Interpolation series for continuous functions on \( \pi \)-adic completions of \( GF(q, x) \)\(^*\)  

by  

CARL G. WAGNER (Knoxville, Tenn.)  

1. Introduction. In 1944 Dieudonné [7] proved an analogue of the Weierstrass Approximation Theorem for continuous functions of a \( p \)-adic variable. In 1958 Mahler [8] sharpened this result by exhibiting a series expansion for continuous functions defined on the \( p \)-adic integers. He showed that every such function \( f \) is the uniform limit of an interpolation series  

\[
f(t) = \sum_{n=0}^{\infty} A_n \binom{t}{n},
\]

where the coefficients \( A_n \) are uniquely determined by  

\[
A_n = \partial^n f(0) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} f(n-k).
\]

In the present paper we choose an irreducible element \( \pi \) from the polynomial ring \( GF(q, x) \) over the finite field \( GF(q) \) and use it to equip the function field \( GF(q, x) \) with a \( \pi \)-adic absolute value. We denote by \( F_\pi \) the completion of \( GF(q, x) \) for this absolute value and by \( I_\pi \) the valuation ring of \( F_\pi \). The aforementioned theorem of Dieudonné may easily be seen to generalize to the case of a locally compact non-archimedean field. Hence, every continuous function \( f: K \rightarrow F_\pi \), where \( K \) is a compact subset of \( F_\pi \), is the uniform limit of some sequence of polynomials over \( F_\pi \). Our aim in this paper is to prove some Mahler type theorems for such functions.  

We mention that Amice [1] has already constructed a certain type of series approximation for continuous functions defined on locally compact non-archimedean fields. In the process, Amice characterized those sequences ("suites très bien réparties") in the domain of a continuous function with respect to which a Newton type interpolation procedure will yield

\* This research was supported in part by the National Science Foundation, under Research Grant GP - 7855.
a uniformly convergent series approximation for that function. In particular, the nonnegative rational integers, ordered in the usual way, constitute such a sequence in the $p$-adic integers, and so Mahler's result appears as a special case of Amice's Interpolation Theorem [1].

In what follows, we exhibit a "suite très bien répartie" in $I_\pi$, denoted \( \{m_i\} \), consisting of a special sequential ordering of $GF[q, x]$. Specializing Amice, we prove (Theorem 4.4) that for every continuous function $f: I_\pi \rightarrow F_\pi$ there exists a unique sequence $\{A_i\}$ in $F_\pi$ such that

\[
f(t) = \sum_{i=0}^{\infty} A_i Q_i(t),
\]

where $Q_i(t)$ is the $i$th Newton interpolation polynomial for the interpolation sequence $\{m_i\}$, and (1.3) converges uniformly on $I_\pi$. We add that $\{A_i\}$ is always a null sequence, i.e., $\lim_{i=\infty} A_i = 0$. Moreover, the above result may be extended to continuous functions $f: K \rightarrow F_\pi$, where $K$ is any compact subset of $F_\pi$, by employing a Urysohn type theorem for totally disconnected spaces due to Dieudonné [7].

We may regard the foregoing approach to constructing function field analogues of Mahler's result as deriving from the observation that the polynomials $\binom{t}{n}$ are the Newton interpolation polynomials for the nonnegative rational integers. From this standpoint, the crucial problem, completely solved by Amice, is that of identifying those sequences in $I_\pi$ for which the associated Newton polynomials yield interpolation series for continuous functions.

If, instead, one regards the sequence $\binom{t}{n}$ merely as an ordered basis of the $Q_\pi$-vector space $Q_\pi[t]$, then one is led to ask which ordered bases of the $F_\pi$-vector space $F_\pi[t]$ yield interpolation series for continuous functions on $I_\pi$. In this connection, it is of interest to recall that the sequence $\binom{t}{n}$ has the further property of being an ordered basis of the $\mathbb{Z}$-module of polynomials over $Q$ which map $\mathbb{Z}$ into $\mathbb{Z}$ (and also of the $\mathbb{Z}_\pi$-module of polynomials over $Q_\pi$ that map $\mathbb{Z}_\pi$ into $\mathbb{Z}_\pi$, where $\mathbb{Z}_\pi$ is the valuation ring of $Q_\pi$).

The function field analogue of the latter property is that of being an ordered basis of the $I_\pi$-module of polynomials over $F_\pi$ that map $I_\pi$ into $I_\pi$. Let $\{H_i(t)\}$ be such a basis. We prove (Theorem 4.5) that for every continuous function $f: I_\pi \rightarrow I_\pi$ there exists a unique null sequence $\{B_i\}$ in $I_\pi$ such that

\[
f(t) = \sum_{i=0}^{\infty} B_i H_i(t),
\]

where (1.4) converges uniformly on $I_\pi$. 
The above theorem may be applied to a sequence of polynomials \( \{G_i(t)/g_i\} \) introduced in 1948 by Carlitz [4]. This leads to the following characterization (Theorem 5.1) of continuous linear operators on the GF\((q)\)-vector space \( I_n \): Let \( f: I_n \rightarrow I_n \) be continuous. If the (unique) interpolation series for \( f \) constructed from the Carlitz polynomials is given by

\[
   f(t) = \sum_{i=0}^{\infty} A_i \frac{G_i(t)}{g_i},
\]

then \( f \) is a linear operator on the GF\((q)\)-vector space \( I_n \) if and only if \( A_i = 0 \) for \( i \neq q^k \), where \( k \geq 0 \).

The author wishes to thank Professor Carlitz for his guidance and encouragement during the preparation of this paper, and of the doctoral dissertation on which it is based.

2. Preliminaries. Let GF\((q)\) be a finite field of cardinality \( q \). Denote by GF\([q, x]\) the ring of polynomials in an indeterminate \( x \) over GF\((q)\), and by GF\((q, x)\) the quotient field of GF\([q, x]\). Let \( \pi \in \text{GF}\([q, x]\) \) be an irreducible polynomial of degree \( d \). Then every nonzero \( \alpha \in \text{GF}(q, x) \) may be written, in essentially unique fashion,

\[
   \alpha = \pi^{n} \frac{m_1}{m_2},
\]

where \( n \) is integral, and \( m_1 \) and \( m_2 \) are polynomials prime to each other and to \( \pi \).

