STRONG INSTABILITY OF STANDING WAVES FOR NONLINEAR KLEIN-GORDON EQUATIONS

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Abstract. The strong instability of ground state standing wave solutions \( e^{i\omega t} \phi_\omega(x) \) for nonlinear Klein-Gordon equations has been known only for the case \( \omega = 0 \). In this paper we prove the strong instability for small frequency \( \omega \).

1. Introduction and Results. We consider the strong instability of the ground state standing wave solutions \( e^{i\omega t} \phi_\omega(x) \) for nonlinear Klein-Gordon equation of the form

\[
\partial_t^2 u - \Delta u + u = |u|^{p-1} u, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^n,
\]

where \( n \geq 3, 1 < p < 1 + 4/(n - 2) \), \( \omega \in (-1, 1) \), and \( \phi_\omega(x) \) is the ground state, i.e., the unique positive radially symmetric solution in \( H^1(\mathbb{R}^n) \) of

\[
-\Delta \phi + (1 - \omega^2) \phi - |\phi|^{p-1} \phi = 0, \quad x \in \mathbb{R}^n
\]

(see Strauss [17] and Berestycki and Lions [2] for the existence, and Kwong [8] for the uniqueness of \( \phi_\omega \)). The stability and instability of the ground state standing waves \( e^{i\omega t} \phi_\omega(x) \) for (1.1) have been studied by many authors. Berestycki and Cazenave [1] proved that \( e^{i\omega t} \phi_\omega(x) \) are strongly unstable when \( \omega = 0 \) and \( 1 < p < 1 + 4/(n - 2) \) (see also Payne and Sattinger [13] and Shatah [15]). Shatah (see [14]) proved that the ground state standing waves \( e^{i\omega t} \phi_\omega(x) \) are orbitally stable when \( 1 < p < 1 + 4/n \) and \( \omega_c < |\omega| < 1 \), where the critical frequency \( \omega_c \) is

\[
\omega_c = \sqrt{\frac{p - 1}{4 - (n - 1)(p - 1)}},
\]

Shatah and Strauss [16] proved that \( e^{i\omega t} \phi_\omega(x) \) are orbitally unstable when \( 1 < p < 1 + 4/n \) and \( |\omega| < \omega_c \) or when \( p \geq 1 + 4/n \) and \( \omega \in (-1, 1) \). For related results for nonlinear Schrödinger equations, see [1, 3, 4, 18, 19], and for general theory of orbital stability and instability of solitary waves, see Grillakis, Shatah and Strauss...
Here, we give the definition of orbital stability/instability and blow-up instability of $e^{i\omega t}\phi_\omega(x)$. Orbital stability refers to stability up to translations and phase shifts. More precisely

**Definition of orbital stability** We say that the standing wave $e^{i\omega t}\phi_\omega(x)$ is orbitally stable for (1.1) if for any $\varepsilon > 0$ there exists $\delta > 0$ such that if $(u_0, u_1) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ satisfies

$$
\inf_{\theta \in \mathbb{R}, y \in \mathbb{R}^n} \|(u(\cdot), u_1(\cdot)) - (e^{i\theta}\phi_\omega(\cdot + y), i\omega e^{i\theta}\phi_\omega(\cdot + y))\|_{H^1 \times L^2} < \delta,
$$

then the solution $u(t, x)$ of (1.1) with data $(u_0, u_1)$ exists globally in time and satisfies

$$
\sup_{t \geq 0} \inf_{\theta \in \mathbb{R}, y \in \mathbb{R}^n} \|(u(t, \cdot), \partial_t u(t, \cdot)) - (e^{i\theta}\phi_\omega(\cdot + y), i\omega e^{i\theta}\phi_\omega(\cdot + y))\|_{H^1 \times L^2} < \varepsilon.
$$

Otherwise, $e^{i\omega t}\phi_\omega(x)$ is said to be orbitally unstable.

