=1,2,\cdots,n)$ will be called *weights*.

Proof. Sufficiency is clear. To prove necessity we bring in new intervals, variables and functions. On first thought, one would want to move $\boldsymbol{\mu}$ into $\mathbf{0}$ and thus go back to the problem previously considered (as described in § 1). However, the intervals *open at* 0 ($]\mathbf{0}, \boldsymbol{\nu}-\boldsymbol{\mu}[$ or $]\mathbf{0}, \boldsymbol{\nu}-\boldsymbol{\mu}]$) would lead to complications and also the necessary extension (cf. Appendix) is easier if $\mathbf{0}$ is in the *interior* of the new domain. So we define

$$\tilde{I} = I - \frac{1}{m}\sigma, \qquad \tilde{z}_j = z_j - \frac{1}{m}\boldsymbol{\sigma},$$

(7)

$$\tilde{\Phi}_j(\tilde{z}_j) = \Phi_j\left(\tilde{z}_j + \frac{1}{m}\boldsymbol{\sigma}\right) = \Phi_j(z_j) \qquad (j = 1, \cdots, m)$$

in order to transform (3) into

$$\left(\tilde{z}_j \in \tilde{I}^n \text{ and } \sum_{j=1}^{m} \tilde{z}_j = \mathbf{0}\right) \Rightarrow \sum_{j=1}^{m} \tilde{\Phi}_j(\tilde{z}_j) = \sigma. \tag{8}$$

Note that $\tilde{I}$ contains 0 as an interior point because of (4). Putting $\tilde{z}_j = \mathbf{0}$ $(j = 1, \cdots, m)$ we obtain $\sum_{j=1}^{m} \tilde{\Phi}_j(\mathbf{0}) = \sigma$ and so the functions defined by

$$\psi_j(\tilde{z}) = \tilde{\Phi}_j(\tilde{z}) - \tilde{\Phi}_j(\mathbf{0}) \tag{9}$$

satisfy

$$\left(\tilde{z}_j \in \tilde{I}^n \text{ and } \sum_{j=1}^{m} \tilde{z}_j = \mathbf{0}\right) \Rightarrow \sum_{j=1}^{m} \psi_j(\tilde{z}_j) = 0, \qquad \psi_j(\mathbf{0}) = 0 \qquad (j = 1, \cdots, m). \tag{10}$$

We first consider (10) within a fixed symmetric subinterval $[-\varepsilon, \varepsilon] \subseteq \tilde{I}$ (with $\varepsilon > 0$). In particular we get

$$\left(\tilde{z}_j \in [-\varepsilon, \varepsilon]^n \text{ and } \sum_{j=1}^{m} \tilde{z}_j = \mathbf{0}\right) \Rightarrow \sum_{j=1}^{m} \psi_j(\tilde{z}_j) = 0, \qquad \psi_j(\mathbf{0}) = 0. \tag{11}$$

Putting $\tilde{z}_1 = y$, $\tilde{z}_2 = -y$, $\tilde{z}_3 = \cdots = \tilde{z}_m = \mathbf{0}$ in (11) we get $\psi_1(y) + \psi_2(-y) = 0$ on $[-\varepsilon, \varepsilon]^n$. Similarly $\psi_j(y) + \psi_k(-y) = 0$ on $[-\varepsilon, \varepsilon]^n$ for all $j \neq k$. This implies (as $m \geqq 3$)

$$\psi_1 = \psi_2 = \cdots = \psi_m = \text{an odd function } \psi \text{ on } [-\varepsilon, \varepsilon]^n. \tag{12}$$

In view of (12), we get from (11), with $\tilde{z}_1 = x$, $\tilde{z}_2 = y$, $\tilde{z}_3 = -x - y$, $\tilde{z}_4 = \cdots = \tilde{z}_m = \mathbf{0}$, the Cauchy equation

$$\psi(x) + \psi(y) = \psi(x + y), \qquad x, y, x + y \in [-\varepsilon, \varepsilon]^n. \tag{13}$$

This ψ can be extended uniquely to a function $\bar{\psi}: \mathbb{R}^n \to \mathbb{R}$ satisfying

$$\bar{\psi}(x) + \bar{\psi}(y) = \bar{\psi}(x + y) \quad \text{all } x, y \in \mathbb{R}^n. \tag{14}$$

This extension theorem is due to Daróczy and Losonczi [5]. (For completeness we include in the appendix a shorter proof, cf. [3], [6], [8]).