Define a function \( v_n: \text{GF}(q, x) - \{0\} \rightarrow \mathbb{Z} \) by

\[
   v_n(\alpha) = n,
\]

where \( n \) is written as in (2.1). It follows that

\[
   v_n(\alpha \beta) = v_n(\alpha) + v_n(\beta) \quad (\alpha \beta \neq 0)
\]

and

\[
   v_n(\alpha + \beta) \geq \min \{v_n(\alpha), v_n(\beta)\} \quad (\alpha, \beta, \alpha + \beta \neq 0).
\]

Fixing a real number \( b \) such that \( 0 < b < 1 \), define the \( \pi \)-adic absolute value \( |\cdot|_\pi \) on GF\((q, x)\) as follows:

\[
   |0|_\pi = 0,
\]

\[
   |\alpha|_\pi = b^{v_n(\alpha)} \quad (\alpha \neq 0).
\]

By familiar methods GF\((q, x)\) may be embedded as a dense subfield in an essentially unique complete field, denoted \( F_\pi \). With respect to the
extended absolute value, \( F_\pi \) is a discrete non-archimedean field. Equipped
with the metric \( d_\pi \), defined by

\[(2.7) \quad d_\pi(a, \beta) = |a - \beta|_\pi,\]

\( F_\pi \) is a metric field. In particular, polynomial functions over \( F_\pi \) are
continuous.

Denote by \( I_\pi \) the valuation ring of \( F_\pi \), i.e.,

\[I_\pi = \{a \in F_\pi : |a|_\pi \leq 1\}\]

Then the valuation ideal

\[(\pi) = \{a \in I_\pi : |a|_\pi < 1\}\]

is maximal and the residue class field \( I_\pi/(\pi) \) is isomorphic to \( GF(q^d) \),
where \( d = \deg \pi \).

Let \( \Gamma \) be a complete set of representatives of \( I_\pi/(\pi) \) in \( I_\pi \). Then every
nonzero \( a \in F_\pi \) may be uniquely represented as a \( \pi \)-series,

\[(2.8) \quad a = \pi^k \sum_{i=0}^{\infty} a_i \pi^i,\]

where \( a_i \in \Gamma, \pi \nmid a_0 \) in \( I_\pi \), and \(|a| = b^k \) \[6\]. In particular, \( \Gamma \) may be taken
to be the set of polynomials in \( GF[q, \pi] \) having degree less than \( d \).

For \( a \in F_\pi \) and \( k \) any integer, let

\[(2.9) \quad B_k(a) = \{\beta \in F_\pi : |\beta - a|_\pi < b^k\} = \{\beta \in F_\pi : |\beta - a|_\pi < b^{k-1}\}.\]

Then the collection \( \{B_k(a) : k \geq 0\} \) is a fundamental system of open-
closed neighborhoods of \( a \); hence \( F_\pi \) is totally disconnected.

Again, let \( \Gamma \) be a complete set of representatives of \( I_\pi/(\pi) \) in \( I_\pi \).
Given \( \varepsilon > 0 \), let \( k \) be a positive integer such that \( b^k < \varepsilon \). Let

\[(2.10) \quad A = \{a \in I_\pi : a = a_0 + a_1 \pi + \ldots + a_{k-1} \pi^{k-1}\}\]

where \( a_i \in \Gamma \). Then \( A \) has \( q^{kd} \) elements and the collection

\[(2.11) \quad \{B_k(a) : a \in A\}\]

is a pairwise disjoint open cover of \( I_\pi \), all of the members of which have
radius less than \( \varepsilon \). It follows that \( I_\pi \) (and, therefore, every closed and
bounded subset of \( F_\pi \)) is compact. (In fact, the Heine-Borel Theorem
holds in all locally compact non-archimedean fields, a result due to
Sch"obe [9].)

In the special case \( \pi = a \), the complete field \( F_\pi \) may be identified
with the field of formal power series over \( GF(q) \), for by (2.8) every nonzero
\( a \in F_\pi \) may be written

\[(2.12) \quad a = \sum_{i=-\infty}^{\infty} a_i \pi^i,\]

C. G. Wagner
where \(a_i \in \text{GF}(q)\), all but a finite number of the \(a_i\) vanish for \(i < 0\), and \(|a|_\infty = b^w\), for \(n\) the smallest integer such that \(a_n \neq 0\).

There would, in fact, be no loss of generality in restricting the investigation we have in mind to the case of \(\pi\)-adic absolute values; for it is known that every locally compact Hausdorff field having nonzero characteristic is topologically isomorphic to a field of formal power series in one indeterminate over some finite field ([10], pp. 12–22). In the case of the fields \(F_n\) we may specialize this result as follows.

**Theorem 2.1.** Let \(F_n\) be the completion of \(GF(q, x)\) for the absolute value \(|\cdot|_\pi\), where \(\pi\) is an irreducible polynomial of degree \(d\). Then \(F_n\) is topologically isomorphic to a field of formal power series in one indeterminate over the finite field \(GF(q^d)\).

**Proof.** In view of representations (2.8) and (2.12), it suffices to show that \(I\), a complete set of representatives of \(I_n(\pi)\) in \(I_n\), may be chosen in such a way that \(I\) is a subfield of \(I_n\).

Let \(a \in I_n\). Since \(I_n(\pi)\) is isomorphic to \(GF(q^d)\), it follows that \(\pi | a^{\pi d} - a\), and hence that

\[
\pi^{(n-1)d} | a^{\pi d} - a^{(n-1)d},
\]

for all natural numbers \(n\). Therefore, the series

\[
(2.13) \quad \alpha + (\alpha^{\pi d} - \alpha) + (\alpha^{2\pi d} - \alpha^{\pi d}) + \ldots
\]

converges, i.e., \(\lim_{n \to \infty} \alpha^{\pi n d}\) exists for all \(a \in I_n\).