**Definition of strong blow-up instability** We say that the standing wave $e^{i\omega t}\phi_\omega(x)$ is strongly blow-up unstable if for any $\varepsilon > 0$ there exists $(u_0, u_1) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)$ such that

$$
\|(u(\cdot), u_1(\cdot)) - (\phi_\omega(\cdot), i\omega \phi_\omega(\cdot))\|_{H^1 \times L^2} < \varepsilon
$$

and the solution $u(t, x)$ of (1.1) with data $(u_0, u_1)$ blows up in a finite time.

From the above definitions of instability, if the standing wave $e^{i\omega t}\phi_\omega(x)$ is strongly blow-up unstable then it is orbitally unstable as well. We note that the strong instability of ground state standing waves $e^{i\omega t}\phi_\omega(x)$ has not been known except for the case of frequency $\omega = 0$.

The main result in this paper is as follows.

**Theorem 1.** Let $n \geq 3$, $1 < p < 1 + 4/(n - 2)$, $\omega \in (-1, 1)$, and $\phi_\omega(x)$ be the unique positive radial symmetric solution of (1.2). If $|\omega| \leq \sqrt{(p - 1)/(p + 3)}$, then the standing wave solution $e^{i\omega t}\phi_\omega(x)$ for (1.1) is strongly blow-up unstable.

**Remark 1** When $p \geq 1 + 4/n$ and the frequency $\omega$ is close to 1 the standing waves $e^{i\omega t}\phi_\omega(x)$ for NLKG (1.1) are again strongly blow-up unstable (forthcoming paper Ohta and Todorova).

**Remark 2** It is an interesting problem whether or not there is a frequency $\omega$ such that the standing wave $e^{i\omega t}\phi_\omega(x)$ is orbitally unstable for (1.1) but not strongly blow-up unstable.

The idea of the proof of Theorem 1 is the following. By the modulation $v(t, x) = e^{-i\omega t}u(t, x)$, the nonlinear Klein-Gordon equation (1.1) is transformed to the following perturbed Schrödinger equation

$$
\partial_t^2 v + 2\omega i\partial_t v - \Delta v + (1 - \omega^2)v = |v|^{p-1}v, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^n.
$$

(1.4)

So, for $\gamma \in \mathbb{R}$ and $m > 0$, we consider

$$
\partial_t^2 u + 2\gamma i\partial_t u - \Delta u + m^2 u = |u|^{p-1}u, \quad (t, x) \in \mathbb{R} \times \mathbb{R}^n,
$$

(1.5)

and the stationary problem

$$
-\Delta \phi + m^2 \phi - |\phi|^{p-1}\phi = 0, \quad x \in \mathbb{R}^n.
$$

(1.6)

Then Theorem 1 follows from
**Theorem 2.** Let \( n \geq 3, 1 < p < 1 + 4/(n-2), \gamma \in \mathbb{R}, m > 0 \) and \( \psi(x) \) be the unique positive radially symmetric solution of (1.6) in \( H^1(\mathbb{R}^n) \). If \( 4\gamma^2 \leq (p-1)m^2 \), then the stationary solution \( \psi(x) \) for (1.5) is strongly blow-up unstable in the following sense. For any \( \lambda > 1 \), the solution \( u(t, x) \) of (1.5) with data \((\lambda \psi, 0)\) blows up in a finite time.

In the next section, we give the proof of Theorem 2. For the special case \( \gamma = 0 \) the result of Theorem 2 was proved by Payne and Sattinger [13]. Their proof is based on the “potential well” arguments. The crucial point in their proof is the invariance under the flow of the set \( \Sigma_1 \) defined by (2.4). However, when the frequency \( \gamma \neq 0 \), new terms appear in (2.5) and (2.6), and we need to modify the argument in [13]. To control those terms, we need not only the invariant set \( \Sigma_1 \) but also introduce another invariant set \( \Sigma_2 \) defined by (2.4) and consider the intersection \( \Sigma_1 \cap \Sigma_2 \) of these two invariant under the flow sets. The restriction for the space dimension \( n \geq 3 \) in Theorems 1 and 2 comes from the variational characterization (2.2) of \( \psi(x) \) with \( j = 2 \). Finally, we note that the functional \( K_2 \) and the invariant set \( \Sigma_2 \) have been used by many authors (see, e.g., [14, 15, 16]), and the idea for using the intersection of appropriate two or more invariant sets was applied by Liu [10, 11] for the generalized Kadomtsev-Petviashvili equations.