We now claim that $\psi_1 = \psi_2 = \cdots = \psi_m = \bar{\psi}$ on $\tilde{I}^n$. We first consider

$$\phi_j(\tilde{z}) = \psi_j(\tilde{z}) - \bar{\psi}(\tilde{z}), \qquad z \in \tilde{I}^n \qquad (j = 1, \cdots, m),$$

and we need to show that $\phi_j = 0$ on $\tilde{I}^n$ for all j. Since $\bar{\psi}$ extends ψ, we already have from (12)

$$\phi_j(x) = 0 \quad \text{for all } x \in [-\varepsilon, \varepsilon]^n. \tag{15}$$

From (10) we get

$$\text{(16)}\qquad \left(\tilde{z}_j \in \tilde{I}^n \text{ and } \sum_{j=1}^{m} \tilde{z}_j = \mathbf{0}\right) \Rightarrow \sum_{j=1}^{m} \phi_j(\tilde{z}_j) = 0.$$

Let $z \in \tilde{I}^n$ be arbitrarily given and we will show that $\phi_j(z) = 0$ for all j. From (1) and the discussion following (3), there exist $z_2, \cdots, z_m \in \tilde{I}^n$ such that $z + \sum_{j=2}^{m} z_j = \mathbf{0}$, i.e. $\sum_{j=2}^{m} z_j = -z$. Therefore the mean of $z_2, \cdots, z_m$, which equals $-z/(m-1)$ is also in $\tilde{I}^n$. By (16) we have

$$\text{(17)}\qquad \phi_1(z) + \phi_2\left(\frac{-1}{m-1}z\right) + \phi_3\left(\frac{-1}{m-1}z\right) + \cdots + \phi_m\left(\frac{-1}{m-1}z\right) = 0.$$

Repeating this, using $(-1/(m-1))^l z$ in place of z, we get a sequence of equations

$$\text{(18)}\qquad \begin{aligned} &\phi_1\left(\left(\frac{-1}{m-1}\right)^{l} z\right) + \phi_2\left(\left(\frac{-1}{m-1}\right)^{l+1} z\right) \\ &\quad + \phi_3\left(\left(\frac{-1}{m-1}\right)^{l+1} z\right) + \cdots + \phi_m\left(\left(\frac{-1}{m-1}\right)^{l+1} z\right) = 0 \end{aligned}$$

for $l = 0, 1, 2, \cdots$. Since (16) is symmetric in the ϕ_j's, (18) remains valid under any permutation of ϕ_j's. For large enough l, $(-1/(m-1))^{l+1} z$ will be in the interval $[-\varepsilon, \varepsilon]^n$ where the ϕ_j's are zero, and so by (18) $\phi_1((-1/(m-1))^l z) = 0$. By symmetry $\phi_j((-1/(m-1))^l z) = 0$ for all $j = 1, \cdots, m$. Using (18) recursively we get $\phi_j((-1/(m-1))^0 z) = \phi_j(z) = 0$ as claimed.

The boundedness (2) implies that $\bar{\psi}$ is bounded below on some rectangle and so (14) yields $\psi_j(z) = \bar{\psi}(z) = \sum_{i=1}^{n} \omega_i \zeta_i$ with appropriate constants (weights) $\omega_1, \cdots, \omega_n$ on $\tilde{I}^n$ (see e.g. [1, pp. 214–216]). This, (9), (7), and (14) give

