Extremals for the families of commuting spherical contractions and their adjoints.

Stefan Richter

joint work with Carl Sundberg

University of Tennessee

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**Topic:** Dilations and extensions of \( d \)-tuples of commuting Hilbert space operators

Example for \( d = 1 \):

**Thm 1.** *(the Sz.-Nagy dilation theorem)*

\[ T \in \mathcal{B}(\mathcal{H}), \|T\| \leq 1 \]

\[ \Rightarrow \exists \ V \in \mathcal{B}(\mathcal{K}), \quad \mathcal{H} \subseteq \mathcal{K}, \]

\[ V\mathcal{H} \subseteq \mathcal{H}, \quad \|V^*x\| = \|x\|, \]

\[ T = V|_{\mathcal{H}}. \]

\[ V = \text{co-isometric extension of } T \]

\[ V = S^* \oplus U, \]

\( S \) unilateral shift of some multiplicity, \( U \) unitary

A study of such \( V \) leads to function theory in \( \mathbb{D} \).
All Hilbert spaces in the following are supposed to be separable.

\[ d \in \mathbb{N}, \ B^d = \{ z \in \mathbb{C}^d : |z| < 1 \} \]

**Defn 2. (Agler) A family** \( \mathcal{F} \) **is a collection**
of \( d \)-tuples \( T = (T_1, .., T_d) \) of Hilbert space operators, \( T_i \in \mathcal{B}(\mathcal{H}) \) **such that**

(a) \( \mathcal{F} \) is bounded,

\[ \exists c > 0 \ \forall T = (T_1, .., T_d) \in \mathcal{F} : \|T_i\| \leq c \ \forall i \]

(b) restrictions to invariant subspaces

\[ T \in \mathcal{F}, \mathcal{M} \subseteq \mathcal{H}, T_i\mathcal{M} \subseteq \mathcal{M} \ \forall i \Rightarrow T|\mathcal{M} \in \mathcal{F} \]

(c) direct sums

\[ T_n \in \mathcal{F} \Rightarrow \bigoplus_n T_n \in \mathcal{F}, \quad T_n = (T_{1n}, .., T_{dn}) \]

(d) unital \(*\)-representations

\[ \pi : \mathcal{B}(\mathcal{H}) \rightarrow \mathcal{B}(\mathcal{K}), \pi(I) = I, \]

\[ T = (T_1, .., T_d) \in \mathcal{F} \]

\[ \Rightarrow \pi(T') = (\pi(T_1), .., \pi(T_d)) \in \mathcal{F}. \]
Examples:

d = 1 :
\( \mathcal{F} = \) contractions, \( T^*T \leq I \)
isometries, \( T^*T = I \)
subnormal contractions

d \geq 1 :
\( \mathcal{F} = \) contractions
commuting contractions
isometries
commuting isometries

\( \mathcal{F} = \) commuting spherical contractions
commuting row contractions (d-contractions)
commuting spherical isometries
**Defn 3.** If \( T = (T_1, \ldots, T_d), \) \( T_i \in \mathcal{B}(\mathcal{H}), \)
\( S = (S_1, \ldots, S_d), \) \( S_i \in \mathcal{B}(\mathcal{K}), \) \( T, S \in \mathcal{F}, \)
then
\[ T \leq S \iff \mathcal{H} \subseteq \mathcal{K}, S\mathcal{H} \subseteq \mathcal{H}, T = S|\mathcal{H} \]
\[ \iff S = \begin{pmatrix} T & X \\ 0 & Y \end{pmatrix} \]
\[ \iff S \text{ extends } T. \]

**Defn 4.** \( T \) is **extremal** for \( \mathcal{F}, \)
\[ \iff S \geq T, S \in \mathcal{F} \Rightarrow S = \begin{pmatrix} T & 0 \\ 0 & Y \end{pmatrix} = T \oplus Y. \]

We will write \( T \in \text{ext}(\mathcal{F}) \)

**Thm 5.** (Agler)
\( \mathcal{F} \) family, \( T \in \mathcal{F} \Rightarrow \exists \) \( S \in \text{ext}(\mathcal{F}) \) \( S \geq T \)
Examples:

$\mathcal{F} =$contractions $\Rightarrow \text{ext}(\mathcal{F}) =$ co-isometries

**Cor 6. (Sz. Nagy)**
Every contraction has a co-isometric extension.

$\mathcal{F} =$isometries $\Rightarrow \text{ext}(\mathcal{F}) =$ unitaries

**Cor 7.** Every isometry is subnormal.

$\mathcal{F} =$subnormal contractions
$\Rightarrow \text{ext}(\mathcal{F}) =$ normal contractions.