Define a function \(w: I_n \to I_n\) by

\[
(2.14) \quad w(\alpha) = \lim_{n \to \infty} \alpha^{\pi n d}.
\]

Then \(w\) is an endomorphism of the ring \(I_n\) with kernel \((\pi)\), and so \(w(I_n)\) is a subfield of \(I_n\) isomorphic to \(GF(q^d)\). By (2.13) and (2.14), it follows that \(w(\alpha) = \alpha (\text{mod} \pi)\); hence we may take \(I = w(I_n)\), as desired.

To conclude this section, we recall that, in addition to the \(\pi\)-adic absolute values, \(GF(q, x)\) admits only one other non-trivial absolute value, \(|\cdot|_\infty\), defined by

\[
(2.15) \quad \frac{|m_1|}{|m_2|_\infty} = b^{\deg m_2 - \deg m_1},
\]

for \(m_1, m_2\) nonzero elements of \(GF[q, x]\) and \(0 < b < 1\) ([6], pp. 45–47). The completion of \(GF(q, x)\) for \(|\cdot|_\infty\), denoted by \(F_\infty\), may be seen to consists of the set of all descending formal power series over \(GF(q)\),

\[
(2.16) \quad \alpha = \sum_{i=-\infty}^{\infty} a_i x^{-i},
\]
where \( a_i \in \text{GF}(q) \), all but a finite number of these coefficients vanish for \( i < 0 \), and \( |\alpha|_\infty = b^s \), \( n \) the smallest integer such that \( a_n \neq 0 \).

In what follows, we shall appeal to the obvious topological isomorphism between \( F_\infty \) and \( F_\infty \) to omit an explicit treatment of the problem of approximating continuous functions in \( F_\infty \). There appears, however, to be no particular advantage in a similar appeal to Theorem 2.1, and so we shall state our results for the fields \( F_\infty \).

3. A special ordering of \( \text{GF}(q, \alpha) \). Let \( \alpha \in \text{GF}(q, \alpha) \) be an irreducible polynomial of degree \( d \). We define a sequential ordering of \( \text{GF}(q, \alpha) \) which has the property of being, in the terminology of Amice [1], “très bien répartie” in \( I_n \). Let \( (a_0, a_1, \ldots, a_{qd-1}) \) be a fixed ordering of the polynomials in \( \text{GF}(q, \alpha) \) of degree \( < d \) such that \( a_0 = 0, a_1 = 1 \), and \( \deg a_i \leq \deg a_j \) for \( 1 \leq i \leq j \). The special sequence \( \{m_n\} \), running through \( \text{GF}(q, \alpha) \), is defined as follows. If

\[
(3.1) \quad n = k_0 + k_1 q^d + \ldots + k_s q^{sd} \quad (0 \leq k_i < q^d),
\]

set

\[
(3.2) \quad m_n = a_{k_0} + a_{k_1} \alpha + \ldots + a_{k_s} \alpha^s.
\]

Theorem 3.1. For any integers \( s \geq 0 \) and \( k \geq 1 \), the set

\[
(3.3) \quad \{m_{i+sq^d} : 0 \leq i < q^{sd}\}
\]

is a complete residue system \( \text{(mod} \pi^k \text{)} \).

Proof. As there is no “overlap” in the \( q^d \)-adic expansions (3.1) of \( i \) and \( q^{sd} \), it follows that

\[
(3.4) \quad m_{i+sq^d} = m_i + m_{sq^d}.
\]

The set \( \{m_i : 0 \leq i < q^d\} \) is a complete residue system \( \text{(mod} \pi^k \text{)} \), and this property is preserved under shifting by the additive constant \( m_{sq^d} \).

Let

\[
(3.5) \quad S_n = \{m_0, m_1, \ldots, m_{n-1}\} \quad (n \geq 1),
\]

and let

\[
(3.6) \quad \ell(a; k, n) = \text{card}(B_k(a) \cap S_n),
\]

with \( a \in I_n \), \( n, k \geq 1 \), and \( B_k(a) \) as in (2.9). Then the following theorem is a straightforward consequence of Theorem 3.1.

Theorem 3.2. For every \( a \in I_n \), and for all positive integers \( n \) and \( k \),

\[
(3.7) \quad \left[ \frac{n}{q^d} \right] \leq \ell(a; k, n) \leq \left[ \frac{n-1}{q^d} \right] + 1.
\]
Furthermore,

\begin{equation}
q(m_n; k, n) = \left[ \frac{n}{q^{kd}} \right].
\end{equation}

We now introduce an ordered basis of the \( F_n \)-vector space \( F_n[t] \), consisting of the Newton interpolation polynomials for the interpolation sequence \( \{m_n\} \), defined by (3.2). Set

\begin{equation}
P_0(t) = 1, \quad P_n(t) = (t - m_0)(t - m_1) \cdots (t - m_{n-1}) \quad (n \geq 1),
\end{equation}

and

\begin{equation}
Q_0(t) = 1, \quad Q_n(t) = P_n(t)/P_n(m_n) \quad (n \geq 1).
\end{equation}

Since \( \text{deg} Q_n(t) = n \), \( \{Q_n(t)\} \) is an ordered basis of the \( F_n \)-vector space \( F_n[t] \). Hence, every polynomial \( g(t) \in F_n[t] \) of degree \( \leq n \) may be written uniquely as

\begin{equation}
g(t) = \sum_{i=0}^{n} A_i Q_i(t).
\end{equation}

To derive a formula for the coefficients \( A_i \), let \( g_r(t) \) be the unique polynomial of degree \( \leq r \) for which \( g_r(m_j) = g(m_j) \) for \( 0 \leq j \leq r \). Then

\begin{equation}
g_r(t) = \sum_{j=0}^{r} A_j Q_j(t) = \sum_{j=0}^{r} \frac{P_{r+1}(t) g(m_j)}{(t - m_j) P_{r+1}(m_j)},
\end{equation}

where the second equality above is the result of Lagrange interpolation. It follows from (3.12) that

\begin{equation}
g_i(t) - g_{i-1}(t) = A_i Q_i(t) = \left( P_i(m_i) \sum_{j=0}^{i} \frac{g(m_j)}{P_{i+1}(m_j)} \right) Q_i(t).
\end{equation}

