INVARIANT SUBSPACES FOR THE BACKWARD SHIFT ON HILBERT SPACES OF ANALYTIC FUNCTIONS WITH REGULAR NORM

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Dedicated to Boris Korenblum on the occasion of his eightieth birthday

Abstract. We investigate the structure of invariant subspaces of backward shift operator $Lf = (f - f(0))/\zeta$ on a large class of abstract Hilbert spaces of analytic functions on the unit disc where the forward shift operator $M_\zeta f = \zeta f$ acts as a contraction. Our main results show that under certain regularity conditions on the norm of such a space, the functions in a nontrivial invariant subspace of L have meromorphic *pseudocontinuations* in the Nevanlinna class of the exterior of the unit disc. We also provide a regularity condition which implies that the subspace itself is contained in the Nevanlinna class of the disc. These results imply that the spectrum of the restriction of L to these subspaces intersects the unit disc in a discrete set and this fact is then applied to prove a general index-one theorem for the forward shift invariant subspaces of the Cauchy dual of the original space. Finally, we give a detailed discussion of the weighted shift operators for which our main results apply.

1. Introduction

Let $\mathbb{D}$ denote the open unit disc in the complex plane and let ζ denote the identity function on $\mathbb{D}$, $\zeta(z) = z,\ z \in \mathbb{D}$. In this paper we shall consider Hilbert spaces of analytic functions on the open unit disc $\mathbb{D}$ that contain the constants and such that the operator M_ζ of multiplication with the function ζ acts as a contractive operator on $\mathcal{H}$, and such that the backward shift defined by the rule

$$Lf(z) = \frac{f(z) - f(0)}{z}$$

acts as a bounded linear operator. We will investigate the structure of the nontrivial invariant subspaces of the backward shift. Our goal is to show that there are quite general regularity conditions on the norm of such a Hilbert space which automatically imply very special properties of the nontrivial invariant subspaces of the backward shift. In order to explain our results let us begin with the basic properties of the spaces to be considered. Throughout the paper we shall assume that $\mathcal{H}$ is a Hilbert space consisting of analytic functions on $\mathbb{D}$ which satisfies the following two axioms:

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(1.1) for each $f \in \mathcal{H}$ we have $\zeta f \in \mathcal{H}$ and $||\zeta f|| \leq ||f||$,

(1.2) the analytic polynomials are dense in $\mathcal{H}$ and for each $\lambda \in \mathbb{D}$ we have that $(\zeta - \lambda)\mathcal{H}$ is closed in $\mathcal{H}$ with $\dim \mathcal{H} \ominus (\zeta - \lambda)\mathcal{H} = \dim \mathcal{H} \cap ((\zeta - \lambda)\mathcal{H})^{\perp} = 1$.

These two properties immediately imply that

(1.3) for each $\lambda \in \mathbb{D}$ the evaluation functional $f \to f(\lambda)$ is continuous on $\mathcal{H}$,

(1.4) for every $\lambda \in \mathbb{D}$ there is a $c_\lambda > 0$ such that $\|\frac{\zeta-\lambda}{1-\overline{\lambda}\zeta} f\| \geq c_\lambda \|f\|$ for all $f \in \mathcal{H}$.

Moreover, from the closed graph theorem we also see that the backward shift L defined above is a bounded linear operator on $\mathcal{H}$. Many common examples of Hilbert spaces of analytic functions satisfy these axioms. Such are the Hardy space, the weighted Bergman spaces, or, more generally, the closure of analytic polynomials in $L^2(\mu)$ for certain measures μ on $\overline{\mathbb{D}}$. Another relevant class of examples are those spaces where M_ζ acts as a weighted shift, i.e. an operator for which there is an orthonormal basis $\{e_n\}$ such that each e_n is mapped to a positive multiple of e_{n+1}. In the case of the operator M_ζ, this immediately implies that each e_n is a multiple of ζ^n and also, that the norm is a weighted l^2-norm of the sequence of Taylor coefficients of the functions in $\mathcal{H}$.

We shall be concerned with the structure of $\mathrm{Lat}(L, \mathcal{H})$, the lattice of invariant subspaces of the backward shift operator. For specific Banach spaces of analytic functions these subspaces have been intensively studied [DSS], [RiSu], [AR], [ARR]. For many examples satisfying (1.1) and (1.2) it turns out that whenever a function belongs to a nontrivial invariant subspace for L, it has a meromorphic pseudocontinuation to the exterior of the unit disc $\mathbb{D}_e = \{z \in \mathbb{C}, |z| > 1\}$.

Definition 1.1. Definition 1.1 A meromorphic function g in $\mathbb{D}_e$ is a pseudocontinuation of the analytic function f in $\mathbb{D}$ if both functions f and g have the same nontangential limits a.e. on the unit circle $\partial\mathbb{D}$.

Recall also that it follows from the Lusin-Privalov uniqueness theorem that a pseudocontinuation is unique if it exists (see e.g. [RS1], Remark 6.2.2). Even more information is available in many of the cases mentioned above. The values of the pseudocontinuation $\tilde{f}$ of a function f in a nontrivial L-invariant subspace $\mathcal{M}$ satisfies the equality

$$\tilde{f}(\lambda) = \frac{\langle \frac{f}{\zeta-\lambda}, h\rangle}{\langle \frac{1}{\zeta-\lambda}, h\rangle}, \quad |\lambda| > 1 \tag{1.5}$$

whenever h is a nonzero function in $\mathcal{M}^\perp$. Following [RS2], the right hand side of (1.5) is called the h-prolongation of f. Thus, for all $h \in \mathcal{M}^\perp \setminus \{0\}$ the h-prolongation of f equals to the pseudocontinuation of f. By the results proved in [ARR] for general Banach spaces of analytic functions, the remarkable fact that the values of the h-prolongations of f coincide for all $h \in \mathcal{M}^\perp \setminus \{0\}$, is actually equivalent to the fact that $(I - \lambda L|\mathcal{M})^{-1}$ exists. Thus, in all of these cases, a point $\alpha \in \mathbb{D} \setminus \{0\}$ belongs to the spectrum of the restriction of L to a nontrivial invariant subspace if and only if the right hand side of (1.5) is not defined at α^{-1}, i.e. $(\zeta-\alpha^{-1})^{-1}$ belongs to that subspace. Clearly, from (1.2) the set of these points is discrete in $\mathbb{D}$, and it easily follows from (1.1) that it actually forms a Blaschke sequence.

The argument outlined here breaks down in the case of an invariant subspace $\mathcal{M} \in \mathrm{Lat}(L, \mathcal{H})$ which, at its turn, contains a nontrivial M_ζ-invariant subspace $\mathcal{N}$. Indeed, in this case the h-prolongation of any $f \in \mathcal{N}$ vanishes identically and thus, it cannot equal a pseudocontinuation of f unless $f = 0$. The existence of such invariant subspaces has been proved recently by J. Esterle [E]. His examples are L-invariant subspaces of Hilbert spaces $\mathcal{H}$ as above where $M_\zeta|\mathcal{H}$ is a contractive weighted shift with highly irregular weights which however, decay to zero arbitrarily slow. As it has been pointed out by Borichev [B], these invariant subspaces $\mathcal{M}$ not only contain functions that fail to satisfy (1.5), but the spectrum of the restriction of L to $\mathcal{M}$ is the closed unit disc. Further details and open questions related to this phenomenon are discussed in [RS2].

The purpose of this paper is to present two abstract conditions on the space $\mathcal{H}$ under which, the invariant subspaces of the backward shift L on $\mathcal{H}$ consist of functions that have of pseudocontinuations to the exterior of the unit disc. Both of our conditions can be seen as regularity assumptions on the norm on $\mathcal{H}$. The first condition is very easy to state and is nothing else than a uniform version of condition (1.4):

(1.6) there is a $c > 0$ such that $||\frac{\zeta-\lambda}{1-\bar{\lambda}\zeta} f|| \geq c||f||$ for all $f \in \mathcal{H}$ and all $\lambda \in \mathbb{D}$.

In Theorem 2.2 below we prove that given any proper invariant subspace for L on $\mathcal{H}$, every function in the subspace has a meromorphic pseudocontinuation in the Nevanlinna class of $\mathbb{D}_e$ such that (1.5) holds for all nonzero $h \in \mathcal{M}^\perp$. In particular, the spectrum of the restriction of L to that subspace intersects $\mathbb{D}$ in a discrete set.

It is well-known that condition (1.6) is satisfied for Bergman spaces with so-called standard weights (also see Lemma 4.2). We mention that it follows from the proof of Proposition 4.10 of [MR] that all Hilbert spaces with "Bergman-type kernels" satisfy condition 1.6 (see [MR] for definitions). Nevertheless we shall see in Section 4 of this paper that condition (1.6) implies that there are sequences (λ_n) in $\mathbb{D}$ with $|\lambda_n| \to 1$ such that the norms of the evaluation functionals at these points grow slower than a negative power of the distance to the boundary. Hence, the condition does not cover weighted shifts with rapidly decreasing weights. A particular class of examples of this type are weighted Bergman spaces with radial weights that decay exponentially to zero near the boundary. Recall that $\mathcal{H}$ is a weighted Bergman space with a radial weight if the norm on $\mathcal{H}$ can be expressed as

$$\|f\|^2 = \int_{\mathbb{D}} |f|^2 d\mu$$

where μ has the form $d\mu = \omega dr \times dt$, for some positive integrable function ω on $[0, 1)$. However, an even stronger result than the conclusion of the above theorem holds true for all such weighted Bergman spaces. Namely, in addition to the existence of pseudocontinuations, nontrivial invariant subspaces for L are always contained in the Nevanlinna class of the unit disc ([ARR], Theorem 1.11). In order to motivate our second abstract assumption it is useful to recall briefly the argument in [ARR] which leads to the proof of this result. Let $\mathcal{M}$ be invariant for L, $f \in \mathcal{M}$ and h be a nonzero function in $\mathcal{M}^\perp$. Then h anihilates all functions of the form $(f -$

$f(\lambda))/(\zeta-\lambda)=(1-\lambda L)^{-1}Lf,\ f\in\mathcal{M},\lambda\in\mathbb{D}$, or equivalently,

$$f(\lambda)\int_{\mathbb{D}}\frac{\overline{h}}{\zeta-\lambda}d\mu=\int_{\mathbb{D}}\frac{f\overline{h}}{\zeta-\lambda}d\mu, \tag{1.7}$$

for all $\lambda\in\mathbb{D}$. Now due to the special form of the measure μ and to the fact that h is analytic it follows immediately that

$$\int_{|\zeta|<|\lambda|}\frac{\overline{h}}{\zeta-\lambda}d\mu=0 \tag{1.8}$$

Given $r\in(0,1)$ we can split the integrals involved in (1.7) according to the regions $\{|\zeta|<r\}$ and $\{|\zeta|>r\}$ and use (1.8) to obtain an equality of the form

$$f(r\lambda)\overline{F_r(\lambda)}=\overline{G_r}(\lambda)+H_r(\lambda),$$

a.e. on $\partial\mathbb{D}$, where F_r,G_r,H_r have uniformly bounded norms in the usual Hardy spaces $H^p,\ 0<p<1$ for almost every $r\in(0,1)$ and $H_r\to 0$ when $r\to 1^-$ in $H^p,\ 0<p<1$. Thus, by letting $r\to 1^-$ we see that there are functions $F,G\in H^p,\ 0<p<1$ such that $fF\in H^p$ and $f\overline{F}=\overline{G}$ a.e. on $\partial\mathbb{D}$, which leads to the desired result. This special representation of the functions in nontrivial L-invariant subspaces is essentially based on the equality (1.8). It turns out that it is possible to formulate an abstract regularity condition for the norm on a Hilbert space of analytic functions which leads to a similar situation.

Let $\mathcal{H}$ be a Hilbert space of analytic functions satisfies the conditions (1.1) and (1.2). The following condition will be the hypothesis of Theorem 2.4.