**Cor 8.** Every subnormal contraction is subnormal.
Commuting spherical isometries (A. Athavale)

\[ \mathcal{F} = \{ T = (T_1, \ldots, T_d) : T_i \leftrightarrow T_j, \quad \sum_{i=1}^{d} \|T_i x\|^2 = \|x\|^2 \quad \forall x \in \mathcal{H} \} \]

\[ \text{ext}(\mathcal{F}) = \{ U = (U_1, \ldots, U_d) : U_i \leftrightarrow U_j, U_i \text{ normal} \quad \sum_{i=1}^{d} \|U_i x\|^2 = \|x\|^2 \quad \forall x \in \mathcal{H} \} \]

= commuting spherical unitaries

The proof follows Attele-Lubin, JFA, 1996.

**Cor 9.** (Athavale, 91) Every commuting spherical isometry is jointly subnormal.
Commuting spherical contractions

Drury 78, Mueller-Vasilescu 93, Arveson 98

\[ F = \{ T = (T_1, \ldots, T_d) : T_i \leftrightarrow T_j, \quad \sum_{i=1}^{d} \|T_i x\|^2 \leq \|x\|^2 \quad \forall x \in \mathcal{H} \} \]

\[ T \in F \iff \sum_{i=1}^{d} T_i^* T_i \leq I, \text{ commuting} \]

\[ \text{ext}(F) = \{ S^* \oplus U \} \]

\[ U = \text{commuting spherical unitary} \]
\[ S = d\text{-shift of some multiplicity} \]

\[ S = M_z \text{ on } H^2_d(\mathcal{D}) = H^2_d \otimes \mathcal{D} \]

\[ H^2_d \subseteq \text{Hol}(\mathbb{B}^d), \text{ Drury-Arveson-Hardy space} \]

\[ \text{defined by reproducing kernel} \]

\[ k_w(z) = \frac{1}{1 - \langle z, w \rangle}, \quad z, w \in \mathbb{B}^d \]
Thm 10. (Richter-Sundberg) Let $T = (T_1, \ldots, T_d)$ be a commuting operator tuple.

Then the following are equivalent

(a) $T \in \text{ext}(\mathcal{F})$

(b) $T = S^* \oplus U$

(c) (1) $\sum_{i=1}^{d} T_i^* T_i = P = a$ projection
(2) $\sum_{i=1}^{d} T_i T_i^* \geq I$
(3) If $x_1, \ldots, x_d \in \mathcal{H}$ with $T_i x_j = T_j x_i$, then $\exists x \in \mathcal{H}$ with $x_i = T_i x$.

(c3) says that the Koszul complex for $T$ is exact at a certain stage.
Note 1: For $d = 1$ (c) becomes

(1) $T^*T = P$, i.e. $T$ is a partial isometry

(2) $TT^* \geq I$, so $T$ is onto

(3) if $x_1 \in \mathcal{H}$, then $\exists x \in \mathcal{H}$ with $x_1 = Tx$, i.e. $T$ is onto

Hence (1)&(2) or (1)&(3) are equivalent to $T^*$ being an isometry.

For $d > 1$ let $T = M_z$ on $H^2(\partial \mathbb{B}^d)$, then (1) and (3) are satisfied, but (2) is not.

Note 2: If $T \in \mathcal{F}$ and
if $\sum_{i=1}^{d} T_i^*T_i \neq$ a projection, then $T \notin \text{ext}(\mathcal{F})$. 
Commuting row contractions (d-contractions)

\( \mathcal{F} = \{ T : T^* \text{ is a commuting spherical contraction} \} \)

\[ = \{ T : T_i \leftrightarrow T_j, \sum_{i=1}^{d} \|T_i^* x\|^2 \leq \|x\|^2 \forall x \} \]

\[ = \{ T : T_i \leftrightarrow T_j, \| \sum_{i=1}^{d} T_i x_i \|^2 \leq \sum_{i=1}^{d} \| x_i \|^2 \ \forall x_i \} \]

