Dobinski's Formula Carl Wagner 2013

Recall that the exponential generating function of the Bell numbers is given by

(1) $$\sum_{n=0}^{\infty} B_n \frac{x^n}{n!} = e^{e^x - 1} = e^{-1}e^{e^x} = e^{-1}\sum_{k=0}^{\infty}\frac{(e^x)^k}{k!} = e^{-1}\sum_{k=0}^{\infty}\frac{e^{kx}}{k!}.$$

Now, by basic analysis, along with formula (1),

(2) $$B_n = D^n e^{e^x - 1}\,|_{x=0} = e^{-1}\sum_{k=0}^{\infty} D^n\left(\frac{e^{kx}}{k!}\right)|_{x=0} = e^{-1}\sum_{k=0}^{\infty}\frac{k^n}{k!}\ .$$

The infinite series representation of B_n given in (2) above is called *Dobinski's Formula.*

A useful variant of this formula is given by

(3) For all $n \in \mathbb{N}$, $\sum_{k=0}^{\infty}\frac{k^n}{k!} = B_n e$. (By contrast, $\sum_{k=0}^{\infty}\frac{k^{\underline{n}}}{k!} = e$ for all $n \in \mathbb{N}$. Do you see why?)

Special cases of (3) include

(4) $$\sum_{k=0}^{\infty}\frac{1}{k!} = B_0 e = e,$$

(5) $$\sum_{k=0}^{\infty}\frac{k}{k!}\ \left(= \sum_{k=1}^{\infty}\frac{1}{(k-1)!} = \sum_{k=0}^{\infty}\frac{1}{k!}\right) = B_1 e = e,$$

(6) $$\sum_{k=0}^{\infty}\frac{k^2}{k!} = B_2 e = 2e, \text{ etc.}$$

It follows from formula (3) that if $p(k)$ is any polynomial in k, then the infinite series

(7) $$\sum_{k=0}^{\infty}\frac{p(k)}{k!}$$

is easily evaluated. Suppose, for example, that

(8) $p(k) = c_0 + c_1 k + c_2 k^2 + \cdots + c_r k^r$. Then, clearly,

(9) $$\sum_{k=0}^{\infty}\frac{c_0 + c_1 k + c_2 k^2 + \cdots + c_r k^r}{k!} = (c_0 B_0 + c_1 B_1 + c_2 B_2 + \cdots + c_r B_r)e.$$