Hence

\begin{equation}
A_i = P_i(m_i) \sum_{j=0}^{i} \frac{g(m_j)}{P_{i+1}(m_j)}.
\end{equation}

The following two theorems imply that the sequence \( \{Q_n(t)\} \) is, in fact, an ordered basis of the \( I_\pi \)-module of polynomials over \( F_n \) that map \( I_\pi \) into itself. In the remainder of the paper the subscript \( \pi \) will be omitted from the symbols \( v_n \) and \( |_n \).

**Theorem 3.3.** For all \( t \in I_\pi \), \( |Q_n(t)| \leq 1 \).

**Proof (Amice [1]).** In virtue of (2.6) it suffices to show that

\begin{equation}
v(P_n(t)) \geq v(P_n(m_n)).
\end{equation}
By (3.7),
\begin{align}
(3.16) \quad v(P_n(t)) &= \sum_{t=0}^{n-1} v(t-m_t) = \sum_{k=1}^{\infty} k[q(t; k, n) - q(t; k+1, n)] \\
&= \sum_{k=1}^{\infty} q(t; k, n) \geq \sum_{k=1}^{\infty} \left[ \frac{n}{q^n} \right].
\end{align}

But, by (3.8),
\begin{align}
(3.17) \quad v(P_n(m_n)) &= \sum_{k=1}^{\infty} q(m_n; k, n) = \sum_{k=1}^{\infty} \left[ \frac{n}{q^n} \right],
\end{align}
from which the desired result follows.

**Theorem 3.4.** Let \( g(t) \in E_n[1] \), and write
\begin{align}
(3.18) \quad g(t) &= \sum_{i=0}^{n} A_i Q_i(t).
\end{align}

Then \( g \) maps \( I_n \) into itself if and only if \( A_i \in I_n \), for \( 0 \leq i \leq n \).

**Proof.** Sufficiency. By Theorem 3.3, \( |Q_i(t)| \leq 1 \) if \( |t| \leq 1 \), so if \( |A_i| \leq 1 \), \( |g(t)| \leq 1 \), since \(| | \) is non-archimedean.

Necessity. By (3.14), it suffices to show that, for all \( j \leq i \),
\begin{align}
(3.19) \quad v(P_i(m_i)) &\geq v(P'_i(m_j)).
\end{align}

By (3.17),
\begin{align}
(3.20) \quad v(P_i(m_i)) &= \sum_{k=1}^{\infty} \left[ \frac{i}{q^n} \right].
\end{align}

We show that
\begin{align}
(3.21) \quad v(P'_i(m_j)) &\leq \sum_{k=1}^{\infty} \left[ \frac{j}{q^n} \right].
\end{align}

Since
\begin{align}
(3.22) \quad P'_i(m_j) &= (m_j - m_0) \cdots (m_j - m_{j-1}) (m_j - m_{j+1}) \cdots (m_j - m_i),
\end{align}
inequality (3.21) is obvious for \( j = i \), so assume that \( j < i \). Denote by \( S(i, j) \) the set \( S_{i+1} - \{m_j\} \). Then
\begin{align}
(3.23) \quad v(P'_i(m_j)) &= \sum_{i=0}^{i} v(m_j - m_i) \\
&= \sum_{k=1}^{\infty} k[\text{card}(B_k(m_j) \cap S(i, j)] - \text{card}(B_{k+1}(m_j) \cap S(i, j)) \]
&= \sum_{k=1}^{\infty} \text{card}(B_k(m_j) \cap S(i, j)) \leq \sum_{k=1}^{\infty} \left[ \frac{i}{q^n} \right],
\end{align}
as desired.
4. Interpolation theorems. We require a preliminary theorem, due to Amice [1], which specifies conditions under which certain finite subsets of \( \{Q_i(t)\} \) are locally constant (mod \( \pi \)). As in the case of a previous theorem, we include, for completeness, a specialized version of the proof given by Amice.

**Theorem 4.1.** Let \( \pi \in \mathbb{F}[q, x] \) be an irreducible polynomial of degree \( d \), and let \( |\pi| = b \). Then, for all \( k \geq 1 \) and for all \( i \) such that \( 0 \leq i \leq q^{kd} - 1 \), if \( t_1, t_2 \in I_\pi \) and \( |t_1 - t_2| \leq b^k \), then

\[
|Q_i(t_1) - Q_i(t_2)| \leq b.
\]

**Proof.** It suffices to show that for all \( i, j \) with \( 0 \leq i, j \leq q^{kd} - 1 \), if \( t \in B_k(m_j) \), then \( |Q_i(t) - Q_i(m_j)| \leq b \). The cases \( (1) \ j < i \) and \( (2) \ j \geq i \) are treated separately.