2. Proof of Theorem 2. For \((u, v) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)\), we define

\[
E(u, v) = \frac{1}{2}||v||_2^2 + \frac{1}{2}||\nabla u||_2^2 + \frac{m^2}{2}||u||_2^2 + \frac{1}{p+1}||u||_{p+1}^{p+1},
\]

\[
Q(u, v) = \text{Im} \int_{\mathbb{R}^n} v(x)\overline{u(x)} \, dx + \gamma||u||_2^2.
\]

The local existence and uniqueness of the Cauchy problem (1.5) in the energy space yields in the following way. Due to Ginibre and Velo [5] we have the local existence and uniqueness of solutions \( u(t, x) \) in the energy space for the NLKG (1.1). Then the modulation \( v(t, x) = e^{-i\omega t}u(t, x) \), implies the local existence and uniqueness in the energy space of the Cauchy problem (1.5). That is, for any data \((u_0, u_1) \in H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)\), there exist \( T = T(||(u_0, u_1)||_{H^1 \times L^2}) > 0 \) and a unique solution \( u(t, x) \) of (1.5) with data \((u_0, u_1)\) such that

\[(u, \partial_t u) \in C([0, T], H^1(\mathbb{R}^n) \times L^2(\mathbb{R}^n)).\]

Moreover, the energy \( E(t) \) and the charge \( Q(t) \) are conserved quantities of (1.5), namely

\[E(u(t), \partial_t u(t)) = E(u_0, u_1), \quad Q(u(t), \partial_t u(t)) = Q(u_0, u_1) \quad (0 \leq t \leq T).\]

To obtain the conservation of energy we multiply the equation (1.5) by \( \overline{\partial_t u} \), integrate over \( \mathbb{R}^n \) and take the real part. To obtain the conservation of charge we multiply (1.5) by \( \pi \), integrate over \( \mathbb{R}^n \) and take the imaginary part.

For \( u \in H^1(\mathbb{R}^n) \), we define the functionals
By exact calculations one can observe that
\[ K_1(u) = \partial_\lambda J(\lambda u)|_{\lambda=1}, \quad K_2(u) = \frac{1}{n} \partial_\lambda J(u(\cdot/\lambda))|_{\lambda=1}. \]

Let \( \psi \) be the bound state of equation (1.6), i.e. the unique positive radially symmetric solution of (1.6).

**Lemma 3.** Let \( n \geq 3 \), \( 1 < p < 1 + 4/(n-2) \), \( m > 0 \). Consider the minimization problems
\[ d_j = \inf \{ J_j(u) : u \in H^1_{\text{rad}}(\mathbb{R}^n) \setminus \{0\}, \ K_j(u) = 0 \}, \quad j = 1, 2 \quad (2.1) \]
and
\[ \tilde{d}_j = \inf \{ J_j(u) : u \in H^1_{\text{rad}}(\mathbb{R}^n) \setminus \{0\}, \ K_j(u) \leq 0 \}, \quad j = 1, 2. \quad (2.2) \]
Then
\[ d_j = \tilde{d}_j, \quad j = 1, 2 \quad (2.3) \]
are attained at the unique positive radially symmetric solution \( \psi(x) \) of (1.6) in \( H^1(\mathbb{R}^n) \). Moreover, \( d_1 = d_2 = J(\psi), \ J'(\psi) = 0 \) and \( K_1(\psi) = K_2(\psi) = 0 \).

**Proof.** It is known that the identity (2.3) holds for \( j = 2 \), and the infimum of the minimization problems is attained at a positive function \( \psi_2(x) \) in \( H^1_{\text{rad}}(\mathbb{R}^n) \) which is a solution of (1.6) (see [15]).