(1.9) There exists a set $A(\mathcal{H})\subseteq(0,1)$ with $\sup A(\mathcal{H})=1$ such that for every $r\in A(\mathcal{H})$ the scalar product on $\mathcal{H}$ can be written as a sum of two scalar products

$$\langle\cdot,\cdot\rangle=\langle\cdot,\cdot\rangle_r+\langle\cdot,\cdot\rangle_{\frac{1}{r}}$$

such that $\frac{1}{r}M_\zeta$ and rL are contractive with respect to the norms induced by $\langle\cdot,\cdot\rangle_r$ and $\langle\cdot,\cdot\rangle_{\frac{1}{r}}$ respectively.

It is not difficult to verify that weighted Bergman spaces with radial weights satisfy this condition if $\|\cdot\|_r,\|\cdot\|_{\frac{1}{r}}$ are defined as integrals over $\{|z|<r\}$ and $\{|z|\geq r\}$ respectively. As we shall see in Section 4 of this paper (1.9) applies to many other examples, especially to weighted shifts. For spaces $\mathcal{H}$ which satisfy (1.9) we prove in Theorem 2.4 that any nontrivial invariant subspace for L on $\mathcal{H}$ is contained in the Nevanlinna class of the unit disc. Moreover, every function in the subspace has a meromorphic pseudocontinuation in the Nevanlinna class of $\mathbb{D}_e$ and the spectrum of the restriction of L to that subspace intersects $\mathbb{D}$ in a discrete set.

One application of Theorems 2.2 and 2.4 concerns the index of ζ-invariant subspaces in the Cauchy dual $\mathcal{H}'$ of $\mathcal{H}$. This is presented in Section 3 of the paper. $\mathcal{H}'$ consists of analytic functions of the form

$$f(\lambda)=\langle g,(1-\overline{\lambda}\zeta)^{-1}\rangle_{\mathcal{H}},\quad\lambda\in\mathbb{D},$$

where $g\in\mathcal{H}$ and $\|f\|_{\mathcal{H}'}=\|g\|_{\mathcal{H}}$. The space $\mathcal{H}'$ satisfies (1.2)-(1.4), and M_ζ is a bounded expansive operator on this space. Given a ζ-invariant subspace $\mathcal{N}$ of $\mathcal{H}'$, the index of $\mathcal{N}$ is defined by

$$\mathrm{ind}\mathcal{N}=\dim\mathcal{N}/\zeta\mathcal{N}=\dim\mathcal{N}\cap(\zeta\mathcal{N})^{\perp}.$$

¿From the results in [Ri] we will derive the fact that whenever the space $\mathcal{H}$ satisfies (1.1), (1.2) and either one of the assumptions (1.6) or (1.9), then every M_ζ-invariant subspace $\mathcal{N}$ of $\mathcal{H}'$ has index one (Corollary 3.1). This is no longer true for the Cauchy duals of the spaces considered by Esterle in [E] which contain ζ-invariant subspaces of arbitrary index. This was pointed out by Borichev, [B]. Since no explicit proof of this fact is available in published form, we have included an outline of his argument, see Proposition 3.2.

In the last section of this paper we present several examples of weighted shift operators that satisfy our conditions. We will consider spaces $\mathcal{H}_w$ where the norm has the form

$$\|\sum_{n=0}^{\infty} a_n\zeta^n\|^2 = \sum_{n=0}^{\infty} |a_n|^2 w_n,$$

with $w = (w_n)_{n\geq 0}$ a fixed nonincreasing sequence of positive numbers with

$$\lim_{n\to\infty} \frac{w_{n+1}}{w_n} = 1.$$

In ([AB], p. 24) it is pointed out that if $\frac{w_{n+1}}{w_n}$ is nondecreasing with n then the norm on $\mathcal{H}_w$ is equivalent to a Bergman space norm with a radial weight and that norm will satisfy (1.9). Hence the conclusion of Theorem 2.4 holds for the space $\mathcal{H}_w$. The conditions we consider in Section 4 go beyond this regularity condition. For example, we show that if we set

$$\alpha_- = \liminf_{n\to\infty}(n+1)(1-\frac{w_{n+1}}{w_n}), \qquad \alpha_+ = \limsup_{n\to\infty}(n+1)(1-\frac{w_{n+1}}{w_n}),$$

then the space $\mathcal{H}$ satisfies (1.6), whenever $\alpha_+ - \alpha_- < 1$ (see Corollary 4.3), while if

$$\sup\{r\in(0,1): \sum_{n=0}^{\infty} \frac{w_n}{r^{2n+2}} \max\{r^2 - \frac{w_{n+1}}{w_n}, 0\} < 1\} = 1$$

then $\mathcal{H}$ satisfies (1.9) (Corollary 4.6). This last condition applies for example to all strictly decreasing weight sequences w with the property that

$$\alpha_+\left(\ln\frac{\alpha_+}{\alpha_-} - (1-\frac{\alpha_-}{\alpha_+})\right)e^{\alpha_+-\alpha_-} < 1,$$

and weight sequences satisfying $1-\frac{w_{n+1}}{w_n} = \frac{\alpha}{n^\gamma} + o(1/n)$ as $n\to\infty$, for some $\alpha > 0$ and $0<\gamma<1$ (Examples 4.8 and 4.9). It also is satisfied for strictly decreasing sequences $\{w_n\}$ such that there is a k_0 such that $\frac{w_{m+1}}{w_m} > \frac{w_{n+1}}{w_n}$ whenever $m \geq n+k_0$ (Lemma 4.7).

2. Main results

This section contains the proofs of the main results announced in the Introduction. We begin with some basic estimates of contractive operators on separable Hilbert spaces which are needed in our arguments. The result below can be found in [ARS] together with a more detailed analysis and further estimates for contractions. For the sake of completeness we have included an outline of the proof. For a complex-valued function u on $\mathbb{D}$ we shall denote the nontangential limit of u at at a point $\zeta\in\partial\mathbb{D}$ by $\mathrm{nt}-\lim_{\lambda\to z} u(\lambda)$. Analogous notations will be used for the nontangential limit superior and limit inferior of real-valued functions at points of $\partial\mathbb{D}$.

Lemma 2.1. *Let $\mathcal{K}$ be a separable Hilbert space and T be a contraction on $\mathcal{K}$. Then for every $x \in \mathcal{K}$ we have*

$$\mathrm{nt} - \limsup_{\lambda \to z} (1 - |\lambda|^2) \|(1 - \lambda T)^{-1} x\|^2 < \infty$$

a.e. on $\partial\mathbb{D}$. If $\lim_{n\to\infty} \|T^n x\| = 0$ then

$$\mathrm{nt} - \lim_{\lambda \to z} (1 - |\lambda|^2) \|(1 - \lambda T)^{-1} x\|^2 = 0,$$

a.e. on $\partial\mathbb{D}$.

Proof. The first part follows immediately from the inequality

$$(1 - |\lambda|^2) \|(1 - \lambda T)^{-1} x\|^2 \le \|(1 - \lambda T)^{-1} x\|^2 - \|\lambda T (1 - \lambda T)^{-1} x\|^2.$$

Indeed, a direct calculation shows that the right hand side of this inequality is a harmonic function of $\lambda \in \mathbb{D}$ which is also nonnegative. Such functions have finite nontangential limits a.e. on $\partial\mathbb{D}$ which gives the desired conclusion. Note also that the first part of the lemma together with the classical construction in the proof of Privalov's theorem (see [K], Chapter 3), imply that the function $u(\lambda) = (1 - |\lambda|^2) \|(1 - \lambda T)^{-1} x\|^2$ has nontangential limits a.e. on $\partial\mathbb{D}$. Then, to see the second part, it suffices to show that for some sequence (r_n) tending to 1 form below we have $u(r_n e^{it}) \to 0$ a.e. To this end, we use Parseval's formula and the hypothesis to obtain

$$\lim_{r \to 1^-} \int_0^{2\pi} u(re^{it}) \frac{dt}{2\pi} = \lim_{r \to 1^-} (1 - r^2) \sum_{n=0}^{\infty} \|T^n x\|^2 r^{2n} = 0$$

which completes the proof. □

With these preparations we can now turn to our first result about invariant subspaces for the backward shift.

Theorem 2.2. *Let $\mathcal{H}$ be a Hilbert space of analytic functions satisfying conditions (1.1), (1.2) and (1.6) and let $\mathcal{M} \in \mathrm{Lat}(L, \mathcal{H})$ be a nontrivial invariant subspace. Then every $f \in \mathcal{M}$ has a meromorphic pseudocontinuation that belongs to the Nevanlinna class of the exterior of the closed unit disc and satisfies (1.5) for all $h \in \mathcal{M}^\perp \setminus \{0\}$. In particular, $\sigma(L|\mathcal{M}) \cap \mathbb{D}$ is a discrete subset of $\mathbb{D}$.*

Proof. We begin with the simple observation that for $f, h \in \mathcal{H}$ the function

$$F_{f,h}(\lambda) = \bar{\lambda} \langle f(1 - \bar{\lambda}\zeta)^{-1}, h \rangle, \quad \lambda \in \mathbb{D}$$

is the complex conjugate of the Cauchy transform of a finite measure on $\partial\mathbb{D}$. Indeed, this follows by an application of the spectral theorem to the minimal unitary dilation of M_ζ. Consequently, $F_{f,h}$ has nontangential limits a.e. on $\partial\mathbb{D}$ which are the same as the nontangential limits of the function $\lambda \mapsto F_{f,h}(\frac{1}{\lambda})$, $|\lambda| > 1$. Clearly, this last function is again a Cauchy transform of a finite measure on $\partial\mathbb{D}$. In particular, it belongs to the Nevanlinna class of the exterior of the closed unit disc. Of course, the reasoning applies to the function $F_{1,h}(\lambda) = \bar{\lambda}\langle (1 - \bar{\lambda}\zeta)^{-1}, h \rangle$, $\lambda \in \mathbb{D}$ and in addition, if $h \ne 0$ the function $F_{1,h}$ does not vanish identically because polynomials are dense in $\mathcal{H}$. Our goal is to prove that each $f \in \mathcal{M}$ has nontangential limits a.e. on $\partial\mathbb{D}$ which satisfy

$$\text{nt-}\lim_{\lambda \to z} f(\lambda) = \text{nt-}\lim_{\lambda \to z} \frac{F_{f,h}(\lambda)}{F_{1,h}(\lambda)}. \tag{2.1}$$

for every $h \in \mathcal{M}^\perp \setminus \{0\}$. By the above considerations we then obtain that f has the pseudocontinuation $\lambda \mapsto F_{f,h}(1/\lambda)/F_{1,h}(1/\lambda),\ |\lambda| > 1$ which is just (1.5). As we have already shown, this function belongs to the Nevanlinna class of the exterior of the closed unit disc.