(1) If \( j < i \), then \( |Q_i(t) - Q_i(m_j)| = |Q_i(t)| \), and so it suffices to show that, for \( t \in B_k(m_j) \),

\[
\begin{align*}
\sum_{r=1}^{\infty} \varepsilon(t; r, i) > \sum_{r=1}^{\infty} \varepsilon(m_j; r, i),
\end{align*}
\]

or, as in (3.16), that

\[
\sum_{r=1}^{\infty} \varepsilon(t; r, i) > \sum_{r=1}^{\infty} \varepsilon(m_j; r, i).
\]

By Theorem 3.2,

\[
\varepsilon(t; r, i) \geq \varepsilon(m_j; r, i) = \left[ \frac{i}{q^d} \right].
\]

When \( r = k \), however, inequality (4.3) is strict, since

\[
\varepsilon(t; k, i) = 1 \quad \text{and} \quad \varepsilon(m_j; k, i) = 0.
\]

(2) Let \( i \leq j \). By hypothesis, \( |t - m_j| \leq b^k \). For all \( r \) with \( 0 \leq r \leq i - 1 < j \), \( m_r \neq m_j(\text{mod } q^k) \), and so

\[
|t - m_j| \leq b |m_j - m_r|,
\]

or

\[
|(t - m_r) - (m_j - m_r)| \leq b |m_j - m_r|,
\]

or

\[
\left| \frac{t - m_r}{m_j - m_r} - 1 \right| \leq b.
\]

Hence, for each \( r \), there is an \( a_r \in I_\pi \) such that

\[
\frac{t - m_r}{m_j - m_r} = 1 + \pi a_r,
\]

(4.7)
and so, there is a $\beta \in I_\pi$ such that
\begin{equation}
\prod_{r=0}^{t-1} \frac{t-m_r}{m_j-m_r} = \frac{Q_\pi(t)}{Q_\pi(m_j)} = 1 + \pi \beta.
\end{equation}
Therefore,
\begin{equation}
\left| \frac{Q_\pi(t)}{Q_\pi(m_j)} - 1 \right| \leq b,
\end{equation}
and, by Theorem 3.3,
\begin{equation}
|Q_\pi(t) - Q_\pi(m_j)| \leq b |Q_\pi(m_j)| \leq b.
\end{equation}

The interpolation theorems announced in the Introduction are included in the following sequence of theorems.

**Theorem 4.2.** Let $\pi, b, a$ be as in Theorem 4.1. Let $f: I_\pi \to I_\pi$ be continuous. Then there is an integer $k \geq 1$ and a continuous function $h: I_\pi \to I_\pi$ such that
\begin{equation}
f(t) = \sum_{i=0}^{q^{kd}-1} f(m_i) \chi_i(t) + \pi h(t),
\end{equation}
where $\chi_i$ is the characteristic function of the set $B_k(m_i)$.

**Proof.** Since $I_\pi$ is compact, $f$ is uniformly continuous. Hence, there is an integer $k \geq 1$ such that, for all $i$, $0 \leq i \leq q^{kd}-1$, if $t \in B_k(m_i)$, then $|f(t) - f(m_i)| \leq b$. Thus, there is a continuous function $h^i: B_k(m_i) \to I_\pi$ such that, for $t \in B_k(m_i)$,
\begin{equation}
f(t) = f(m_i) + \pi h^i(t).
\end{equation}
Since the sets $B_k(m_i)$ are a pairwise disjoint open-closed cover of $I_\pi$, (4.11) may be gotten by setting $h(t) = h^i(t)$ for $t \in B_k(m_i)$.

**Theorem 4.3.** Let $f: I_\pi \to I_\pi$ be continuous. Then there is an integer $k \geq 1$, a continuous function $f_1: I_\pi \to I_\pi$, and a sequence $\{a_i: 0 \leq i \leq q^{kd}-1\}$ in $I_\pi$ such that
\begin{equation}
f(t) = \sum_{i=0}^{q^{kd}-1} a_i Q_\pi(t) + \pi f_1(t).
\end{equation}

**Proof.** Using the uniform continuity of $f$, determine $k$ as in Theorem 4.2. By Theorem 4.1, this $k$ is also associated with the uniform continuity of the functions $Q_\pi(t)$, $0 \leq i \leq q^{kd}-1$. Applying Theorem 4.2 to these functions, we get
\begin{equation}
Q_\pi(t) = \sum_{j=0}^{q^{kd}-1} Q_\pi(m_j) \chi_j(t) + \pi h_\pi(t).
\end{equation}
Since \( Q_j(m_j) = 0 \) when \( j < i \), system (4.14) is triangular. Solving for the functions \( x_i(t) \) in terms of the \( Q_i(t) \) and the error functions \( h_i(t) \), and substituting in (4.11), we get (4.13), where \( f_i(t) \) is expressed in terms of the error functions \( h_i(t) \).

**Theorem 4.4.** Let \( f: I_n \to I_n \) be continuous. Then there is a unique sequence \( \{A_i\} \) in \( F_n \) such that:

\[
(4.15) \quad f(t) = \sum_{i=0}^{\infty} A_i Q_i(t),
\]

where (4.15) converges uniformly on \( I_n \). Moreover, for all \( i, |A_i| \leq 1 \) and \( \lim_{i \to \infty} A_i = 0 \).

**Proof.** By Theorem 4.3 there is an integer \( k_0 \geq 1 \), a sequence \( \{a_i^0: 0 \leq i \leq q^{k_0} - 1\} \), and a continuous function \( f_i: I_n \to I_n \) such that

\[
(4.16) \quad f(t) = \sum_{i=0}^{q^{k_0} - 1} a_i^0 Q_i(t) + \pi f_i(t).
\]

Similarly, we may write

\[
(4.17) \quad f_i(t) = \sum_{i=0}^{q^{k_0} - 1} a_i^i Q_i(t) + \pi f_i(t).
\]

Iterating and substituting in (4.16) at each stage, we get

\[
(4.18) \quad f(t) = \sum_{i=0}^{M_{n-1} - 1} (a_i^0 + \pi a_i^1 + \ldots + \pi^{n-1} a_i^{n-1}) Q_i(t) + \pi f_n(t),
\]

where

\[
(4.19) \quad M_{n-1} = \max\{q^{k_0}, q^{k_1}, \ldots, q^{k_{n-1}}\}.
\]

Define the sequence \( \{A_i\} \) by

\[
(4.20) \quad A_i = \sum_{j=0}^{\infty} a_i^j.
\]

The series (4.20) converges to an element of \( I_n \), for \( |a_i^0| \leq 1 \). Also

\[
(4.21) \quad \lim_{i \to \infty} A_i = 0,
\]

for if \( i \geq M_{n-1} \), then \( a_i^0 = a_i^1 = \ldots = a_i^{n-1} = 0 \), and so \( |A_i| \leq b^n \).