Below, we prove Lemma 3 for the case \( j = 1 \). First, we show that the identity (2.3) holds for \( j = 1 \). By the definition of \( d_1 \) and \( \tilde{d}_1 \) we have \( \tilde{d}_1 \leq d_1 \). On the other hand, for any \( v \in H^1_{\text{rad}}(\mathbb{R}^n) \) satisfying \( K_1(v) < 0 \), there exists \( \lambda_0 \in (0, 1) \) such that \( K_1(\lambda_0 v) = 0 \), because \( K_1(\lambda v) = K_2(v) < 0 \) for \( \lambda = 1 \) and \( K_1(\lambda v) > 0 \) for \( \lambda \) close to 0. Then, we have \( d_1 \leq J_1(\lambda_0 v) = \lambda_0^2 J_1(v) < J_2(v) \), which implies \( d_1 \leq \tilde{d}_1 \). Thus, the identity \( d_1 = \tilde{d}_1 \) holds. Next, we show that the infimum of the minimization problem (2.2) for \( j = 1 \) is attained at a positive function \( \psi_1(x) \) in \( H^1_{\text{rad}}(\mathbb{R}^n) \). Let \( \{v_k\} \) be a minimizing sequence for (2.2) with \( j = 1 \). Then \( \{v_k\} \) is bounded in \( H^1_{\text{rad}}(\mathbb{R}^n) \). Therefore, there exist a subsequence of \( \{v_k\} \) (we still denote it by the same letter) and \( v_0 \in H^1_{\text{rad}}(\mathbb{R}^n) \) such that \( v_k \rightharpoonup v_0 \) weakly in \( H^1_{\text{rad}}(\mathbb{R}^n) \) and \( v_k \to v_0 \) strongly in \( L^{p+1}(\mathbb{R}^n) \). The last convergence is because of the compactness of the embedding \( H^1_{\text{rad}}(\mathbb{R}^n) \hookrightarrow L^q_{\text{rad}}(\mathbb{R}^n) \) for \( 2 < q < 2 + 4/(n-2) \) (see [17]). We show that \( v_0 \neq 0 \). Suppose that \( v_0 = 0 \). Then, from \( K_1(v_k) \leq 0 \), together with the strong convergence \( v_k \to v_0 \) in \( L^{p+1}(\mathbb{R}^n) \), it follows that \( v_k \to 0 \) strongly in \( H^1(\mathbb{R}^n) \). However, by \( K_1(v_k) \leq 0 \) and the Sobolev inequality, we have
\[ \|\nabla v_k\|_2^2 + m^2 \|v_k\|_2^2 \leq \|v_k\|_{p+1}^{p+1} \leq C_0(\|\nabla v_k\|^2_2 + m^2 \|v_k\|^2_2)^{(p+1)/2}. \]
Since $v_k \neq 0$, we have $\|\nabla v_k\|_2^2 + m^2 \|v_k\|_2^2 \geq C_0^{-2/(p-1)}$, which contradicts to the strong convergence $v_k \to 0$ in $H^1(\mathbb{R}^n)$. Thus, we see that $v_0 \in H^1(\mathbb{R}^n) \setminus \{0\}$. Therefore, by the lower semicontinuity of the norm in $H^1_{rad}(\mathbb{R}^n)$, together with the strong convergence $v_k \to 0$ in $L^{p+1}(\mathbb{R}^n)$, we have

$$K_1(v_0) \leq \lim\inf_{k \to \infty} K_1(v_k) \leq 0, \quad d_1 \leq J_1(v_0) \leq \lim\inf_{k \to \infty} J_1(v_k) = d_1.$$ 