We start with the fact that for $|\lambda| < 1$

$$\langle (f - f(\lambda))(\zeta - \lambda)^{-1}, h\rangle = \langle (1 - \lambda L)^{-1} Lf, h\rangle = 0$$

which implies

$$\begin{aligned}\bar\lambda\langle (f - f(\lambda))(1 - \bar\lambda\zeta)^{-1}, h\rangle &= F_{f,h}(\lambda) - f(\lambda)F_{1,h}(\lambda)\\ &= (1 - |\lambda|^2)\langle \frac{f - f(\lambda)}{(\zeta - \lambda)(1 - \bar\lambda\zeta)}, h\rangle.\end{aligned}$$

Let us now estimate the last expression above. By (1.6) we have

$$\left\|\frac{f - f(\lambda)}{\zeta - \lambda}\right\| \le \frac{1}{c}\left\|\frac{f - f(\lambda)}{1 - \bar\lambda\zeta}\right\|$$

which leads to

$$\left|\langle \frac{f - f(\lambda)}{(\zeta - \lambda)(1 - \bar\lambda\zeta)}, h\rangle\right| \le \frac{1}{c}\left\|\frac{f - f(\lambda)}{1 - \bar\lambda\zeta}\right\| \left\|(1 - \lambda M_\zeta^*)^{-1}h\right\|.$$

Thus,

$$\begin{aligned}&|F_{f,h}(\lambda) - f(\lambda)F_{1,h}(\lambda)| \qquad (2.2)\\ &\quad\le \frac{1}{c}(1 - |\lambda|^2)\left\|\frac{f - f(\lambda)}{1 - \bar\lambda\zeta}\right\| \left\|(1 - \lambda M_\zeta^*)^{-1}h\right\|\\ &\quad\le \frac{1}{c}(1 - |\lambda|^2)\left(\left\|\frac{f}{1 - \bar\lambda\zeta}\right\| + |f(\lambda)|\left\|\frac{1}{1 - \bar\lambda\zeta}\right\|\right)\left\|(1 - \lambda M_\zeta^*)^{-1}h\right\|\end{aligned}$$

From the first part of Lemma 2.1 it follows that

$$\operatorname{nt-}\limsup_{\lambda\to z}\sqrt{1 - |\lambda|^2}\left\|\frac{g}{1 - \bar\lambda\zeta}\right\| < \infty$$

a.e. on $\partial\mathbb{D}$ and for all $g \in \mathcal{H}$. Also, we claim that

$$(1 - |\lambda|^2)||(I - \lambda M_\zeta^*)^{-1}h||^2 \to 0$$

as λ approaches a.e. $z \in \partial\mathbb{D}$ nontangentially. This follows immediately from the second part of Lemma 2.1 once we show that $\|(M_\zeta^*)^n h\| \to 0$ when $n \to \infty$. But this last condition is verified whenever h is a finite linear combination of reproducing kernels for $\mathcal{H}$ and since this set is dense in $\mathcal{H}$ and M_ζ^* is a contraction, the condition holds for all $h \in \mathcal{H}$.

¿From these observations we conclude that there is a subset E of $\partial\mathbb{D}$ of full measure such that for each $z \in E$ we have:

1) $F_{f,h}$ has a nontangential limit at z,
2) $F_{1,h}$ has a nonzero nontangential limit at z,
3) $\operatorname{nt-}\lim_{\lambda\to z}(1 - |\lambda|^2)||(I - \lambda M_\zeta^*)^{-1}h||^2 = 0$.
4) $\operatorname{nt-}\limsup_{\lambda\to z}\sqrt{1 - |\lambda|^2}\|(1 - \bar\lambda\zeta)^{-1}f\| < \infty$
5) $\operatorname{nt-}\limsup_{\lambda\to z}\sqrt{1 - |\lambda|^2}\|\frac{1}{1-\bar\lambda\zeta}\| < \infty$.

We will now use (2.2) to show that

$$\text{nt-}\limsup_{\lambda\to z}|f(\lambda)|<\infty$$

for each point $z\in E$. Indeed, if λ_n converges nontangentially to $z\in E$ such that $|f(\lambda_n)|\to\infty$, then the left hand side of (2.2) is $O(|f(\lambda_n)|)$, while the right hand side is $o(|f(\lambda_n)|)$. But then from (2.2) we see that (2.1) holds at every point of E.

To prove the assertion about the spectrum of $L|\mathcal{M}$ we use Corollary 2.3 and Proposition 2.8 of [ARR]. According to these results we have for $|\lambda|>1$ that $((1-\lambda L)|\mathcal{M})^{-1}$ exists if and only if the value

$$c_\lambda(f,g)=\langle\zeta f(\zeta-\lambda)^{-1},g\rangle/\langle\lambda(\zeta-\lambda)^{-1},g\rangle$$

does not depend on the choice of $g\in\mathcal{M}^\perp$. Since $\frac{\zeta}{\zeta-\lambda}=1+\frac{\lambda}{\zeta-\lambda}$ we see that $c_\lambda(f,g)=\frac{F_{f,g}(\frac{1}{\lambda})}{F_{1,g}(\frac{1}{\lambda})}$, whenever $F_{1,g}(\frac{1}{\lambda})\neq 0$. But the above reasoning shows that whenever λ is not a pole of the meromorphic pseudocontinuation of f to the exterior disc, $c_\lambda(f,g)$ equals the value of this pseudocontinuation at λ. Since the pseudocontinuation is unique the result follows. $\square$

The assumption that polynomials are dense in $\mathcal{H}$ is not really necessary for the above argument. In fact, the same reasoning applies whenever $\mathcal{M}$ is a nontrivial L-invariant subspace that does not contain all polynomials.

Let us now turn to our second result. As pointed out in the Introduction, Theorem 2.2 can be improved for Hilbert spaces of analytic functions that satisfy (1.9). In fact, in addition to the existence of pseudocontinuations we shall show that the subspaces under consideration are always contained in the Nevanlinna class of the unit disc.

Let us begin with some simple observations derived from the condition (1.9).

Proposition 2.3. *Let $\mathcal{H}$ be a Hilbert space of analytic functions in $\mathbb{D}$ satisfying (1.1), (1.2) and (1.9). Then*
(i) For all $r\in A(\mathcal{H})$ the operator $\frac{1}{r}M_\zeta$ is expansive with respect to the norm induced by $\langle\cdot,\cdot\rangle_{\frac{1}{r}}$.
(ii) There exists an equivalent Hilbert-space norm $\|\ \|_1$ on $\mathcal{H}$ such that $(\mathcal{H},\|\ \|_1)$ satisfies (1.9) for all r in some set $A_1(\mathcal{H})\subset(0,1)$, $\sup A_1(\mathcal{H})=1$ such that

$$\liminf_{\substack{r\to1^-\\ r\in A_1(\mathcal{H})}}\langle f,f\rangle_{1,r}>0$$

for all $f\in\mathcal{H}, f\neq 0$.

Proof. (i) Since rL is contractive with respect to the norm induced by $\langle\cdot,\cdot\rangle_{\frac{1}{r}}$ we have

$$\frac{1}{r^2}\langle\zeta f,\zeta f\rangle_{\frac{1}{r}}\geq\langle L\zeta f,L\zeta f\rangle_{\frac{1}{r}}=\langle f,f\rangle_{\frac{1}{r}}.$$

(ii) Let $r_0\in(0,1)$ be fixed and set

$$\|f\|_1^2=\|f\|^2+\int_{\partial\mathbb{D}}|f(r_0z)|^2\frac{|dz|}{2\pi}.$$

By (1.3) it follows immediately that $\|\cdot\|_1$ is equivalent to the original norm and clearly, (1.9) is satisfied with $A_1(\mathcal{H})=A(\mathcal{H})\cap(r_0,1)$ and

$$\langle f,g\rangle_{1,r}=\langle f,g\rangle_r+\int_{\partial\mathbb{D}}f(r_0z)\overline{g(r_0z)}\frac{|dz|}{2\pi},\quad \langle f,g\rangle_{1,\frac{1}{r}}=\langle f,g\rangle_{\frac{1}{r}}.$$

The fact that this new norm satisfies the condition in the statement is straightforward. □

Theorem 2.4. *Let $\mathcal{H}$ be a Hilbert space of analytic functions satisfying conditions (1.1),(1.2) and (1.9). Let $\mathcal{M} \in \mathrm{Lat}(L, \mathcal{H})$ be a nontrivial invariant subspace for L. Then $\mathcal{M}$ is contained in the Nevanlinna class and every $f \in \mathcal{M}$ has a meromorphic pseudocontinuation that belongs to the Nevanlinna class of the exterior of the closed unit disc. Furthermore, (1.5) holds for all $h \in \mathcal{M}^\perp \setminus \{0\}$ and $\sigma(L|\mathcal{M}) \cap \mathbb{D}$ is a discrete subset of $\mathbb{D}$.*

Proof. We shall first show that $\mathcal{M}$ is contained in the Nevanlinna class of the unit disc. Let $f \in \mathcal{M}$ and $h \in \mathcal{M}^\perp, h \neq 0$ and denote by $\|\cdot\|_r,\ \|\cdot\|_{\frac{1}{r}}$ the norms induced by $\langle\cdot,\cdot\rangle_r$ and $\langle\cdot,\cdot\rangle_{\frac{1}{r}}$ respectively. By Proposition 2.3 (ii) we can assume without loss of generality that $\inf\{\|h\|_r,\ r \in A(\mathcal{H})\} > 0$. For $|\lambda| < 1$ we have

$$\langle (f - f(\lambda))(\zeta - \lambda)^{-1}, h\rangle = \langle (1 - \lambda L)^{-1} L f, h\rangle = 0.$$

If $r = |\lambda| \in A(\mathcal{H})$ use (1.9) to write

$$0 = \langle (f - f(\lambda))(\zeta - \lambda)^{-1}, h\rangle_r + \langle (f - f(\lambda))(\zeta - \lambda)^{-1}, h\rangle_{\frac{1}{r}}. \tag{2.3}$$

We claim that

$$\langle (f - f(\lambda))(\zeta - \lambda)^{-1}, h\rangle_r = \lim_{\xi \to \lambda} \langle (f - f(\xi))(\zeta - \xi)^{-1}, h\rangle_r.$$

Indeed, if we denote by $\mathcal{H}_r$ the completion of $\mathcal{H}$ with respect to $\langle\cdot,\cdot\rangle_r$ then (1.9) together with von Neumann's inequality imply that every function h, analytic in a disc $\{|z| < r'\}$ with $r' > r$, multiplies $\mathcal{H}$ into $\mathcal{H}_r$. Moreover, the norm of the induced operator does not exceed the supremum norm of h on the disc of radius r. Thus, since f is analytic in $\mathbb{D}$ and $(f - f(\xi))(\zeta - \xi)^{-1}$ converges uniformly to $(f - f(\lambda))(\zeta - \lambda)^{-1}$ on $\{|z| \le r\}$ when $\xi \to \lambda$, we obtain that $(f - f(\xi))(\zeta - \xi)^{-1} \cdot 1$ converges in $\mathcal{H}_r$ to $(f - f(\lambda))(\zeta - \lambda)^{-1} \cdot 1$ when $\xi \to \lambda$.

Using the claim and (2.3) we see that

$$\langle (f - f(\lambda))(\zeta - \lambda)^{-1}, h\rangle_{\frac{1}{r}} = \lim_{\xi \to \lambda} \langle (f - f(\xi))(\zeta - \xi)^{-1}, h\rangle_{\frac{1}{r}}$$

and thus, we can write

$$\begin{aligned} &\lim_{\rho \to 1^+} \langle (f - f(\rho\lambda))(\zeta - \rho\lambda)^{-1}, h\rangle_r + \lim_{t \to 1^-} \langle (f - f(t\lambda))(\zeta - t\lambda)^{-1}, h\rangle_{\frac{1}{r}} \\ &= \lim_{\rho \to 1^+} \langle f(\zeta - \rho\lambda)^{-1}, h\rangle_r - f(\lambda) \lim_{\rho \to 1^+} \langle (\zeta - \rho\lambda)^{-1}, h\rangle_r \\ &+ \lim_{t \to 1^-} \langle (1 - t\lambda L)^{-1} L f, h\rangle_{\frac{1}{r}} = 0. \end{aligned} \tag{2.4}$$

Let F_r, G_r, H_r be defined in the unit disc by

$$\begin{aligned} &\overline{F_r(z)} = \langle f(\zeta - \frac{r}{\bar z})^{-1}, h\rangle_r, \quad \overline{G_r(z)} = \langle (\zeta - \frac{r}{\bar z})^{-1}, h\rangle_r, \\ &H_r(z) = \langle (1 - rzL)^{-1} L f, h\rangle_{\frac{1}{r}}. \end{aligned}$$

From the hypothesis (1.9) and the general theory of unitary dilations applied to $M_{\zeta/r}$ and rL in the respective inner product spaces, it follows that F_r, G_r, H_r are the Cauchy transforms of finite measures μ_r, ν_r, ω_r on $\partial\mathbb{D}$ whose total variations satisfy

$$|\mu_r| \le c\|f\|_r\|h\|_r, \quad |\nu_r| \le c\|1\|_r\|h\|_r \quad |\omega_r| \le c\|f\|_{\frac{1}{r}}\|h\|_{\frac{1}{r}}$$

for some absolute constant $c > 0$. This implies (see [D], Theorem 3.5) that $F_r, G_r, H_r \in H^p$ for every $0 < p < 1$ and that there exists an absolute constant $c_p > 0$ depending only on p with

$$\max\{\|F_r\|_p^p, \|H_r\|_p^p\} \le c_p\|f\|^p\|h\|^p \text{ and } \|G_r\|_p^p \le c_p\|h\|^p. \tag{2.5}$$