Let \( k \geq M_{n-1} - 1 \). Then

\[
(4.22) \quad \left| \sum_{i=0}^{k} A_i Q_i(t) - \sum_{i=0}^{M_{n-1} - 1} (a_i^0 + \pi a_i^1 + \ldots + \pi^{n-1} a_i^{n-1}) Q_i(t) \right| \leq \max \left\{ \left| \sum_{i=M_{n-1} - 1}^{k} A_i Q_i(t) \right|, \left| \sum_{i=0}^{M_{n-1} - 1} (A_i - (a_i^0 + \ldots + \pi^{n-1} a_i^{n-1})) Q_i(t) \right| \right\} \leq b^n,
\]
and by (4.18)

\[ |f(t) - \sum_{i=0}^{n} A_i Q_i(t)| \leq \delta^n. \]

Hence (4.15) converges uniformly to \( f \) on \( I_n \). The coefficients \( A_i \) are uniquely determined by \( f \), since for each \( n \geq 0 \), the finite sum \( \sum_{i=0}^{n} A_i Q_i(t) \) is the unique polynomial of degree \( \leq n \) which takes the same values as \( f \) on the set \( \{m_0, \ldots, m_n\} \). Hence, by (3.14),

\[ A_i = P_i(m_i) \sum_{j=0}^{i} \frac{f(m_j)}{P_{i+1}'(m_j)}. \]

In the slightly more general case of a continuous function \( f: I_n \to F_n \), the boundedness of \( f \) implies the existence of an integer \( k \geq 0 \) such that \( \pi^k f: I_n \to I_n \). Hence

\[ \pi^k f(t) = \sum_{i=0}^{\infty} \left( P_i(m_i) \sum_{j=0}^{i} \frac{\pi^k f(m_j)}{P_{i+1}'(m_j)} \right) Q_i(t), \]

and so

\[ f(t) = \sum_{i=0}^{\infty} A_i Q_i(t), \]

where \( A_i \) is defined by (4.25).

In the case of a continuous function \( f: B_{\epsilon}(0) \to F_n \), where \( \epsilon < 0 \), define \( g: I_n \to F_n \) by \( g(t) = f(\pi^k t) \). Then by (4.21) and (4.27), we have, for all \( t \in I_n \),

\[ f(\pi^k t) = g(t) = \sum_{i=0}^{\infty} \left( P_i(m_i) \sum_{j=0}^{i} \frac{f(\pi^k m_j)}{P_{i+1}'(m_j)} \right) Q_i(t). \]

Hence, for all \( t \in B_{\epsilon}(0) \),

\[ f(t) = f(\pi^k(\pi^{-k} t)) = \sum_{i=0}^{\infty} \left( P_i(m_i) \sum_{j=0}^{i} \frac{f(\pi^k m_j)}{P_{i+1}'(m_j)} \right) Q_i(\pi^{-k} t). \]

It follows that every continuous function \( f: K \to F_n \), where \( K \) is a compact subset of \( F_n \), has a series expansion of the form (4.29), for \( K \subseteq B_{\epsilon}(0) \) for some \( \epsilon \leq 0 \) and, by a theorem of Dieudonné ([7], p. 82), any such \( f \) has a continuous extension to \( B_{\epsilon}(0) \).
Theorem 4.5. Let \( \{H_i(t)\} \) be an ordered basis of the \( I_n \)-module of polynomials over \( F_n \) that map \( I_n \) into itself. Let \( f: I_n \rightarrow I_n \) be continuous. Then there exists a unique null sequence \( \{B_i\} \) in \( I_n \) such that

\[
f(t) = \sum_{i=0}^{\infty} B_i H_i(t),
\]

where (4.30) converges uniformly on \( I_n \).

Proof. By Theorem 4.4,

\[
f(t) = \sum_{i=0}^{\infty} A_i Q_i(t),
\]

where \( A_i \in I_n \) and \( \lim_{i \to \infty} A_i = 0 \). By Theorem 3.3, for all \( j \geq 0 \), \( Q_j(t) \) may be written uniquely as

\[
Q_j(t) = \sum_{i=0}^{n_j} D_i H_i(t),
\]

where \( D_i \in I_n \). Set

\[
B_i = \sum_{j=0}^{\infty} A_i D_i.
\]

Since \( \lim_{i \to \infty} A_i = 0 \) and \( |D_i| \leq 1 \), (4.33) converges to an element of \( I_n \).

Moreover, \( \lim_{i \to \infty} B_i = 0 \), for, given any integer \( k \geq 0 \), let \( r \) be such that

\[
|A_j| \leq b^k \text{ if } j \geq r.
\]

Let \( i > \max\{n_0, \ldots, n_r\} \). Then \( D_i = 0 \) if \( j < r \), and so \( |B_i| \leq b^k \).

If \( k \geq 0 \), let \( r \) be such that \( |A_j| \leq b^k \) for \( j \geq r \) and

\[
\sum_{j=0}^{n} A_j Q_j(t) - f(t) \leq b^k
\]

for \( s \geq r \). If \( n \geq \max\{n_0, \ldots, n_r\} \), then

\[
\left| \sum_{i=0}^{n} B_i H_i(t) - \sum_{j=0}^{r} A_j Q_j(t) \right| = \left| \sum_{i=r+1}^{\infty} A_i \sum_{j=0}^{n} D_i H_i(t) \right| \leq b^k.
\]

Then (4.34) and (4.35) yield (4.30).