Hence, $v_0$ attains the infimum of (2.2) for $j = 1$. Since $\psi_1 := |v_0|$ also attains (2.2) for $j = 1$, we see that (2.2) for $j = 1$ is attained at a positive function $\psi_1(x)$ in $H^1_{rad}(\mathbb{R}^n)$, and $K_1(\psi_1) = 0$ and $J(\psi_1) = d_1$. Next, we show that $\psi_1$ is a solution of (1.6). Since $\psi_1$ attains (2.1) for $j = 1$, there exists a Lagrange multiplier $\lambda \in \mathbb{R}$ such that $J'(\psi_1) = \lambda K_1'(\psi_1)$. Then, we have

$$0 = K_1(\psi_1) = \langle J'(\psi_1), \psi_1 \rangle = \lambda_1(K_1'(\psi_1), \psi_1)$$

$$= \lambda_1(2\|\nabla \psi_1\|_2^2 + 2m^2\|\psi_1\|_2^2 - (p + 1)\|\psi_1\|_{p+1}^p)$$

$$= -(p - 1)\lambda_1\|\psi_1\|_{p+1}^p,$$

where in the last identity we used that $K_1(\psi_1) = 0$. This implies $\lambda_1 = 0$. So, we have $J'(\psi_1) = 0$, namely the positive function $\psi_1 \in H^1_{rad}(\mathbb{R}^n)$ is a solution of (1.6).

Finally, since the ground state $\psi(x)$ is the unique positive solution of (1.6) in $H^1_{rad}(\mathbb{R}^n)$, we have $\psi_1 = \psi_2 = \psi$. The identities $K_j(\psi) = K_j(\psi_j) = 0$ for $j = 1, 2$ lead to $d_j = J(\psi_j) = J(\psi)$ for $j = 1, 2$ which imply $d_1 = d_2$. \hfill $\square$

Denote by $d = d_1 = d_2$ and $\Sigma = \Sigma_1 \cap \Sigma_2$, where

$$\Sigma_j = \{(u, v) \in H^1_{rad}(\mathbb{R}^n) \times L^2_{rad}(\mathbb{R}^n) : E(u, v) < d, K_j(u) < 0\}, \quad j = 1, 2.$$

(2.4)

Note that $(\lambda \psi, 0) \in \Sigma$ for any $\lambda > 1$.

**Lemma 4.** The set $\Sigma$ is invariant under the flow of (1.5). That is, if $(u_0, u_1) \in \Sigma$, then the solution $u(t, x)$ of (1.5) with data $(u_0, u_1)$ satisfies $(u(t, \cdot), \partial_t u(t, \cdot)) \in \Sigma$ for any $t \in [0, T^*)$, where $T^*$ is the life span of the solution $u(t, x)$.

**Proof.** It is enough to prove that the sets $\Sigma_j$ $(j = 1, 2)$ are invariant under the flow of (1.5). From the conservation of energy, we have $E(u(t), \partial_t u(t)) = E(u_0, u_1) < d$ for any $t \in [0, T^*)$. Thus, to conclude the proof, we have only to show that $K_j(u(t)) < 0$ for any $t \in [0, T^*)$. Suppose that there exists $t_0 \in (0, T^*)$ such that $K_j(u(t_0)) = 0$ and $K_j(u(t)) < 0$ for $t \in [0, t_0]$. Then, it follows from Lemma 3 that $J_j(u(t)) \geq d_j > 0$ for $t \in [0, t_0]$. Thus, we see that $u(t_0) \neq 0$. Since $K_j(u(t_0)) = 0$ and $u(t_0) \neq 0$, it follows from the definition of $d_j$ that $d_j \leq J(u(t_0)) \leq E(u(t_0), \partial_t u(t_0)) < d_j$, which is a contradiction. This completes the proof. \hfill $\square$

For $\lambda > 1$, let $u_\lambda$ be the solution of (1.5) with data $(\lambda \psi, 0)$ where $\psi$ is the ground state of (1.6). Let $T_\lambda$ be the life span of $u_\lambda$. Denote by $E_\lambda$ and $Q_\lambda$ the energy and the charge of the solution $u_\lambda$, respectively. Let

$$I_\lambda(t) = \frac{1}{2}\|u_\lambda(t, \cdot)\|_2^2, \quad 0 \leq t < T_\lambda.$$ 

The key lemma is the following lower estimate for the second derivative $I_\lambda''(t)$.