With these considerations we can rewrite (2.4) as

$$\overline{F_r(z)} - f(rz)\overline{G_r(z)} + H_r(z) = 0$$

a.e. on $\partial\mathbb{D}$ and from (2.5) we deduce that

$$\int_{\partial\mathbb{D}} |f(rz)G_r(z)|^p \frac{|dz|}{2\pi} \le \|F_r\|_p^p + \|H_r\|_p^p \le 2c_p\|f\|^p\|h\|^p$$

whenever $0 < p < 1$ and $r \in A(\mathcal{H})$. Also from (2.5) we see that $\{G_r,\ r \in A(\mathcal{H})\}$ is a normal family. Note that from our assumptions it follows that there is a sequence $\{r_n\}$ in $A(\mathcal{H})$ converging to 1 such that G_{r_n} converges uniformly on compacts to a nonzero function $G \in H^p$. Otherwise, we would have $G_r(z) \to 0$ uniformly on compacts when $r \to 1^-, r \in A(\mathcal{H})$ which implies that $\langle q, h\rangle_r \to 0,\ r \to 1^-, r \in A(\mathcal{H})$, for every polynomial q. Since polynomials are dense in $\mathcal{H}$ this contradicts the fact that $\inf\{\|h\|_r,\ r \in A(\mathcal{H})\} > 0$. Thus, if G is as above, then for $0 < \rho < 1$ fixed but arbitrary and for all $0 < p < 1$ we deduce from (2.5) that

$$\int_{\partial\mathbb{D}} |f(\rho z)G(\rho z)|^p \frac{|dz|}{2\pi} \le \lim_{n\to\infty} \int_{\partial\mathbb{D}} |f(\rho r_n z)G_{r_n}(\rho z)|^p \frac{|dz|}{2\pi} \le 2c_p\|f\|^p\|h\|^p$$

which shows that $fG \in H^p$, and hence f belongs to the Nevanlinna class.

To prove the remaining part of the statement, we proceed precisely as in the proof of the preceding theorem. We are going to show first that for $f \in \mathcal{M},\ h \in \mathcal{M}^\perp,\ h \ne 0$ we have

$$\mathrm{nt} - \lim_{\lambda\to z} \langle (f - f(\lambda))(1 - \bar\lambda\zeta)^{-1}, h\rangle = 0 \tag{2.6}$$

a.e. on $\partial\mathbb{D}$. Recall from the considerations made at the beginning of the proof of Theorem 2.2 that the functions

$$F_{f,h}(\lambda) = \bar\lambda\langle (1 - \bar\lambda\zeta)^{-1}f, h\rangle, \quad F_{1,h}(\lambda) = \bar\lambda\langle (1 - \bar\lambda\zeta)^{-1}, h\rangle, \quad \lambda \in \mathbb{D}$$

are complex conjugates of Cauchy transforms of finite measures on $\partial\mathbb{D}$ and hence, they have nontangential limits a.e. on $\partial\mathbb{D}$. Then (2.6) will imply that the nontangential boundary values of f (which exist a.e. by the previous argument) satisfy

$$f(z) = \mathrm{nt} - \lim_{\lambda\to z} F_{f,h}(z)/F_{1,h}(z)$$

a.e. on $\partial\mathbb{D}$. The same arguments as in the proof of Theorem 2.2 show that the right hand side equals the nontangential limit from the outside of the unit disc of a quotient of two Cauchy transforms of finite measures on $\partial\mathbb{D}$ such that the denominator does not vanish identically, because h cannot annihilate all polynomials. Consequently, f has a pseudocontinuation that belongs to the Nevanlinna class of the exterior of $\mathbb{D}$ and satisfies (1.5). Finally, the fact that $\sigma(L|\mathcal{M}) \cap \mathbb{D}$ is a discrete subset of $\mathbb{D}$ follows with the same argument as in the proof of Theorem 2.2. Thus, it remains to prove that (2.6) holds.

It follows immediately from what we have already proved that the nontangential limit in (2.6) exists for almost every $z \in \partial\mathbb{D}$. In order to prove (2.6) it will be sufficient to show that there exists a sequence $\{r_n\}$ tending to 1 from below such that

$$\lim_{n\to\infty} \langle (f - f(r_n z))(1 - r_n \bar{z}\zeta)^{-1}, h\rangle = 0 \tag{2.7}$$

for a.e. $z \in \partial\mathbb{D}$. As in the proof of Theorem 2.2 we use the fact that $h \in \mathcal{M}^\perp$ to conclude that

$$\langle (f - f(\lambda))(1 - \bar{\lambda}\zeta)^{-1}, h\rangle = (1 - |\lambda|^2)\langle (f - f(\lambda))(\zeta - \lambda)^{-1}, (1 - \lambda M_\zeta^*)^{-1}h\rangle.$$

For $r \in A(\mathcal{H}), r > 1/3$ and $|\lambda| = \rho = \frac{3r-1}{2}$ we use (1.9) to write

$$\begin{aligned}
&|(1 - |\lambda|^2)\langle (f - f(\lambda))(\zeta - \lambda)^{-1}, (1 - \lambda M_\zeta^*)^{-1}h\rangle| \\
&\qquad \le |(1 - |\lambda|^2)\langle (f - f(\lambda))(\zeta - \lambda)^{-1}, (1 - \lambda M_\zeta^*)^{-1}h\rangle_r| \\
&\qquad + |(1 - |\lambda|^2)\langle (f - f(\lambda))(\zeta - \lambda)^{-1}, (1 - \lambda M_\zeta^*)^{-1}h\rangle_{\frac{1}{r}}| \\
&\qquad = \tau_1(\lambda) + \tau_2(\lambda).
\end{aligned} \tag{2.8}$$

We shall conclude the proof by showing that there is a nonzero H^∞-function u such that

$$\liminf_{\rho\to 1^-} \int_{|\lambda|=\rho} [\tau_1(\lambda) + |u(\lambda)|\tau_2(\lambda)]|d\lambda| = 0. \tag{2.9}$$

For $r \in A(\mathcal{H}), r > 1/3$ we can use the Cauchy-Schwarz inequality to obtain for $\rho = \frac{3r-1}{2}$

$$\begin{aligned}
&(1-\rho^2)\int_{|\lambda|=\rho} |\langle (f - f(\lambda))(\zeta - \lambda)^{-1}, (1 - \lambda M_\zeta^*)^{-1}h\rangle_{\frac{1}{r}}| \frac{|d\lambda|}{2\pi\rho} \\
&\quad \le (1-\rho^2)\left(\int_{|\lambda|=\rho} \|(1 - \frac{\lambda}{r} rL)^{-1}Lf\|_{\frac{1}{r}}^2 \frac{|d\lambda|}{2\pi\rho} \int_{|\lambda|=\rho} \|(1 - \lambda M_\zeta^*)^{-1}h\|^2 \frac{|d\lambda|}{2\pi\rho}\right)^{1/2}.
\end{aligned}$$

Since rL is contractive with respect to $\|\cdot\|_{\frac{1}{r}}$ as in the proof of Lemma 2.1 we have by Parseval's formula

$$\int_{|\lambda|=\rho} \|(1 - \frac{\lambda}{r} rL)^{-1}Lf\|_{\frac{1}{r}}^2 \frac{|d\lambda|}{2\pi\rho} \le (r^2 - \rho^2)^{-1}\|f\|_{\frac{1}{r}}^2 \le 9(1-\rho)^{-1}\|f\|^2$$

and with the same reasoning we obtain

$$\lim_{\rho\to 1^-} (1-\rho^2)\int_{|\lambda|=\rho} \|(1 - \lambda M_\zeta^*)^{-1}h\|^2 \frac{|d\lambda|}{2\pi\rho} = 0.$$

Thus, we conclude from above that

$$\int_{|\lambda|=\rho} (\tau_1(\lambda) d|\lambda| = (1-\rho^2)\int_{|\lambda|=\rho} |\langle (f - f(\lambda))(\zeta - \lambda)^{-1}, (1 - \lambda M_\zeta^*)^{-1}h\rangle_{\frac{1}{r}}| \frac{|d\lambda|}{2\pi\rho} \to 0$$

when $r \in A(\mathcal{H})$ and $r \to 1^-$.

The estimation of the other term in (2.9) is a little more subtle, but very similar. Let $u \in H^\infty$, $u \ne 0$ with $uf \in H^\infty$ and $\|u\|_\infty, \|uf\|_\infty \le 1$. For $r \in A(\mathcal{H}), r > 1/3$

let $\rho' = \frac{1+r}{2}$ and use the fact that the function $\lambda \mapsto |u(\lambda)|^2 \|(f - f(\lambda))(\zeta - \lambda)^{-1}\|_r^2$ is subharmonic to obtain

$$\begin{aligned}
\int_{|\lambda|=\rho} &|u(\lambda)|^2 \|(f - f(\lambda))(\zeta - \lambda)^{-1}\|_r^2 \frac{|d\lambda|}{2\pi\rho} \\
&\leq \int_{|z|=\rho'} |u(z)|^2 \|(f - f(z))(\zeta - z)^{-1}\|_r^2 \frac{|dz|}{2\pi\rho'} \\
&\leq 2 \int_{|z|=\rho'} \|f(\zeta - z)^{-1}\|_r^2 + \|(\zeta - z)^{-1}\|_r^2 \frac{|dz|}{2\pi\rho'} \\
&\leq 2 \frac{\|f\|^2 + \|1\|^2}{\rho'^2 - r^2},
\end{aligned}$$

where the last step follows again by Parseval's formula and from the fact that $M_{\zeta/r}$ is contractive with respect to $\|\cdot\|_r$. Another use of the Cauchy Schwarz inequality yields the estimate

$$\begin{aligned}
(1 - \rho^2) &\int_{|\lambda|=\rho} |u(\lambda)\langle (f - f(\lambda))(\zeta - \lambda)^{-1}, (1 - \lambda M_\zeta^*)^{-1} h\rangle_r| \frac{|d\lambda|}{2\pi\rho} \\
&\leq \sqrt{2}(\|f\| + \|1\|) \frac{1 - \rho^2}{(\rho'^2 - r^2)^{1/2}} \left(\int_{|\lambda|=\rho} \|(1 - \lambda M_\zeta^*)^{-1} h\|^2 \frac{|d\lambda|}{2\pi\rho} \right)^{1/2} \to 0
\end{aligned}$$

as $r \to 1^-$ with the same reasoning as above. As pointed out before, this implies (2.6) and Theorem 2.4 is completely proved. □

3. The index of ζ-invariant subspaces in the Cauchy dual

Both of our main theorems theorems can be applied to study invariant subspaces for the operator of multiplication by ζ in the Cauchy dual space of $\mathcal{H}$. The Cauchy dual of $\mathcal{H}$ was defined in the Introduction. An alternate way to define it is as the completion of the set of polynomials with respect to the norm

$$\|p\|' = \sup \left\{ \left| \int_{\partial\mathbb{D}} p(z)\overline{q(z)} \frac{|dz|}{2\pi} \right|, \ q \text{ is a polynomial and } \|q\|_{\mathcal{H}} \leq 1 \right\},$$

that is, $\mathcal{H}'$ is the dual of $\mathcal{H}$ with respect to the H^2-pairing

$$\langle f, g\rangle_2 = \lim_{r\to 1^-} \int_{\partial\mathbb{D}} f(rz)\overline{g(rz)} \frac{|dz|}{2\pi}. \tag{3.1}$$

A third equivalent way to define the space $\mathcal{H}'$ is to use the Cowen-Douglas model for the operator L (see [Ri]). More precisely, one checks that each $\lambda \in \mathbb{D}$ is an eigenvalue of multiplicity one for L. Since the corresponding eigenvectors $(1-\bar{\lambda}\zeta)^{-1}$ span $\mathcal{H}$, the general theory of such operators implies that L^* is unitarily equivalent to multiplication by ζ on a Hilbert space of analytic functions and with the correct normalization it turns out that this space coincides with $\mathcal{H}'$. In many cases, the space $\mathcal{H}'$ can be found explicitly. For example, if $M_\zeta|\mathcal{H}$ is a weighted shift, i.e. when $\mathcal{H}$ can be identified with a weighted l^2-space, its Cauchy dual is a weighted l^2-space as well, whose weights are just the reciprocals of the original ones. Moreover, under our hypothesis, the space $\mathcal{H}'$ is a much smaller space that is usually contained in the Hardy space H^2. This happens for example, if the constant functions belong to

$\zeta\mathcal{H}^\perp$ (see also [ARR], Section 5 for a more detailed discussion of the Cauchy dual). The most relevant fact for our purposes is that the operators $L|\mathcal{H}$ and $M_\zeta^*|\mathcal{H}'$ are unitarily equivalent.