Moreover, \( \{B_i\} \), as defined in (4.33), is the only null sequence in \( I_n \) for which (4.30) holds. For suppose that

\[
f(t) = \sum_{i=0}^{\infty} C_i H_i(t),
\]

where \( \lim_{i \to \infty} C_i = 0 \). For all \( i \geq 0 \), write

\[
H_i(t) = \sum_{j=0}^{n_i} D_i Q_j(t),
\]

for \( s \geq r \). If \( n \geq \max\{n_0, \ldots, n_r\} \), then

\[
\left| \sum_{i=0}^{n} B_i H_i(t) - \sum_{j=0}^{r} A_j Q_j(t) \right| = \left| \sum_{i=r+1}^{\infty} A_i \sum_{j=0}^{n} D_i H_i(t) \right| \leq b^k.
\]

Then (4.34) and (4.35) yield (4.30).
where $E_j^T I_n$. A repetition of the preceding argument yields

\begin{equation}
(4.38) \quad f(t) = \sum_{j=0}^{\infty} Q_j(t) \sum_{i=0}^{\infty} C_i E_j^T.
\end{equation}

By Theorem 4.4, however,

\begin{equation}
(4.39) \quad \sum_{i=0}^{\infty} C_i E_j^T = A_j \quad (j \geq 0),
\end{equation}

where $A_j$ is defined by (4.25). Since $\{C_i\}$ and $\{A_j\}$ are null sequences, the equations (4.39) may be written matrically,

\begin{equation}
(4.40) \quad MC = A,
\end{equation}

where $C$ and $A$ are the infinite column vectors $[C_0, C_1, \ldots]^T$ and $[A_0, A_1, \ldots]^T$ and $M$ is the column-finite matrix $[m_{rs}]$, where

\begin{equation}
(4.41) \quad m_{rs} = E_r^T _s \quad (r, s \geq 0),
\end{equation}

and $E_r^T _s$ is defined by (4.39).

Using (4.32) and (4.37) the matrix $M$ may be seen to possess the two-sided inverse $Q = [q_{rs}]$, where

\begin{equation}
(4.42) \quad q_{rs} = D_r^s \quad (r, s \geq 0),
\end{equation}

and $D_r^s$ is defined by (4.32). Hence the relation (4.40) determines $C$ uniquely, and

\begin{equation}
(4.43) \quad C_i = B_i = \sum_{j=0}^{\infty} A_j D_j^T.
\end{equation}

We stress that Theorem 4.5 asserts the uniqueness of the coefficients $B_i$ on the assumption that $\{B_i\}$ is null. The unqualified uniqueness of these coefficients (which we have been able to prove only in special cases) is equivalent to the assertion that a series

\begin{equation}
(4.44) \quad \sum_{i=0}^{\infty} O_i H_i(t)
\end{equation}

converges uniformly on $I_n$ only if $\{C_i\}$ is null.

5. Applications. Define the sequence of polynomials $\psi_r(t)$ over $GF[q, x]$ by

\begin{equation}
(5.1) \quad \psi_r(t) = \prod_{\deg m < r} (t - m), \quad \psi_0(t) = t,
\end{equation}

where the product in (5.1) extends over all polynomials $m \in GF[q, x]$ (including 0) having degree $< r$. It follows [3] that

\begin{equation}
(5.2) \quad \psi_r(t) = \sum_{i=0}^{r} (-1)^{r-i} \binom{r}{i} q^i t^i,
\end{equation}
where
\[
\left[ r \right] = \frac{F_r}{L_r L'_{r-1}} , \quad \left[ 0 \right] = \frac{F_r}{L_r} , \quad \left[ r \right] = 1 ,
\]
and
\[
F_r = [r][r-1]^2 \ldots [1]^{m-1} , \quad F_0 = 1 , \quad L_r = [r][r-1] \ldots [1] , \quad L_0 = 1 ,
\]
\[
\left[ r \right] = a^r - a .
\]

Let \( K \) be any extension field of \( GF(q, a) \). By (5.2), the functions associated to the polynomials \( \psi_r(t) \) are linear operators on the \( GF(q) \)-vector space \( K \). Furthermore, \( \psi_r(x) = \psi_r(m) = F_r \), for \( m \) monic of degree \( r \), so that \( F_r \) is the product of all monic polynomials in \( GF[q, a] \) of degree \( r \).

On the other hand, \( L_r \) may be seen to be the l.c.m. of all polynomials in \( GF[q, a] \) of degree \( r \) [2].

Following Carlitz [4], we define \( g_k \in GF[q, a] \), and polynomials \( G_k(t) \), \( G'_k(t) \) over \( GF[q, a] \). Let \( k \) be a positive integer, and write
\[
k = e_0 + e_1 q + \ldots + e_s q^s \quad (0 \leq e_i < q).
\]
Define \( g_k \) by
\[
g_k = F_{e_0} \ldots F_{e_s} , \quad g_0 = 1 ,
\]
and \( G_k(t) \) and \( G'_k(t) \) by
\[
G_k(t) = \psi_{e_0}(t) \ldots \psi_{e_s}(t) , \quad G_0(t) = 1
\]
and
\[
G'_k(t) = \prod_{i=0}^s G'_{e_i}(t) ,
\]
where
\[
G'_{e_i}(t) = \begin{cases} \psi_{e_i}(t) & \text{for } 0 \leq e < q-1, \\ \psi_{e_i}(t) - F_{e_i} & \text{for } e = q-1 . \end{cases}
\]

Let \( K \) be any extension field of \( GF(q, a) \). Since \( \deg G_k(t) = \deg G'_k(t) = n \), the sequences \( \{ G_n(t)/g_n \} \) and \( \{ G'_n(t)/g_n \} \) are ordered bases of the \( K \)-vector space \( K[t] \). Indeed, for any \( f(t) \in K[t] \) of degree \( \leq n \), we have [4] the unique representations
\[
f(t) = \sum_{i=0}^n A_i \frac{G_i(t)}{g_i}
\]
and
\[
f(t) = \sum_{i=0}^s A'_i \frac{G'_i(t)}{g_i} .
\]
where \( A_i \) is uniquely determined by choosing any \( r \) such that \( i < q^r \), and setting
\[
A_i = (-1)^r \sum_{\text{deg } m < r} \frac{G^r_{q^r-1-t}(m)}{g^{q^r-1-t}} f(m) \quad (m \in \text{GF}[q, x]),
\]
and \( A^*_{i} \) is uniquely determined by choosing any \( r \) such that \( n < q^r \), and setting
\[
A^*_{i} = (-1)^r \sum_{\text{deg } m < r} \frac{G^r_{q^r-1-t}(m)}{g^{q^r-1-t}} f(m) \quad (m \in \text{GF}[q, x]).
\]

Note the difference between the defining conditions for \( r \) in (5.12) and (5.13).