**Lemma 5.** For any $\lambda > 1$, there exists a constant $a_\lambda > 0$ such that

$$I_\lambda''(t) \geq \frac{p+3}{2}\|\partial_t u_\lambda(t, \cdot)\|_2^2 + a_\lambda, \quad 0 \leq t < T_\lambda.$$
Proof. We have
\[ I''_0(t) = \text{Re} \int_{\mathbb{R}^n} \partial_t u_\lambda(t, x) \overline{u_\lambda(t, x)} \, dx. \]
By standard approximation arguments we can prove that \( I''_0(t) \) exists in \([0, T_\lambda]\) and
\[
I''_\lambda(t) = \|\partial_t u_\lambda(t, \cdot)\|_2^2 + \text{Re} \int_{\mathbb{R}^n} \partial_t^2 u_\lambda(t, x) \overline{u_\lambda(t, x)} \, dx \\
= \|\partial_t u_\lambda(t, \cdot)\|_2^2 + 2\gamma \text{Im} \int_{\mathbb{R}^n} \partial_t u_\lambda(t, x) u_\lambda(t, x) \, dx - K_1(u_\lambda(t)). \quad (2.5)
\]
Since
\[
2\gamma \text{Im} \int_{\mathbb{R}^n} \partial_t u_\lambda(t, x) \overline{u_\lambda(t, x)} \, dx = 2\gamma Q_\lambda - 2\gamma^2 \|u_\lambda(t, \cdot)\|_2^2,
\]
and
\[
-K_1(u_\lambda(t)) = \frac{p+1}{2} \|\partial_t u_\lambda(t, \cdot)\|_2^2 + (p+1) J_1(u_\lambda(t)) - 2\gamma^2 \|u_\lambda(t, \cdot)\|_2^2 - (p+1) E_\lambda, \quad (2.6)
\]
we obtain
\[
I''_\lambda(t) = \frac{p+3}{2} \|\partial_t u_\lambda(t, \cdot)\|_2^2 + (p+1) J_1(u_\lambda(t)) - 2\gamma^2 \|u_\lambda(t, \cdot)\|_2^2 - (p+1) E_\lambda + 2\gamma Q_\lambda.
\]
Here, we note that for any \( \lambda > 1 \) we have
\[
E_\lambda = J(\lambda \psi) < d, \quad \gamma Q_\lambda = \lambda^2 2\gamma^2 \|\psi\|_2^2 > \gamma^2 \|\psi\|_2^2.
\]
By Lemma 3 it follows the identity \( J_1(\psi) = J_2(\psi) = d \) which implies
\[
\|\psi\|_2^2 = \frac{(n+2) - (n-2)p}{(p-1)m^2} d.
\]
Thus, we have
\[
(p+1) E_\lambda - 2\gamma Q_\lambda < (p+1) d - 2\gamma^2 \|\psi\|_2^2 \\
= \left( \frac{p-1}{2} - \frac{2\gamma^2}{m^2} \right) \frac{2(p+1)}{p-1} d + \frac{2\gamma^2}{m^2} nd. \quad (2.7)
\]
Since for \( \lambda > 1 \) the data \((\lambda \psi, 0)\) are in \( \Sigma \), by Lemma 4 the solution \( u_\lambda(t, x) \) of (1.5) with data \((\lambda \psi, 0)\) remains in \( \Sigma \) for any \( 0 \leq t < T_\lambda \). Because of the variational definition of \( d = d_1 = d_2 \) due to Lemma 3 we have
\[
J_1(u_\lambda(t)) = \frac{p-1}{2(p+1)} (\|\nabla u_\lambda(t, \cdot)\|_2^2 + m^2 \|u_\lambda(t, \cdot)\|_2^2) \geq d
\]
and
\[
J_2(u_\lambda(t)) = \frac{1}{n} \|\nabla u_\lambda(t, \cdot)\|_2^2 \geq d
\]
for any \( 0 \leq t < T_\lambda \). Then we estimate the term \((p+1) J_1(u_\lambda(t)) - 2\gamma^2 \|u_\lambda(t, \cdot)\|_2^2\) in the right hand side of the identity (2.6) in a following way
\[
(p+1) J_1(u_\lambda(t)) - 2\gamma^2 \|u_\lambda(t, \cdot)\|_2^2 \\
= \left( \frac{p-1}{2} \right) \left( \|\nabla u_\lambda(t, \cdot)\|_2^2 + m^2 \|u_\lambda(t, \cdot)\|_2^2 \right) - 2\gamma^2 \|u_\lambda(t, \cdot)\|_2^2 \\
= \left( \frac{p-1}{2} - \frac{2\gamma^2}{m^2} \right) \|\nabla u_\lambda(t, \cdot)\|_2^2 + \left( \frac{m^2}{2} \|\nabla u_\lambda(t, \cdot)\|_2^2 \right) + \frac{2\gamma^2}{m^2} \|\nabla u_\lambda(t, \cdot)\|_2^2 \\
\geq \left( \frac{p-1}{2} \right) \frac{2(p+1)}{p-1} d + \frac{2\gamma^2}{m^2} nd. \quad (2.8)
\]
where the assumption $4\gamma^2 \leq (p - 1)m^2$ was used. Finally, using the estimate (2.7) together with (2.8) we can rewrite (2.6) in the form