\( T \in \mathcal{F} \Leftrightarrow (T_1, .., T_d) : \mathcal{H} \oplus .. \oplus \mathcal{H} \rightarrow \mathcal{H} \text{ is contractive commutative} \)

\[ \Rightarrow T^* = S^* \oplus U|\mathcal{H} \]

(by Mueller/Vasilescu-Arveson)

\[ \Rightarrow T = P_{\mathcal{H}}(S \oplus U^*)|\mathcal{H}, \]

\( \mathcal{H} = \text{co-invariant for} \ S \oplus U^* \)

\( \text{ext}(\mathcal{F}) = ? \)
\[ D_* = (I - \sum_{i=1}^{d} T_i T_i^*)^{1/2} \]

**Thm 11. (easy)**

(a) If \( D_* = 0 \), then \( T \in \text{ext}(F) \) spherical co-isometries

(b) If \( D_* \) is onto, then \( T \notin \text{ext}(F) \).

Thus, if \( \exists c < 1 \exists \sum_i \|T_i^*x\|^2 \leq c\|x\|^2 \), then \( T \notin \text{ext}(F) \).

(c) If \( D_* \) is a projection, then

\[ T \notin \text{ext}(F) \]

\[ \Leftrightarrow \exists x_1, \ldots, x_d \in \text{ran} \ D_*, \sum_{i=1}^{d} \|x_i\|^2 > 0 \]

with \( T_i x_j = T_j x_i \).
If $S = M_z = d$-shift, then

$$D_*^2 = I - \sum_{i=1}^{d} S_i S_i^*$$

is a rank 1 projection, and

$$\text{ran } D_* = \text{constants},$$

hence $S \in \text{ext}(\mathcal{F})$.

**Cor 12.** If $d > 1$, then $\{S \oplus U^*\} \subsetneq \text{ext}(\mathcal{F})$

$U = \text{spherical unitary}$

Recall

$$\{S^* \oplus U\} = \text{ext}(\mathcal{F}^*)$$

Example for $\neq$:

$$T = (M_z^*, H^2(\partial \mathbb{B}^d)) \in \text{ext}(\mathcal{F}).$$
\[ D_* = (I - \sum_{i=1}^{d} T_i T_i^*)^{1/2} \]

**Thm 13. (R-S)**

If \( T \in \mathcal{F} \) and if \( D_* \) has rank one, i.e.

\[ D_* = u \otimes u \]

for some \( u \neq 0 \), then

\( T \in \text{ext}(\mathcal{F}) \iff \dim \text{span}\{u, T_1 u, \ldots, T_d u\} \geq 3 \)
If $S = (M_z, H_d^2) = \text{the } d\text{-shift}$, if $\mathcal{M}$ is invariant for $S$, $\mathcal{M} \neq H_d^2$ then

$$T = P_{\mathcal{M} \bot} S|\mathcal{M} \bot \in \mathcal{F},$$

and $D_*$ has rank 1:

$$D_*^2 = I_{\mathcal{M} \bot} - \sum_{i=1}^{d} P_{\mathcal{M} \bot} S_i S_i^* P_{\mathcal{M} \bot}$$

$$= P_{\mathcal{M} \bot} (I - \sum S_i S_i^*) P_{\mathcal{M} \bot}$$

$$= u \otimes u,$$

$$u = P_{\mathcal{M} \bot} 1 \neq 0$$
Cor 14. If $\mathcal{M} \neq H^2_d$, if

$$\mathcal{L} = \{a + \sum_{i=1}^{d} b_i z_i\},$$

then $T = P_{\mathcal{M}^\perp S} | \mathcal{M}^\perp \notin \text{ext}(\mathcal{F})$ if and only if

$$\dim \mathcal{M} \cap \mathcal{L} \in \{d - 1, d\}$$

In fact, in this case, if $T \notin \text{ext}(\mathcal{F})$, then

$$T_i = a_i I + b_i S$$

for some $S$ with rank $(I - SS^*) = 1$.

The Theorem can be used to produce examples

$$T \in \text{ext}(\mathcal{F}) \text{ but } D_\ast \neq \text{ a projection.}$$

Thus THM 11(c) does not characterize all extremals.