Given $\mathcal{N} \in \mathrm{Lat}(M_\zeta, \mathcal{H}')$, the lattice of M_ζ invariant subspaces of $\mathcal{H}'$, such that $\mathcal{N} \neq \{0\}$, we define the index of $\mathcal{N}$ as

$$\mathrm{ind}\mathcal{N} = \dim\mathcal{N}/\zeta\mathcal{N} = \dim\mathcal{N} \cap (\zeta\mathcal{N})^\perp.$$

For many examples it is shown in [ARR] that the index of such an invariant subspace is always one. The same holds true for the Cauchy duals of the abstract spaces $\mathcal{H}$ considered in the previous Section and this can be deduced from the results in [Ri] and the unitary equivalence of the operators $L|\mathcal{H}$ and $M_\zeta^*|\mathcal{H}'$.

Corollary 3.1. *Suppose that $\mathcal{H}$ satisfies the hypotheses of either Theorem 2.2 or Theorem 2.4. Then every nonzero invariant subspace for M_ζ on the Cauchy dual $\mathcal{H}'$ has index one.*

Proof. If $\mathcal{N}$ is such a subspace of $\mathcal{H}'$, the restriction of $(\mathcal{M}_\zeta|\mathcal{H}')^*$ to $\mathcal{N}^\perp$ is unitarily equivalent to the restriction of L to a nontrivial invariant subspace denoted by $\mathcal{M}$. Now if $\mathcal{H}$ satisfies the hypothesis of Theorem 2.2 or the one of Theorem 2.4 we can conclude that the set $\sigma(L|\mathcal{N}) \cap \mathbb{D}$ is discrete in $\mathbb{D}$ and hence, the same holds for the spectrum of the restriction of M_ζ^* to $\mathcal{N}^\perp$. By Theorem 4.5 in [Ri] this implies that $\mathcal{N}$ has index one. □

A completely different situation occurs in the spaces constructed recently by Esterle in [E]. In order to discuss in more detail some consequences of his work, let us introduce the following notation. For $f \in \mathcal{H}, g \in \mathcal{H}'$ let $f * g$ be the sequence obtained by the convolution of the sequences of Taylor coefficients of f and g. In other words, the n-th term $(f * g)_n,\ n \in \mathbb{Z}$, of $f * g$ is given by

$$(f * g)_n = \langle M_\zeta^n f, g\rangle_2,\ \text{for } n \geq 0 \quad \text{and} \quad (f * g)_n = \langle L^{|n|} f, g\rangle_2,\ \text{for } n < 0.$$

Now Theorem 4.10 in [E] together with its proof implies the existence of Hilbert spaces of analytic functions $\mathcal{H}$ which satisfy (1.1) and (1.2) and share the following remarkable property:

(3.2) For any array w_{ij} of sequences with finite support there exist $f_i \in \mathcal{H}, g_j \in \mathcal{H}'$ such that $f_i * g_j = w_{ij}$.

In fact, Esterle's theorem shows that $\mathcal{H}$ can be chosen such that $M_\zeta|\mathcal{H}$ is a weighted shift and that the weights $\|\zeta^n\|^{1/2}$ involved in the definition of the norm, decay arbitrarily slow to zero. Moreover, (3.2) actually holds even for arrays of infinite sequences w_{ij} that belong to a certain weighted l^1-space (see Proposition 4.6, and Theorems 4.5 and 4.2 in [E]).

The proof of the following result was kindly communicated to us by Alexander Borichev.

Proposition 3.2. *Let $\mathcal{H}$ be a Hilbert space of analytic functions that satisfies (1.1), (1.2) and (3.2). Then* $\mathrm{Lat}(M_\zeta, \mathcal{H}')$ *contains invariant subspaces of arbitrary index.*

Proof. According to the results in [E], there exist functions $f_i \in \mathcal{H},\ g_j \in \mathcal{H}'$ which solve the system of equations

$$f_i * g_j = \delta_{ij} e_0,$$

where $\delta_{ij} = 0$ if $i \neq j$ and $\delta_{ii} = 1$, and $e_0 = (\delta_{0n})$. Then all functions g_j belong to the invariant subspace

$$\mathcal{N}_0 = \{g \in \mathcal{H}',\ (f_i * g)_n = 0,\ n \leq -1, i = 1, 2, ..\}.$$

The point is that no nontrivial linear combination of the functions g_j can belong to $M_\zeta \mathcal{N}_0$. Indeed, $\sum \alpha_j g_j \in M_\zeta \mathcal{N}_0$ implies that

$$0 = (f_i * \sum \alpha_j g_j)_0 = \sum \alpha_j \delta_{ij}$$

for all i, that is, all coefficients α_j vanish. This shows that $[g_j]$ are linearly independent in $\mathcal{N}_0/M_\zeta \mathcal{N}_0$, i.e. $\operatorname{ind}\mathcal{N}_0 = \infty$. Also, given any positive integer N, let $\mathcal{N}$ be the invariant subspace generated by the functions $g_j,\ j = 1, 2, \ldots, N$, that is, the closed span of all polynomial multiples of these functions. Then it is easy to see that $\operatorname{ind}\mathcal{N} \leq N$ and by the above argument we have $\operatorname{ind}\mathcal{N} = N$ which completes the proof. □

4. Weighted shift operators and conditions (1.6) and (1.9)

Recall that an operator T on a separable Hilbert space is called a weighted shift if there exists an orthonormal basis $\{e_n : n \geq 0\}$ of the space such that $Te_n = \alpha_n e_{n+1},\ n \geq 0$, where $\alpha_n > 0$. It is clear that for spaces $\mathcal{H}$ of analytic functions in the unit disc the fact that $M_\zeta|\mathcal{H}$ is a weighted shift means that the space $\mathcal{H}$ consists of power series in the unit disc whose coefficients belong to a given weighted l^2-space, i.e. the norm of the function $f = \sum_{n\geq 0} a_n \zeta^n$ is given by

$$\|f\|_w^2 = \sum_{n=0}^{\infty} |a_n|^2 w_n.$$

To stress the dependence on the weight sequence we will denote this space of analytic functions by $\mathcal{H}_w$. In this paper we shall assume that the weight sequence $w = (w_n)_{n\geq 0}$ is strictly positive, non-increasing, that it satisfies $\lim_{n\to\infty} \frac{w_{n+1}}{w_n} = 1$, and that $w_0 = 1$. It is then a simple exercise to show that $\mathcal{H}_w$ satisfies the conditions (1.1) and (1.2). If the weights w_n are bounded below then $\mathcal{H}_w$ is just $H^2(\mathbb{D})$ with an equivalent norm, while if $w_n \to 0,\ n \to \infty$ it follows immediately by the dominated convergence theorem that $\zeta^n f \to 0$ for each $f \in \mathcal{H}_w$. We want to discuss some conditions on the weight sequence w under which the assumptions (1.6) or (1.9) hold for the space $\mathcal{H}_w$.

Proposition 4.1. *Let $v = (v_n)_{n\geq 0}$ be a weight sequence as above and suppose that $M_\zeta|\mathcal{H}_v$ satisfies (1.6) with the constant $c \in (0, 1)$, i.e.*

$$\|\frac{\zeta - \lambda}{1 - \bar{\lambda}\zeta} f\|_v \geq c\|f\|_v, \qquad f \in \mathcal{H}_v,\ \lambda \in \mathbb{D}.$$

Furthermore, let $w = (w_n)_{n\geq 0}$ be a weight sequence for which there exists a non-negative integer n_0 and a constant $b > 0$ such that

$$1 - \frac{v_{n+1}}{v_n} \leq 1 - \frac{w_{n+1}}{w_n} \leq b\left(1 - \frac{v_{n+1}}{v_n}\right)$$

for all $n \geq n_0$. If $b(1 - c^2) < 1$, then

$$\|\frac{\zeta - \lambda}{1 - \bar{\lambda}\zeta} \zeta^{n_0} f\|_w^2 \geq (1 - b(1 - c^2))\|\zeta^{n_0} f\|_w^2 \quad \textit{for all } f \in \mathcal{H}_w.$$

In particular, $\mathcal{H}_w$ satisfies (1.6).

Proof. Write $b_\lambda = \frac{\zeta-\lambda}{1-\overline{\lambda}\zeta}$, then for $h \in \mathcal{H}_w$

$$||h||_w^2 - ||b_\lambda h||_w^2 = (1-|\lambda|^2)\|D_w(1-\overline{\lambda}M_\zeta)^{-1}h\|_w^2,$$

where D_w denotes the positive square root of the operator $1-M_\zeta^* M_\zeta$ acting on $\mathcal{H}_w$. Of course, a similar identity holds in $\mathcal{H}_v$. Hence we can restate the first hypothesis as

$$(1-|\lambda|^2)\|D_v(1-\overline{\lambda}M_\zeta)^{-1}f\|_v^2 \le (1-c^2)||f||_v^2$$

for all $f \in \mathcal{H}_v$, $\lambda \in \mathbb{D}$, and we must show that

$$(1-|\lambda|^2)\|D_w(1-\overline{\lambda}M_\zeta)^{-1}\zeta^{n_0}f\|_w^2 \le b(1-c^2)||\zeta^{n_0}f||_w^2$$

for all $f \in \mathcal{H}_w$, $\lambda \in \mathbb{D}$.

We write $f = \sum_{n\ge 0} a_n\zeta^n \in \mathcal{H}_w$, then

$$\begin{aligned}(1-|\lambda|^2)&\|D_w(1-\overline{\lambda}M_\zeta)^{-1}M_\zeta^{n_0}f\|_w^2\\ &= (1-|\lambda|^2)\sum_{n=n_0}^{\infty}(w_n-w_{n+1})\left|\sum_{k=n_0}^{n} a_{k-n_0}\overline{\lambda}^{n-k}\right|^2 ..1\end{aligned} \tag{4}$$

¿From the second assumption we have for all $n \ge n_0$

$$w_n - w_{n+1} = \frac{w_n}{v_n}v_n\left(1-\frac{w_{n+1}}{w_n}\right) \le b\frac{w_n}{v_n}v_n\left(1-\frac{v_{n+1}}{v_n}\right) = b\frac{w_n}{v_n}(v_n-v_{n+1}).$$

Note that the first inequality of the second assumption says that (w_n/v_n) is nonincreasing for $n \ge n_0$. Hence we can use (4.1) to obtain

$$\begin{aligned}(1-|\lambda|^2)&\|D_w(1-\overline{\lambda}M_\zeta)^{-1}M_\zeta^{n_0}f\|_w^2\\ &\le b(1-|\lambda|^2)\sum_{n=n_0}^{\infty}(v_n-v_{n+1})\left(\sum_{k=n_0}^{n}|a_{k-n_0}|\sqrt{\frac{w_k}{v_k}}|\lambda|^{n-k}\right)^2.\end{aligned}$$

If we denote by $g = \sum_{k\ge n_0}|a_{k-n_0}|\sqrt{\frac{w_k}{v_k}}\zeta^k$, then $g \in \mathcal{H}_v$, $\|g\|_v = \|\zeta^{n_0}f\|_w$ and the right hand side of the last inequality can be written as

$$b(1-|\lambda|^2)\|D_v(1-|\lambda|M_\zeta)^{-1}g\|_v^2 \le b(1-c^2)\|g\|_v^2 = b(1-c^2)\|\zeta^{n_0}f\|_w^2.$$