$$\begin{aligned} \Phi_j(z) - \Phi_j(\mathbf{0}) &= \tilde{\Phi}_j\left(z - \frac{1}{m}\boldsymbol{\sigma}\right) - \tilde{\Phi}_j\left(-\frac{1}{m}\boldsymbol{\sigma}\right) = \bar{\psi}\left(z - \frac{1}{m}\boldsymbol{\sigma}\right) - \bar{\psi}\left(-\frac{1}{m}\boldsymbol{\sigma}\right) \\ &= \bar{\psi}(z) = \sum_{i=1}^{n} \omega_i \zeta_i \end{aligned}$$

that is, (6) holds with $\beta_j = \Phi_j(0)$. The functions Φ_j given by (6) satisfy (3) if and only if the constants satisfy (5). This proves the theorem. □

We note that the "weights" ω_i may be negative. It is easy to show that these weights are nonnegative if and only if $\Phi_j(z) \geqq \Phi_j(x)$ for all $z = (\zeta_1, \cdots, \zeta_n)$ and $x = (\xi_1, \cdots, \xi_n)$ with $\zeta_i \geqq \xi_i$, $i = 1, \cdots, n$. We remark also that *the above theorem continues to hold, without essential change in the proof, if the bounds* μ *and* ν *are replaced by possibly different bounds* μ_i *and* ν_i, $1 \leqq i \leqq n$, *on each individual's assignments.*

3. We next investigate aggregation methods which supplement the hypothesis of Theorem 1 by (a) conditions requiring that aggregation respect unanimity among the individuals and (b) a narrowing of the range of the functions Φ_j to I, the same interval where individuals assign values to the variables in question.

It turns out that very weak unanimity conditions have rather substantial consequences:

THEOREM 2. *Let* $m \geqq 3$ *and suppose that an aggregation method satisfies* (2) *and* (3). *If* $\Phi_j(\boldsymbol{\sigma}/m) = \sigma/m$ $(j = 1, \cdots, m)$ *then, and only then, for all* $z = (\zeta_1, \cdots, \zeta_n) \in I^n$,

$$\text{(19)}\qquad \Phi_j(z) = \sum_{i=1}^{n} \omega_i \zeta_i + \beta \qquad (j = 1, \cdots, m),$$

where

$$\beta=\left(1-\sum_{i=1}^{n}\omega_i\right)\frac{\sigma}{m}; \tag{20}$$

and if, for some $\alpha \in I$ *with* $\alpha \neq \sigma/m$, $\Phi_j(\boldsymbol{\alpha})=\alpha$, $j=1,\cdots,m$, *then, and only then, for all* $z=(\zeta_1,\cdots,\zeta_n)\in I^n$,

$$\Phi_j(z)=\sum_{i=1}^{n}\omega_i\zeta_i \qquad (j=1,\cdots,m), \tag{21}$$

where $\sum_{i=1}^{n}\omega_i=1$.

Proof. Assume

$$\phi_j(\boldsymbol{\alpha})=\alpha \quad \text{for a fixed } \alpha\in I.$$

From Theorem 1, (6)

$$\alpha=\sum_{i=1}^{n}\omega_i\alpha+\beta_j \qquad (j=1,2,\cdots,m).$$

Thus

$$\beta_1=\beta_2=\cdots=\beta_m=\beta=\left(1-\sum_{i=1}^{n}\omega_i\right)\alpha.$$

Comparison with (5) gives

$$\left(1-\sum_{i=1}^{n}\omega_i\right)(m\alpha-\sigma)=0.$$

So there are two cases. Either $\alpha=\sigma/m$ and then we have (19) and (20). Or $\alpha\neq\sigma/m$, in which case

$$\sum_{i=1}^{n}\omega_i=1.$$

Thus $\beta_1=\beta_2=\cdots=\beta_m=0$, so that (6) goes over into (21). □

We consider next the effect of specifying, in place of (2), the stronger (and more natural) condition

$$\Phi_j: I^n\to I, \qquad 1\leqq j\leqq m \tag{22}$$

(in fact, it suffices to posit this range restriction for $j\in J$, where J is some nonempty subset of $\{1,\cdots,m\}$).