An important property of the polynomials \( G_i(t)/g_i \) and \( G^*_i(t)/g_i \) is the fact that for all \( m \in \text{GF}[q, x] \), \( G_i(m)/g_i \in \text{GF}[q, x] \) and \( G^*_i(m)/g_i \in \text{GF}[q, x] \) [4]. With (5.12) and (5.13), this implies that \( \{G_i(t)/g_i\} \) and \( \{G^*_i(t)/g_i\} \) are, in fact, ordered bases of the \( \text{GF}[q, x] \)-module of polynomials over \( \text{GF}(q, x) \) that map \( \text{GF}[q, x] \) into itself.

Moreover, since \( \text{GF}[q, x] \) is dense in \( I_\infty \) and the polynomials \( G_i(t)/g_i \) and \( G^*_i(t)/g_i \) are, by an earlier observation, continuous functions, it follows that \( a \in I_\infty \) implies that \( G_i(a)/g_i \) and \( G^*_i(a)/g_i \in I_\infty \). With (5.12) and (5.13) this implies that \( \{G_i(t)/g_i\} \) and \( \{G^*_i(t)/g_i\} \) are ordered bases of the \( I_\infty \)-module of polynomials over \( F_\infty \) that map \( I_\infty \) into itself.

Hence, by Theorem 4.5, for every continuous function \( f: I_\infty \to I_\infty \), there exist null sequences \( \{B_i\} \) and \( \{B^*_i\} \) in \( I_\infty \) such that
\[
f(t) = \sum_{i=0}^{\infty} B_i \frac{G_i(t)}{g_i}
\]
and
\[
f(t) = \sum_{i=0}^{\infty} B^*_i \frac{G^*_i(t)}{g_i},
\]
where (5.14) and (5.15) converge uniformly on \( I_\infty \).

The coefficients \( B_i \) in (5.14) are uniquely determined by \( f \). For if \( n \) is any positive integer, the finite sum
\[
\sum_{i=0}^{q^n-1} B_i \frac{G_i(t)}{g_i}
\]
is the unique polynomial of degree \( \leq q^n - 1 \) which takes the same values as \( f \) on the set \( q \times 1 \) of all polynomials in \( \text{GF}[q, x] \) of degree \( < n \). Hence, by (5.12),
\[
B_i = (-1)^r \sum_{\text{deg } m < r} \frac{G^r_{q^r-1-t}(m)}{g^{q^r-1-t}} f(m) \quad (i < q^r).
\]
The question of the unconditional uniqueness of the coefficients $B^k_i$ remains open.

Interpolation series of the type which appears in (5.14) may be used to characterize continuous linear operators on the GF(q)-vector space $I_n$.

**Theorem 5.1.** Let $f: I_n \to I_n$ be continuous. If the (unique) interpolation series for $f$ constructed from the Carlitz polynomials is given by

$$ f(t) = \sum_{i=0}^{\infty} A_i \frac{G_i(t)}{g_i}, $$

then $f$ is a linear operator on the GF(q)-vector space $I_n$ if and only if $A_i = 0$ for $i \neq q^k$, where $k \geq 0$.

**Proof.** Sufficiency. If $A_i = 0$ for $i \neq q^k$, where $k \geq 0$, (5.18) becomes

$$ f(t) = \sum_{k=0}^{\infty} A_{q^k} \frac{\psi_k(t)}{E_k}. $$

Since, by (5.2), the partial sums of (5.19) are linear operators, it follows immediately that $f$ is a linear operator.

**Necessity.** We require the following identities [4]:

$$ G_i(\lambda t) = \lambda^i G_i(t) \quad (\lambda \in \text{GF}(q)), $$

$$ G_i(t_1 + t_2) = \sum_{j=0}^{i} \binom{i}{j} G_j(t_1) G_{i-j}(t_2). $$

Let $\lambda \in \text{GF}(q)$ be a primitive root of unity. Then (5.18), (5.20), and $f(\lambda t) = \lambda f(t)$, yield

$$ \sum_{i=0}^{\infty} \lambda A_i \frac{G_i(t)}{g_i} = \sum_{i=0}^{\infty} \lambda^i A_i \frac{G_i(t)}{g_i}, $$

and so $A_i = 0$, unless $i \equiv 1 \pmod{q-1}$.

From (5.18), (5.21), and $f(t_1 + t_2) = f(t_1) + f(t_2)$, we infer that

$$ \sum_{i=0}^{\infty} A_i \frac{G_i(t_1)}{g_i} + \sum_{i=0}^{\infty} A_i \frac{G_i(t_2)}{g_i} = \sum_{i=0}^{\infty} G_i(t_1) \sum_{j=0}^{\infty} \frac{g_j}{g_i} \binom{i}{j} A_j G_{j-i}(t_2). $$

Equating coefficients of $G_0(t_2)$, we see that $A_0 = 0$. Equating coefficients of $G_i(t_1)/g_i$ for $i > 0$, and subtracting $A_0$, we get

$$ \sum_{j=i+1}^{\infty} \frac{g_j}{g_i} \binom{i}{j} A_j G_{j-i}(t_2) = 0. $$
Hence, for all $i,j$ with $1 \leq i < j$,

\[(i) A_j = 0.\]

It follows that $A_j = 0$ unless $j = p^i$, where $p$ is the characteristic of $\text{GF}(q)$. Since $p^i \equiv 1 \pmod{q-1}$, we must have $p^i = q^k$, where $k \geq 0$.

References


DUKE UNIVERSITY

THE UNIVERSITY OF TENNESSEE

Received on 27. 2. 1970