This concludes the proof of the first assertion of the proposition. The second assertion follows from the fact that M_ζ is bounded below and contractive on $\mathcal{H}_w$, hence the operator $M_\zeta|\mathcal{H}_w$ is similar to $M_\zeta|\zeta^{n_0}\mathcal{H}_w$. □

In order to obtain simple conditions for a weight sequence $(w_n)_{n\ge 0}$ such that (1.6) holds in $\mathcal{H}_w$, we are going to apply the above result with a special choice of the sequence $(v_n)_{n\ge 0}$. For $\beta > -1$ we let v^β be defined by

$$v_n^\beta = (\beta+1)\int_0^1 r^n(1-r)^\beta dr, \quad n \ge 0.$$

Using integration by parts we find that

$$\frac{n+1}{\beta+1}\left(1-\frac{v_{n+1}^\beta}{v_n^\beta}\right) = \frac{\int_0^1 r^{n+1}(1-r)^\beta dr}{\int_0^1 r^n(1-r)^\beta dr} = \frac{v_{n+1}^\beta}{v_n^\beta} < 1$$

and hence

$$1 - \frac{v_{n+1}^\beta}{v_n^\beta} = (1 + \frac{n+1}{\beta+1})^{-1} = \frac{\beta+1}{\beta+n+2}. \tag{4.2}$$

Finally, a direct computation shows that the norm in $\mathcal{H}_{v^\beta}$ is given by

$$\|f\|_{v^\beta}^2 = (\beta+1)\int_{\mathbb{D}} |f(z)|^2(1-|z|^2)^\beta dA(z).$$

Lemma 4.2. *If $f \in \mathcal{H}_{v^\beta}$ and $\lambda \in \mathbb{D}$ then*

$$\|\frac{\zeta-\lambda}{1-\overline{\lambda}\zeta} f\|_{v^\beta}^2 \geq \frac{1}{\beta+2}\|f\|_{v^\beta}^2.$$

Proof. Note first that $v_{n+1}^\beta / v_n^\beta$ increases with n, hence,

$$\|zf\|_{v^\beta}^2 \geq v_1^\beta \|f\|_{v^\beta}^2 = \frac{1}{\beta+2}\|f\|_{v^\beta}^2.$$

Further, as before use the notation $b_\lambda(z) = \frac{\lambda-z}{1-\overline{\lambda}z}$ and a straightforward change of variable to obtain

$$\begin{aligned}
\|b_\lambda f\|_{v^\beta}^2 &= (\beta+1)\int_{\mathbb{D}} |(b_\lambda f)(z)|^2(1-|z|^2)^\beta dA(z) \\
&= (\beta+1)\int_{\mathbb{D}} |zf(b_\lambda(z))|^2 |b_\lambda'(z)|^2 (1-|b_\lambda(z)|^2)^\beta dA(z) \\
&\geq \frac{\beta+1}{\beta+2}\int_{\mathbb{D}} |f(b_\lambda(z))|^2 |b_\lambda'(z)|^2 (1-|b_\lambda(z)|^2)^\beta dA(z) = \frac{1}{\beta+2}\|f\|_{v^\beta}^2,
\end{aligned}$$

which gives the desired result. □

Corollary 4.3. *For a weight sequence $w = (w_n)_{n\geq 0}$ let*

$$\alpha_-(w) = \liminf_{n\to\infty}(n+1)(1-\frac{w_{n+1}}{w_n}), \qquad \alpha_+(w) = \limsup_{n\to\infty}(n+1)(1-\frac{w_{n+1}}{w_n}).$$

If $\alpha_+(w) < \infty$ and $\alpha_+(w) - \alpha_-(w) < 1$, then $\mathcal{H}_w$ satisfies (1.6).

Proof. In the case $\alpha_-(w) > 0$ the result follows by a direct application of Proposition 4.1 with $v = v^\beta$, $\beta = \alpha_-(w) - \varepsilon - 1$, where $\varepsilon > 0$ is sufficiently small. Indeed, if $0 < \varepsilon < \alpha_-(w)$ and β as above, then there exists n_0 dependent on ε and such that for all $n \geq n_0$ we have $\frac{\alpha_-(w)-\varepsilon}{n+1} < \varepsilon$ and by (4.2)

$$\begin{aligned}
1 - \frac{v_{n+1}^\beta}{v_n^\beta} &= \frac{\alpha_-(w)-\varepsilon}{n+1+\alpha_-(w)-\varepsilon} < \frac{\alpha_-(w)-\varepsilon}{n+1} \leq 1 - \frac{w_{n+1}}{w_n} \\
&\leq \frac{\alpha_+(w)+\varepsilon}{n+1} = \frac{\alpha_+(w)+\varepsilon}{\alpha_-(w)-\varepsilon}(1+\frac{\alpha_-(w)-\varepsilon}{n+1})(1-\frac{v_{n+1}^\beta}{v_n^\beta}) \\
&\leq \frac{\alpha_+(w)+\varepsilon}{\alpha_-(w)-\varepsilon}(1+\varepsilon)(1-\frac{v_{n+1}^\beta}{v_n^\beta}).
\end{aligned}$$

We set $b = \frac{\alpha_+(w)+\varepsilon}{\alpha_-(w)-\varepsilon}(1+\varepsilon)$ and note that Lemma 4.2 implies that we need to verify that $b(1-c^2) = \frac{\alpha_+(w)+\varepsilon}{\alpha_-(w)-\varepsilon+1}(1+\varepsilon) < 1$. The hypothesis implies that this is true for sufficiently small ε.

Next suppose $\alpha_-(w) = 0$. Then the hypothesis implies $\alpha_+(w) < 1$, and we can choose $\varepsilon > 0$ such that $\delta = 1 - (\alpha_+(w) + 2\varepsilon) > 0$. For $0 < t < \varepsilon$ define w^t by

$$w_n^t = w_n \prod_{k=0}^{n-1} (1 - \frac{t}{k+1}), \quad n \geq 1, \ w_0^t = 1$$

and note that

$$1 - \frac{w_{n+1}^t}{w_n^t} = \frac{t}{n+1} + (1 - \frac{t}{n+1})(1 - \frac{w_{n+1}}{w_n}).$$

Set $\beta = t - 1$, then by (4.2) we have for all n

$$1 - \frac{v_{n+1}^\beta}{v_n^\beta} < \frac{t}{n+1} < 1 - \frac{w_{n+1}^t}{w_n^t}.$$

Furthermore, there exists an n_0 which depends on ε, but is independent of t such that for all $n \geq n_0$

$$\begin{aligned} 1 - \frac{w_{n+1}^t}{w_n^t} &\leq \frac{t + \alpha_+(w) + \varepsilon}{n+1} = \frac{t + \alpha_+(w) + \varepsilon}{t} (1 + \frac{t}{n+1})(1 - \frac{v_{n+1}^\beta}{v_n^\beta}) \\ &\leq \frac{\alpha_+(w) + 2\varepsilon}{t} (1+t)(1 - \frac{v_{n+1}^\beta}{v_n^\beta}). \end{aligned}$$

Thus, in this case we can use Proposition 4.1 with $b = \frac{\alpha_+(w)+2\varepsilon}{t}(1+t)$ and $c^2 = 1/(t+1)$. We obtain $1 - b(1-c^2) = 1 - (\alpha_+(w) + 2\varepsilon) = \delta$. Hence we have for every polynomial f

$$\|\frac{\zeta - \lambda}{1 - \overline{\lambda}\zeta} \zeta^{n_0} f\|_{w^t}^2 \geq \delta \|\zeta^{n_0} f\|_{w^t}^2.$$

Since n_0 and δ do not depend on t we can let $t \to 0$ to obtain

$$\|\frac{\zeta - \lambda}{1 - \overline{\lambda}\zeta} \zeta^{n_0} f\|_{w}^2 \geq \delta \|\zeta^{n_0} f\|_{w}^2$$

for all polynomials f and the result follows. $\square$

Before we turn to our second assumption, let us discuss some cases where (1.6) does not hold. We shall use the following simple observation which yields a necessary condition for a space $\mathcal{H}$ to satisfy (1.6). If k_λ is the reproducing kernel in $\mathcal{H}$ and $\mathcal{H}$ satisfies (1.1), (1.2) and (1.6) then there is a positive constant C such that

$$\Delta \log k_\lambda(\lambda) \leq C(1 - |\lambda|^2)^{-2}, \tag{4.3}$$

where Δ denotes the Laplace operator. Indeed, the condition (1.6) implies that for every $f \in \mathcal{H}$ with $f(\lambda) = 0$ the function $g = f(1 - \overline{\lambda}\zeta)/(\zeta - \lambda)$ belongs to $\mathcal{H}$ and satisfies $\|g\| \leq C\|f\|$. This shows that

$$\sup\{(1 - |\lambda|^2)|f'(\lambda), \ f \in \mathcal{H}, f(\lambda) = 0, \|f\| \leq 1\} \leq C k_\lambda(\lambda).$$

Now a direct computation shows that the function $h_\lambda = \frac{\partial}{\partial \overline{\lambda}} k_\lambda - \frac{\partial}{\partial \overline{\lambda}} k_\lambda(\lambda) \frac{k_\lambda}{k_\lambda(\lambda)}$ satisfies $h_\lambda(\lambda) = 0$ and

$$\langle f, h_\lambda \rangle = f'(\lambda)$$

whenever $f \in \mathcal{H}$ with $f(\lambda) = 0$. Thus, the supremum above equals

$$(1-|\lambda|^2)\|h_\lambda\| = (1-|\lambda|^2)\left(\Delta k_\lambda(\lambda) - \frac{|\frac{\partial}{\partial\overline{\lambda}} k_\lambda(\lambda)|^2}{k_\lambda(\lambda)}\right)^{1/2},$$

and our inequality can be rewritten as

$$\frac{\Delta k_\lambda(\lambda)}{k_\lambda(\lambda)} - \frac{|\frac{\partial}{\partial\overline{\lambda}} k_\lambda(\lambda)|^2}{k_\lambda(\lambda)^2} \le C(1-|\lambda|^2)^{-2}$$

which is just (4.3). It is interesting to note that the quantity $\Delta \log k_\lambda(\lambda)$ involved in (4.3) is a unitary invariant for spaces of analytic functions which is also called the Cowen-Douglas curvature. Note also that

$$(1-|\lambda|^2)^{-2} = -\frac{1}{4}\Delta \log(1-|\lambda|^2)$$

so that by (4.3), we can conclude that $-C' \log(1-|\lambda|^2) - \log k_\lambda(\lambda)$ is subharmonic in $\mathbb{D}$. By the maximum principle, this implies that there are sequences (λ_n) with $|\lambda_n| \to 1$ such that $\limsup n \to \infty(C' \log(1-|\lambda_n|^2) + \log k_{\lambda_n}(\lambda_n) < \infty$. Thus, if $\mathcal{H}$ satisfies (1.6) there are sequences (λ_n) as above such that $k_{\lambda_n}(\lambda_n)$ grows slower than a negative power of the distance to the boundary. This cannot happen for example, if $\mathcal{H} = \mathcal{H}_w$ and $w_n = o(n^{-\alpha})$, $n \to \infty$ for all $\alpha > 0$. The verification of this last statement is immediate and will be omitted.

Our next goal is to examine the condition (1.9). Given a weighted shift, that is, a space $\mathcal{H}_w$ as above, there is a simple attempt to produce a decomposition of the norm as prescribed in (1.9). We set for $f = \sum_{n\ge 0} a_n \zeta^n \in \mathcal{H}_w$

$$\|f\|_r^2 = \sum_{n\ge 0} |a_n|^2 r^{2n+2} w_n, \quad \|f\|_{\frac{1}{r}}^2 = \|f\|_w^2 - \|f\|_r^2 = \sum_{n\ge 0} |a_n|^2 (1-r^{2n+2}) w_n.$$

Of course, $\|\cdot\|_r$ satisfies the requirement in (1.9) since

$$\|\zeta f\|_r^2 = \sum_{n\ge 0} |a_n|^2 r^{2n+4} \frac{w_{n+1}}{w_n} w_n \le r^2 \|f\|_r^2,$$

while $\|\cdot\|_{\frac{1}{r}}$ might not satisfy that (1.9). A sufficient condition for this is given below.