THEOREM 3. *If* $m\geqq 3$, *an aggregation method satisfies* (22) *and* (3) *if, and only if, the aggregation functions* Φ_j *are of the form*

$$\Phi_j(z)=\sum_{i=1}^{n}\omega_i\zeta_i+\beta_j \qquad (j=1,2,\cdots,m), \tag{23}$$

where

$$\sum_{j=1}^{m}\beta_j=\left(1-\sum_{i=1}^{n}\omega_i\right)\sigma, \tag{24}$$

and for each $j\in J$ [*where* (22) *is supposed to hold*]

$$\mu-\mu\Sigma^{**}-\nu\Sigma^{*}<_1\beta_j<_2\nu-\nu\Sigma^{**}-\mu\Sigma^{*} \tag{25}$$

where $\Sigma^* = \Sigma^* \omega_i$ *denotes the sum of the negative weights and* $\Sigma^{**} = \Sigma^{**}\omega_i$ *denotes the sum of the positive weights, and the two inequality symbols* $<_1$ *and* $<_2$ *are either* $<$ *or* $\leqq$ *according to the type of the interval* I:

(A) *When* $I = [\mu, \nu]$, $<_1$ *is* $\leqq$ *and* $<_2$ *is* $\leqq$.

(B) *When* $I = [\mu, \nu[$,

(i) $<_1$ *is* $\leqq$ *and* $<_2$ *is* $\leqq$ *if* $\Sigma^* \neq 0$ *and* $\Sigma^{**} \neq 0$,
(ii) $<_1$ *is* $\leqq$ *and* $<_2$ *is* $\leqq$ *if* $\Sigma^* = 0$ *and* $\Sigma^{**} \neq 0$,
(iii) $<_1$ *is* $\leqq$ *and* $<_2$ *is* $<$ *if* $\Sigma^* \neq 0$ *and* $\Sigma^{**} = 0$,
(iv) $<_1$ *is* $\leqq$ *and* $<_2$ *is* $<$ *if* $\Sigma^* = 0$ *and* $\Sigma^{**} = 0$.

(C) *When* $I =]\mu, \nu]$,

(i) $<_1$ *is* $\leqq$ *and* $<_2$ *is* $\leqq$ *if* $\Sigma^* \neq 0$ *and* $\Sigma^{**} \neq 0$,
(ii) $<_1$ *is* $\leqq$ *and* $<_2$ *is* $\leqq$ *if* $\Sigma^* = 0$ *and* $\Sigma^{**} \neq 0$,
(iii) $<_1$ *is* $<$ *and* $<_2$ *is* $\leqq$ *if* $\Sigma^* \neq 0$ *and* $\Sigma^{**} = 0$,
(iv) $<_1$ *is* $<$ *and* $<_2$ *is* $\leqq$ *if* $\Sigma^* = 0$ *and* $\Sigma^{**} = 0$.

(D) *When* $I =]\mu, \nu[$,

(i) $<_1$ *is* $\leqq$ *and* $<_2$ *is* $\leqq$ *if* $\Sigma^* \neq 0$ *and* $\Sigma^{**} \neq 0$,
(ii) $<_1$ *is* $\leqq$ *and* $<_2$ *is* $\leqq$ *if* $\Sigma^* = 0$ *and* $\Sigma^{**} \neq 0$,
(iii) $<_1$ *is* $\leqq$ *and* $<_2$ *is* $\leqq$ *if* $\Sigma^* \neq 0$ *and* $\Sigma^{**} = 0$
(iv) $<_1$ *is* $<$ *and* $<_2$ *is* $<$ *if* $\Sigma^* = 0$ *and* $\Sigma^{**} = 0$.

Proof. The specification (22) implies (2) as $J \neq \varnothing$ and so, by Theorem 1, (22) and (3) imply (23) and (24). What remains to be done is to examine what relations between the constants ω_i's and β_j's in (23) should correspond to the requirement $\Phi_j(I^n) \subseteq I$ in (22). We analyze the case (B) when $I = [\mu, \nu[$ in full and omit the details for the cases (A), (C) and (D).