$$I'(\lambda(t) \geq \frac{p+3}{2} \|\partial_t u_\lambda(t, \cdot)\|^2_2 + a_\lambda, \quad 0 \leq t < T_\lambda,$$

where

$$a_\lambda = \left(\frac{p-1}{2} - \frac{2\gamma^2}{m^2}\right) \left(\frac{2(p+1)}{p-1} \gamma + \frac{2\gamma^2}{m^2} d + (p+1)E_\lambda + 2\gamma Q_\lambda > 0.\right.$$

This completes the proof. \hfill \Box

Now the proof of Theorem 2 follows from Lemma 5 and concavity arguments due to Levine [9] as in Payne and Sattinger [13]. For the sake of completeness, we give the proof.

**Proof of Theorem 2.** Assume that the life span $T_\lambda = \infty$. By Lemma 5, we have $I'(\lambda(t) \geq a_\lambda > 0$ for any $t \in (0, \infty)$. This implies that there exists $t_1 \in (0, \infty)$ such that $I'(\lambda(t) > 0$ for any $t \in [t_1, \infty)$ and as well $I(\lambda(t) > 0$ for any $t \in [t_1, \infty)$. Let $\alpha = (p - 1)/4$. Then by using Lemma 5 we obtain the following estimate

$$I'(\lambda(t)I(\lambda(t) - (\alpha + 1)I(\lambda(t))^2 \geq \frac{p+3}{4} \left\|\partial_t u_\lambda(t)\|^2_2 \|u_\lambda(t)\|_2^2 - \left(\Re \int_{\mathbb{R}^n} \partial_t u_\lambda(t, x)\overline{u_\lambda(t, x)} dx\right)^2\right\| \geq 0.$$

Thus, for $t \in [t_1, \infty)$, we have

$$(I(\lambda(t)^{-\alpha})' = -\alpha I(\lambda(t)^{-\alpha-1}I'(\lambda(t) < 0,\,$n

$$(I(\lambda(t)^{-\alpha})'' = -\alpha I(\lambda(t)^{-\alpha-2}\{I'(\lambda(t)I(\lambda(t) - (\alpha + 1)I(\lambda(t))^2\} \leq 0.$$

Therefore,

$$I(\lambda(t)^{-\alpha} \leq I(\lambda(t_1)^{-\alpha} - \alpha I(\lambda(t_1)^{-\alpha-1}I'(\lambda(t_1)(t - t_1) \quad t \in [t_1, \infty),$$

so there exists $t_2 \in (t_1, \infty)$ such that $I(\lambda(t_2)^{-\alpha} \leq 0$. However, this is a contradiction. This completes the proof. \hfill \Box

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