Proposition 4.4. *Suppose that*

$$\alpha_+(w) = \limsup_{n\to\infty} (n+1)(1 - \frac{w_{n+1}}{w_n}) < 1.$$

Then there exists an equivalent Hilbert space norm on $\mathcal{H}_w$ that satisfies (1.9).

Proof. If $\alpha_+(w) < 1$, there is a positive integer n_0 such that

$$\frac{w_{n+1}}{w_n} \ge \frac{n}{n+1}, \quad n \ge n_0 \ge 1.$$

If we set $v_n = w_{n+n_0}$, then the norms on $\mathcal{H}_w$ and $\mathcal{H}_v$ are equivalent, and $\frac{v_{n+1}}{v_n} \ge \frac{n+n_0}{n+n_0+1} \ge \frac{n+1}{n+2}$. Moreover, if we apply the above decomposition on $\mathcal{H}_v$ it suffices to show that $r\|Lf\|_{\frac{1}{r}} \le \|f\|_{\frac{1}{r}}$. For $f = \sum_{n\ge 0} a_n \zeta^n \in \mathcal{H}_v$ we have

$$r^2 \|Lf\|_{\frac{1}{r}}^2 = r^2 \sum_{n=1} (1-r^{2n}) v_{n-1} |a_n|^2 \le \sup_{n\ge 1} \frac{(r^2 - r^{2n+2}) v_{n-1}}{(1-r^{2n+2}) v_n} \|f\|_{\frac{1}{r}}^2.$$

Recall that $\frac{v_{n-1}}{v_n} \leq \frac{n+1}{n}$ and use also the inequality

$$\frac{r^2 - r^{2n+2}}{1 - r^{2n+2}} = 1 - \left(\sum_{k=0}^{n} r^{2k}\right)^{-1} \leq 1 - \frac{1}{n+1}$$

to obtain that

$$\sup_{n\geq 1} \frac{(r^2 - r^{2n+2})v_{n-1}}{(1 - r^{2n+2})v_n} \leq 1$$

and the result follows. □

Let us now describe an alternative way to produce a decomposition of the form (1.9). If $\mathcal{H}$ is any Hilbert space of analytic functions satisfying (1.1),(1.2) and in addition, if $\mathcal{H} \ominus \zeta\mathcal{H}$ consists of constant functions then the following identity holds for all $f \in \mathcal{H}$ and all $0 \leq r < 1$

$$\|f\|^2 = \int_{|\lambda|=r} |f(\lambda)|^2 \frac{|d\lambda|}{2\pi r} + \int_{|\lambda|=r} \langle (M_\zeta^* M_\zeta - r^2)L(1 - \lambda L)^{-1} f, L(1 - \lambda L)^{-1} f\rangle \frac{|d\lambda|}{2\pi r}.$$

A proof of this equality can be found in ([ARi], Lemma 2.2). Now write

$$M_\zeta^* M_\zeta - r^2 = D_r^+ - D_r^-$$

with positive operators D_r^+, D_r^- and set

$$\|f\|^2_{\frac{1}{r}} = c(r) \int_{|\lambda|=r} |f(\lambda)|^2 \frac{|d\lambda|}{2\pi r} + \int_{|\lambda|=r} \langle D_r^+ L(1-\lambda L)^{-1} f, L(1-\lambda L)^{-1} f\rangle \frac{|d\lambda|}{2\pi r}, \tag{4.4}$$

where $0 < c(r) < 1$ and

$$\|f\|_r^2 = \|f\|^2 - \|f\|^2_{\frac{1}{r}}. \tag{4.5}$$

While (4.4) clearly defines a norm on $\mathcal{H}$ this is not necessarily the case with the expression in (4.5).

Proposition 4.5. *Let $A(\mathcal{H})$ be the set of all $r \in (0, 1)$ for which there exists $0 < c(r) < 1$ such that the sesquilinear form defined by (4.5) is positive definite. If $\sup A(\mathcal{H}) = 1$, then $\mathcal{H}$ satisfies (1.9).*

Proof. We use first the equalities

$$L(1 - \lambda L)^{-1} f = Lf + \lambda L^2(1 - \lambda L)^{-1} f$$

together with the Parseval formula to conclude that

$$\int_{|\lambda|=r} \langle D_r^+ L(1 - \lambda L)^{-1} f, L(1 - \lambda L)^{-1} f\rangle \frac{|d\lambda|}{2\pi r} = \langle D_r^+ Lf, Lf\rangle \\ + r^2 \int_{|\lambda|=r} \langle D_r^+ L(1 - \lambda L)^{-1} Lf, L(1 - \lambda L)^{-1} Lf\rangle \frac{|d\lambda|}{2\pi r}.$$

This immediately implies that

$$r^2 \|Lf\|^2_{\frac{1}{r}} \leq \|f\|^2_{\frac{1}{r}}.$$

Similarly, we have

$$L(1 - \lambda L)^{-1} M_\zeta f = f + \lambda L(1 - \lambda L)^{-1} f$$

and by Parseval's formula

$$\int_{|\lambda|=r} \langle D_r^- L(1-\lambda L)^{-1} M_\zeta f, L(1-\lambda L)^{-1} M_\zeta f\rangle \frac{|d\lambda|}{2\pi r} = \langle D_r^- f, f\rangle$$
$$+ r^2 \int_{|\lambda|=r} \langle D_r^- L(1-\lambda L)^{-1} f, L(1-\lambda L)^{-1} f\rangle \frac{|d\lambda|}{2\pi r}.$$

This gives the inequality

$$\|M_\zeta f\|_r^2 \le r^2 \|f\|_r^2$$

and the result follows. □

The expressions considered in (4.4) and (4.5) are easily computed in the case of a weighted shift. For example, a direct computation shows that on any space $\mathcal{H}_w$ the operators D_r^+ and D_r^- are diagonal with respect to the orthonormal basis $\{e_n = w_n^{-1/2}\zeta^n : n \ge 0\}$ and

$$D_r^+ e_n = \max\{\frac{w_{n+1}}{w_n} - r^2, 0\} e_n \quad D_r^- e_n = \max\{r^2 - \frac{w_{n+1}}{w_n}, 0\} e_n.$$

With this setup it is now relatively easy to show that if $\frac{w_{n+1}}{w_n}$ is nondecreasing, then w satisfies the hypothesis of Proposition 4.5 and so $\mathcal{H}_w$ satisfies (1.9) (see Corollary 4.10). We shall prove a stronger result.

Corollary 4.6. *For a weight sequence w let*

$$\rho(w) = \sup\{r \in (0,1) : \sum_{n=0}^{\infty} \frac{w_n}{r^{2n+2}} \max\{r^2 - \frac{w_{n+1}}{w_n}, 0\} < 1\}.$$

If $\rho(w) = 1$, then $\mathcal{H}_w$ satisfies (1.9).

Proof. We want to apply Proposition 4.5. In order to estimate $\|f\|_r$ as defined by (4.4) and (4.5) note first that for $f = \sum_{n\ge 0} a_n \zeta^n \in \mathcal{H}_w$ we have

$$L(1-\lambda L)^{-1} f(z) = \frac{f(z) - f(\lambda)}{z - \lambda} = \sum_{k\ge 0} z^k \sum_{n \ge k+1} a_n \lambda^{n-k-1}.$$

Then, as pointed out above,

$$\langle D_r^- L(1-\lambda L)^{-1} f, L(1-\lambda L)^{-1} f\rangle = \sum_{n=0}^{\infty} w_k \max\{r^2 - \frac{w_{k+1}}{w_k}, 0\} \left| \sum_{n\ge k+1} a_n \lambda^{n-k-1} \right|^2$$

and by the Parseval formula

$$\int_{|\lambda|=r} \langle D_r^- L(1-\lambda L)^{-1} f, L(1-\lambda L)^{-1} f\rangle \frac{|d\lambda|}{2\pi r}$$
$$= \sum_{k=0}^{\infty} w_k \max\{r^2 - \frac{w_{k+1}}{w_k}, 0\} \sum_{n \ge k+1} |a_n|^2 r^{2n-2k-2}$$
$$= \sum_{n=1}^{\infty} |a_n|^2 r^{2n} \sum_{k=0}^{n-1} \frac{w_k}{r^{2k+2}} \max\{r^2 - \frac{w_{k+1}}{w_k}, 0\}$$
$$\le \left(\sum_{k=0}^{\infty} \frac{w_k}{r^{2k+2}} \max\{r^2 - \frac{w_{k+1}}{w_k}, 0\} \right) \sum_{n=1}^{\infty} |a_n|^2 r^{2n}.$$

If we now set

$$A(\mathcal{H}) = \{r \in (0,1) : \sum_{n=0}^{\infty} \frac{w_n}{r^{2n+2}} \max\{r^2 - \frac{w_{n+1}}{w_n}, 0\} < 1\}$$

and

$$c(r) = \frac{1 + \sum_{n=0}^{\infty} \frac{w_n}{r^{2n+2}} \max\{r^2 - \frac{w_{n+1}}{w_n}, 0\}}{2} < 1$$

for $r \in A(\mathcal{H})$, then the conditions in Proposition 4.5 are satisfied. Indeed, by assumption we know that $\sup A(\mathcal{H}) = 1$ and from above we have for $r \in A(\mathcal{H})$

$$\|f\|_r^2 \geq c(r) \sum_{n=0}^{\infty} |a_n|^2 r^{2n} - (2c(r) - 1)) \sum_{n=0}^{\infty} |a_n|^2 r^{2n} = (1 - c(r)) \int_{|\lambda|=r} |f(\lambda)|^2 \frac{|d\lambda|}{2\pi r}.$$

The result now follows by an application of Proposition 4.5. □

Some situations where this condition applies are described below. We will fix a strictly decreasing weight sequence w and for $n \geq 0$ set

$$\gamma_n = \inf\{1 - \frac{w_{k+1}}{w_k} : k \leq n\}$$
$$\delta_n = \sup\{1 - \frac{w_{k+1}}{w_k} : k \geq n\}.$$

Then $\{\delta_n\}$ and $\{\gamma_n\}$ are nonincreasing sequences which satisfy

$$0 < \gamma_n \leq 1 - \frac{w_{n+1}}{w_n} \leq \delta_n \to 0$$

as $n \to \infty$. Furthermore set $m_n = \sup\{k : \gamma_n \leq \delta_k\}$. Note that the definitions imply that for all n we have $n \leq m_n < \infty$ and $\delta_n - \gamma_n \geq \delta_k - \gamma_n \geq 0$ for all $k = n, ..., m_n$.

Lemma 4.7. *Let w be a strictly decreasing weight sequence such that*

$$\limsup_{n\to\infty} (m_n - n)\gamma_n < \infty \quad \textit{and}$$

$$\limsup_{n\to\infty} e^{(m_n - n)\gamma_n} \sum_{k=n+1}^{m_n} \delta_k - \gamma_n < 1.$$

Then $\mathcal{H}_w$ satisfies (1.9).

In particular, if w is a strictly decreasing weight sequence such that

$$\limsup_{n\to\infty} (m_n - n)\delta_n < 1,$$

then $\mathcal{H}_w$ satisfies (1.9).

Clearly this implies that $\mathcal{H}_w$ satisfies (1.9) whenever w is strictly decreasing and there is a $k \geq 0$ such that $m_n \leq n + k$ for all n.