With $I = [\mu, \nu[$, the range of Φ_j over I^n given by (23) is:

(i) $]\nu\Sigma^* + \mu\Sigma^{**}, \mu\Sigma^* + \nu\Sigma^{**}[+ \beta_j$ if $\Sigma^* \neq 0$ and $\Sigma^{**} \neq 0$,

(ii) $[\mu\Sigma^{**}, \nu\Sigma^{**}[+ \beta_j$ if $\Sigma^* = 0$ and $\Sigma^{**} \neq 0$,

(iii) $]\nu\Sigma^*, \mu\Sigma^*] + \beta_j$ if $\Sigma^* \neq 0$ and $\Sigma^{**} = 0$

and

(iv) $\{0\} + \beta_j$ if $\Sigma^* = 0$ and $\Sigma^{**} = 0$.

For each case, the inclusion of the range of Φ_j by $[\mu, \nu[$ corresponds to the inequalities (25) under (B) (i)–(iv). □

Remark. For each Φ_j given by (23) on I^n, the range is an interval of length $(\nu - \mu)(\Sigma^{**} - \Sigma^*) = (\nu - \mu)\sum_{i=1}^n |\omega_i|$. The inequalities (25) imply in particular that $\mu - \mu\Sigma^{**}\omega_i - \nu\Sigma^*\omega_i \leqq \nu - \nu\Sigma^{**}\omega_i - \mu\Sigma^*\omega_i$ which is equivalent to

$$(\nu - \mu)\sum_{i=1}^{n} |\omega_i| \leqq \nu - \mu$$

and reflects the fact that, if the range of Φ_j is to be in I, its length must not exceed that of I. Since $\nu - \mu > 0$, we get

$$\sum_{i=1}^{n} |\omega_i| \leqq 1, \tag{26}$$

and it implies in particular that $\sum_{i=1}^n \omega_i \leqq 1$.

If $\sum_{i=1}^{n} \omega_i < 1$, we may rewrite (23) as

$$\Phi_j(z) = \sum_{i=1}^{n} \omega_i(\zeta_i - \sigma_j) + \sigma_j, \qquad j = 1, \cdots, m, \tag{27}$$

where $\sigma_j = \beta_j/(1 - \sum_{i=1}^{n} \omega_i)$ and, by (24), $\sum_{j=1}^{m} \sigma_j = \sigma$. The aggregation functions Φ_j might arise in practice in the above form (27), if our group of n individuals are advisors to some external decisionmaker whose preferred allocations, prior to consulting with the group, are given by the numbers $\sigma_1, \cdots, \sigma_m$. (Note that (19) with (20) is the special case $\sigma_j = \sigma/m$ of (27).)

If $\sum_{i=1}^{n} \omega_i = 1$ in (23), we obtain from (26) that $\Sigma^* \omega_i = 0$ and so all weights are nonnegative. Furthermore (25) gives $\beta_j = 0$ for all $j \in J$. In conclusion, Theorem 2 and Theorem 3 can be combined to give the following characterization of aggregation by ordinary weighted arithmetic means:

THEOREM 4. *Let $m \geqq 3$ and $I = [\mu, \nu]$, $[\mu, \nu[$, $]\mu, \nu]$ or $]\mu, \nu[$ be an interval where $\mu < \nu$ are constants satisfying* (1). *Then a sequence of functions $\Phi_j: I^n \to I$ satisfies* (3) *and $\Phi_j(\alpha) = \alpha$ for some $\alpha \in I$ where $\alpha \neq \sigma/m$ if, and only if, there exists a sequence of weights $\omega_1, \cdots, \omega_n$, nonnegative with sum* 1, *such that, for all $z = (\zeta_1, \cdots, \zeta_n) \in I^n$,*

$$\Phi_j(z) = \sum_{i=1}^{n} \omega_i \zeta_i \qquad (j = 1, \cdots, m).$$

Several proofs of the above have previously appeared in the literature for the special case $I = [0, \sigma]$ and $\alpha = 0$. See [3], [4] and [7, Thm. 6.4].

Appendix.

PROPOSITION. *If, for $\psi: [-\varepsilon, \varepsilon]^n \to \mathbb{R}$*

$$\psi(x + y) = \psi(x) + \psi(y) \text{ whenever } x, y, x + y \in [-\varepsilon, \varepsilon]^n, \tag{28}$$

then there exists a $\bar{\psi}: \mathbb{R}^n \to \mathbb{R}$ satisfying

$$\bar{\psi}(x + y) = \bar{\psi}(x) + \bar{\psi}(y) \quad \text{for all } x, y \in \mathbb{R}^n, \tag{29}$$

and

$$\bar{\psi}(x) = \psi(x) \quad \text{for all } x \in [-\varepsilon, \varepsilon]^n. \tag{30}$$

Proof. From (28), $\psi(kx) = k\psi(x)$ for $kx \in [-\varepsilon, \varepsilon]^n$, that is,

$$\psi\left(\frac{1}{k} z\right) = \frac{1}{k} \psi(z) \quad \text{for all } z \in [-\varepsilon, \varepsilon]^n. \tag{31}$$

Let $x \in \mathbb{R}^n$ be arbitrary. There exists a positive integer k such that $u = x/k \in [-\varepsilon, \varepsilon]^n$. Define

$$\bar{\psi}(x) = k\psi(u) = k\psi\left(\frac{1}{k} x\right) \qquad (x \in \mathbb{R}^n, u \in [-\varepsilon, \varepsilon]^n). \tag{32}$$

This definition is unambiguous: If $x = ku = lv$ $(u, v \in [-\varepsilon, \varepsilon]^n)$ then, by (31),

$$l\psi(v) = kl\psi\left(\frac{v}{k}\right) = kl\psi\left(\frac{u}{l}\right) = k\psi(u),$$

as asserted. Note that (32) implies (30) for $k = 1$.

Finally, we prove (32) by choosing, for given $x, y \in \mathbb{R}^n$, an integer k so that x/k, y/k, $(x+y)/k$ are all in $[-\varepsilon, \varepsilon]^n$. By (32) and (28)

$$\bar{\psi}(x+y) = k\psi\left[\frac{1}{k}(x+y)\right] = k\psi\left(\frac{1}{k}x\right) + k\psi\left(\frac{1}{k}y\right) = \bar{\psi}(x) + \bar{\psi}(y)$$

for all $x, y \in \mathbb{R}^n$. □

REFERENCES

[1] J. ACZÉL, *Lectures on Functional Equations and their Applications*, Academic Press, New York–London, 1966.

[2] J. ACZÉL AND C. WAGNER, *A characterization of weighted arithmetic means*, SIAM J. Alg. Discrete Meth., 1 (1980), pp. 259–260.

[3] ———, *Rational group decisionmaking generalized: The case of several unknown functions*, C.R. Math. Rep. Acad. Sci. Canada, 3 (1981), pp. 138–142.

[4] J. ACZÉL, P. L. KANNAPPAN, C. T. NG AND C. WAGNER, *Functional equations and inequalities in rational group decisionmaking*, General Inequalities 3: Proc. Third International Conference on General Inequalities in Oberwolfach, 1981, to appear.

[5] Z. DARÓCZY AND L. LOSONCZI, *Über Erweiterungen der auf einer Punktmenge additiven Funktionen*, Publi. Math. Debrecen, 14 (1967), 239–245.

[6] J. G. DHOMBRES AND R. GER, *Conditional Cauchy Equations*, Glasnik Mat., 13 (33) (1978), pp. 39–62. (In Russian.)

[7] K. LEHRER AND C. WAGNER, *Rational Consensus in Science and Society*, Reidel Publishing, Boston–Dordrecht, 1981.

[8] C. T. NG, *Representation for measures of information with the branching property*, Inform. Control, 25 (1974), pp. 45–56.

[9] C. WAGNER, *Allocation, Lehrer models, and the consensus of probabilities*, Theory and Decision, 14 (1982), pp. 207–220.