Proof. We are going to apply Corollary 4.6. Let $0 < R < 1$, we have to show that there is an r, $R < r < 1$ such that

$$\sum_{k=0}^{\infty} \frac{w_k}{r^{2k+2}} \max\{r^2 - \frac{w_{k+1}}{w_k}, 0\} < 1. \tag{4.6}$$

The hypothesis implies that there is $\delta > 0$ and a positive integer N_0 such that

$$e^{(1+\delta)(m_n-n)\gamma_n} \sum_{k=n+1}^{m_n} (\delta_k - \gamma_n) < 1 \text{ for all } n \geq N_0. \tag{4.7}$$

Let $0 < \varepsilon < 1$ such that $-\ln(1-x) \leq (1+\delta)x$ for all $0 \leq x \leq \varepsilon$, and $N_1 \geq N_0$ such that $\gamma_n < \varepsilon$ for all $n \geq N_1$. Then

$$(1-\gamma_n)^{-1} = e^{-\ln(1-\gamma_n)} \leq e^{(1+\delta)\gamma_n} \text{ for all } n \geq N_1. \tag{4.8}$$

Since $\frac{w_{n+1}}{w_n} < 1$ for all n and $\{\frac{w_{n+1}}{w_n}\} \to 1$ we can find an index $n \geq N_1$ such that $\frac{w_{n+1}}{w_n} > R^2$ and $\frac{w_{k+1}}{w_k} \leq \frac{w_{n+1}}{w_n}$ for all $k \leq n$. We will verify (4.6) for r defined by $r^2 = \frac{w_{n+1}}{w_n} = 1 - \gamma_n$.

We have

$$\begin{aligned}
&\sum_{k=0}^{\infty} \frac{w_k}{r^{2k+2}} \max\{r^2 - \frac{w_{k+1}}{w_k}, 0\} \\
&\quad = \sum_{k=0}^{n} \frac{w_k}{r^{2k+2}} \max\{r^2 - \frac{w_{k+1}}{w_k}, 0\} + \sum_{k=n+1}^{m_n} \frac{w_k}{r^{2k+2}} \max\{r^2 - \frac{w_{k+1}}{w_k}, 0\} \\
&\quad \leq 1 - \frac{w_{n+1}}{r^{2n+2}} + \frac{w_{n+1}}{r^{2m_n+2}} \sum_{k=n+1}^{m_n} (\delta_k - \gamma_n).
\end{aligned}$$

Since $r^{-2} = (1-\gamma_n)^{-1}$ we have by (4.8) and (4.7)

$$r^{-2(m_n-n)} \sum_{k=n+1}^{m_n} (\delta_k - \gamma_n) \leq e^{(1+\delta)(m_n-n)\gamma_n} \sum_{k=n+1}^{m_n} (\delta_k - \gamma_n) < 1.$$

This verifies the first part of the lemma. To see the final part we will verify the hypothesis of the first part. It is clear that the first limsup is finite. Furthermore, we note that the hypothesis implies that for all sufficiently large n we have $(m_n - n)\delta_n \leq a < 1$. Hence

$$e^{(m_n-n)\gamma_n} \sum_{k=n+1}^{m_n} (\delta_k - \gamma_n) \leq e^{a\frac{\gamma_n}{\delta_n}} (m_n - n)(\delta_n - \gamma_n) \leq e^{ax} a(1-x),$$

where $0 \leq x = \frac{\gamma_n}{\delta_n} \leq 1$. It is easy to see that this last expression must be less than 1. $\square$

Example 4.8. Let $\alpha_+(w)$ and $\alpha_-(w)$ be as defined in Corollary 4.3. If $0 < \alpha_- \leq \alpha_+ < \infty$, and if

$$\alpha_+ \left(\ln \frac{\alpha_+}{\alpha_-} - (1 - \frac{\alpha_-}{\alpha_+}) \right) e^{\alpha_+ - \alpha_-} < 1,$$

then $\mathcal{H}_w$ satisfies the conclusion of Theorem 2.4. In fact, there is a weight sequence $\tilde{w}$ such that $\mathcal{H}_{\tilde{w}}$ satisfies (1.9) and the norms on $\mathcal{H}_w$ and $\mathcal{H}_{\tilde{w}}$ are equivalent.

There is some overlap with Corollary 4.3. In particular, we note that if α_- is sufficiently large, then this method applies to more weight sequences than Corollary 4.3 and gives a stronger conclusion.

Proof. Let $\varepsilon > 0$ such that $\alpha_- - \varepsilon > 0$. There is some N such that for all $k \geq N$ we have

$$0 < \frac{\alpha_- - \varepsilon}{k+1} \leq 1 - \frac{w_{k+1}}{w_k} \leq \frac{\alpha_+ + \varepsilon}{k+1}.$$

Thus, by changing at most finitely many of the w_n we may assume that $\{w_n\}$ is strictly decreasing. The changed weight sequence will produce an equivalent norm on $\mathcal{H}_w$.

Now, if $n \geq N$ and $n \leq k \leq m_n$, then

$$\frac{\alpha_- - \varepsilon}{n+1} \leq \gamma_n \leq \delta_k \leq \frac{\alpha_+ + \varepsilon}{k+1}.$$

Thus, $1 \leq \frac{m_n+1}{n+1} \leq \frac{\alpha_+ + \varepsilon}{\alpha_- - \varepsilon}$ and

$$\begin{aligned}(m_n - n)\gamma_n &\leq (m_n - n)\frac{\alpha_+ + \varepsilon}{m_n + 1} \\ &= (1 - \frac{n+1}{m_n+1})(\alpha_+ + \varepsilon) \\ &\leq \alpha_+ - \alpha_- + 2\varepsilon.\end{aligned}$$

Furthermore,

$$\begin{aligned}\sum_{k=n+1}^{m_n} (\delta_k - \gamma_n) &\leq \int_{n+1}^{m_n+1} \frac{\alpha_+ + \varepsilon}{x} dx - (m_n - n)\frac{\alpha_- - \varepsilon}{n+1} \\ &= (\alpha_+ + \varepsilon) \ln \frac{m_n + 1}{n+1} - (\alpha_- - \varepsilon)(\frac{m_n+1}{n+1} - 1) \\ &\leq (\alpha_+ + \varepsilon) \ln \frac{\alpha_+ + \varepsilon}{\alpha_- - \varepsilon} - (\alpha_- - \varepsilon)(\frac{\alpha_+ + \varepsilon}{\alpha_- - \varepsilon} - 1) \\ &= \ln \frac{\alpha_+ + \varepsilon}{\alpha_- - \varepsilon} - (1 - \frac{\alpha_- - \varepsilon}{\alpha_+ + \varepsilon})\end{aligned}$$

since the function $f(x) = (\alpha_+ + \varepsilon) \ln x - (\alpha_- - \varepsilon)(x - 1)$ is increasing for $1 \leq x \leq \frac{\alpha_+ + \varepsilon}{\alpha_- - \varepsilon}$. Thus by choosing ε sufficiently small the hypothesis implies

$$\limsup_{n\to\infty} (m_n - n)\gamma_n < \infty$$

and

$$\limsup_{n\to\infty} e^{(m_n - n)\gamma_n} \sum_{k=n+1}^{m_n} (\delta_k - \gamma_n) < 1.$$

Hence the result follows from Lemma 4.7. □

An example of exponentially decreasing weights is given by the following.

Example 4.9. Let $\alpha > 0$ and $0 < \gamma < 1$ such that

$$1 - \frac{w_{n+1}}{w_n} = \frac{\alpha}{n^\gamma} + o(1/n) \quad \text{as } n \to \infty,$$

then $\mathcal{H}_w$ satisfies the conclusion of Theorem 2.4.

We omit the verification as it is similar to Example 4.8. One can use the second part of Lemma 4.7. We mention that one can show that such weights are of the general form $w_n = b_n e^{-\frac{\alpha}{1-\gamma} n^{1-\gamma}}$, $b_n = e^{\sum_{k=0}^n \frac{\varepsilon_k}{k+1}}$, where $|\varepsilon_k| \to 0$.

Corollary 4.10. *If $M_\zeta|\mathcal{H}_w$ is a hyponormal weighted shift operator, then $\mathcal{H}_w$ satisfies (1.9).*

Proof. It is well-known that under our assumptions on the weights w the operator $M_\zeta|\mathcal{H}_w$ is hyponormal, if and only if w_{n+1}/w_n is nondecreasing, [C], p. 55. In the notation of Lemma 4.7 that implies that $m_n(w) = n$ for each n. Thus, if w_n is also strictly decreasing, then the result follows from Lemma 4.7. Having observed this, it becomes clear that the general case follows directly from Corollary 4.6. □

As pointed out in the introduction, an alternative proof of this result can be found in [AB].

Corollary 4.11. *Let $u = (u_n)_{n\geq 0}$ be a sequence of positive numbers such that the sequence $w = (1/u_n)_{n\geq 0}$ of reciprocals satisfies the hypothesis of Lemma 4.7, then every nonzero invariant subspace of $M_\zeta|\mathcal{H}_u$ has index 1.*

Proof. This follows immediately from Corollary 3.1 and Lemma 4.7. □

We point out that in the particular case mentioned in Corollary 4.10 it follows that the reproducing kernel for the Cauchy dual of $\mathcal{H}_w$ is a so-called Pick kernel (or complete NP kernel). In this case the conclusion of Corollary 4.11 was known, [MT].

REFERENCES

[AB] Abakumov, E., Borichev, A. *Shift invariant subspaces with arbitrary indices in l^p spaces.* J. Funct. Anal. 188 (2002), no. 1, 1–26.

[ARi] Aleman, A., Richter, S. *Simply invariant subspaces of H^2 of some multiply connected regions.* Integral Equations Operator Theory **24** (1996), no. 2, 127–155.

[AR] Aleman,A., Ross, W. T. *The backward shift on weighted Bergman spaces.* Michigan Math. J. **43** (1996), no. 2, 291–319.

[ARR] Aleman,A., Richter, S., Ross, W. T. *Pseudocontinuations and the backward shift.* Indiana Univ. Math. J. **47** (1998), no. 1, 223–276.

[ARS] Aleman A.; Richter, S.; Sundberg, C. *Analytic contractions and nontangential boundary behavior.* Preprint 2004.

[B] Borichev, A. *On convolution equations with restrictions on supports.* Algebra i Analiz **14** (2002), no. 2, 1–10; St. Petersburg Math. J. **14** (2003), no. 2, 179–187.

[C] Conway, J.B., *The theory of subnormal operators.* Amer. Math. Soc. (1991), Mathematical Surveys and Monographs, Vol. 36.

[DSS] Douglas, R. G., Shapiro, H. S., Shields, A. L. *Cyclic vectors and invariant subspaces for the backward shift operator.* Ann. Inst. Fourier (Grenoble) **20** 1970 fasc. 1, 37–76.

[D] Duren, Peter L. *Theory of H^p spaces.* Pure and Applied Mathematics, Vol. 38 Academic Press, New York-London 1970.

[E] Esterle, J. *Toeplitz operators on weighted Hardy spaces.* Algebra i Analiz **14** (2002), no. 2, 92–116; St. Petersburg Math. J. **14** (2003), no. 2, 251–272.

[K] Koosis, Paul. *Introduction to H_p spaces.* London Mathematical Society Lecture Note Series, 40. Cambridge University Press, Cambridge-New York, 1980.

[MR] McCullough, S., Richter, S., *Bergman-type reproducing kernels, contractive divisors, and dilations.* J. Funct. Anal. **190** (2002), 447-480.

[MT] McCullough, S., Trent, T., *Invariant subspaces and Nevanlinna Pick kernels.* J. Funct. Anal. **178** (2000), 226-249.

[Ri] Richter, S. *Invariant subspaces in Banach spaces of analytic functions.* Trans. Amer. Math. Soc. **304** (1987), no. 2, 585–616.

[RiSu] Richter, S., Sundberg, C., *Invariant subspaces of the Dirichlet shift and pseudocontinuations.*,Trans. Amer. Math. Soc. **341** (1994), 863-879.

[RS1] Ross, W. T.; Shapiro, H. S.; *Generalized analytic continuation.* Amer. Math. Soc. 2002, University Lecture series, Vol. 25.

[RS2] Ross, W. T.; Shapiro, H. S.; *Prolongations and cyclic vectors.* Computational Methods and Function Theory, to appear.

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