WEAK PRODUCTS OF DIRICHLET FUNCTIONS

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Abstract. For a Hilbert space $\mathcal{H}$ of functions let $\mathcal{H} \odot \mathcal{H}$ be the space of weak products of functions in $\mathcal{H}$, i.e. all functions h that can be written as $h = \sum_{i=1}^{\infty} f_i g_i$ for some $f_i, g_i \in \mathcal{H}$ with $\sum_{i=1}^{\infty} \|f_i\| \|g_i\| < \infty$. Let D denote the Dirichlet space of the unit circle ∂D, i.e. the nontangential limits of functions $f \in \operatorname{Hol}(\mathbb{D})$ with $\int_{\mathbb{D}} |f'|^2 dA < \infty$ and let D_h be the harmonic Dirichlet space, which consists of functions of the form $f + \overline{g}$ for $f, g \in D$.

In this paper we show that every real-valued function in $D_h \odot D_h$ is a single product of two functions in D_h and that the Cauchy projection is a bounded operator from $D_h \odot D_h$ onto $D \odot D$. It follows that $D \odot D$ consists exactly of the H^1-functions whose real and imaginary parts are single products of D_h-functions. The dual space of $D \odot D$ was characterized by Arcozzi, Rochberg, Sawyer and Wick in [ARSW10] and the result implies the characterization of the dual of $D_h \odot D_h$. We will show that the characterization of the dual of $D_h \odot D_h$ also follows from results of Mazya and Verbitsky, [MV02]. Thus, we will establish a precise connection between the results of [ARSW10] and [MV02].

We also prove some general results about weak product spaces of analytic functions. For example, we show that $\mathcal{H} \odot \mathcal{H} \subseteq \mathcal{H}(k^2)$, where $\mathcal{H}(k^2)$ is the space of analytic functions whose reproducing kernel is the square of the reproducing kernel of $\mathcal{H}$. Furthermore, we will show that some of the above mentioned results for the Dirichlet space D hold for the superharmonically weighted Dirichlet spaces $D(\mu)$.

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1. Introduction

Let $d \geq 1$, $\Omega \subseteq \mathbb{C}^d$ be an open, connected, and nonempty set, and let $\mathcal{H} \subseteq \operatorname{Hol}(\Omega)$ be a reproducing kernel Hilbert space.

That means that $\mathcal{H}$ consists entirely of analytic functions on Ω and for each $z \in \Omega$ there is a $k_z \in \mathcal{H}$ such that $f(z) = \langle f, k_z \rangle$ for each $f \in \mathcal{H}$. The weak product of $\mathcal{H}$ is denoted by $\mathcal{H} \odot \mathcal{H}$ and it is defined to be the collection of all functions $h \in \operatorname{Hol}(\Omega)$ such that there are sequences $\{f_i\}_{i\geq 1}, \{g_i\}_{i\geq 1} \subseteq \mathcal{H}$ with $\sum_{i=1}^\infty \|f_i\|\|g_i\| < \infty$ and $h(z) = \sum_{i=1}^\infty f_i(z)g_i(z)$ for all $z \in \Omega$. Note that whenever $\sum_{i=1}^\infty \|f_i\|\|g_i\| < \infty$, then

$$\sum_{i=1}^\infty |f_i(z)||g_i(z)| \leq \|k_z\|^2 \sum_{i=1}^\infty \|f_i\|\|g_i\| < \infty,$$

thus the series will converge locally uniformly to the analytic function h.

We define a norm on $\mathcal{H} \odot \mathcal{H}$ by

$$\|h\|_* = \inf\{\sum_{i=1}^\infty \|f_i\|\|g_i\| : h(z) = \sum_{i=1}^\infty f_i(z)g_i(z) \text{ for all } z \in \Omega\}.$$

It is easy to see that $\mathcal{H} \odot \mathcal{H}$ is a Banach space of analytic functions on Ω such that point evaluations are continuous. We have included a proof of the completeness in the next section. From the remark in the previous paragraph it follows that

$$|h(z)| \leq \|k_z\|^2 \|h\|_* \quad \text{for all } h \in \mathcal{H} \odot \mathcal{H}, z \in \Omega.$$

Weak products first appeared in a paper by Coifman, Rochberg and Weiss [CRW76]. They arise in connection with Hankel operators and they have been used in an intermediate step in the proof of the H^1-BMO and the L^1_a-Bergman-Bloch space duality. The paper [ARSW10] (p. 22-24) contains an excellent motivation for the study of weak products and a summary of results about such spaces, also see [ARSW11]. Here we just mention that if $\mathcal{H}$ is the Hardy space H^2 of the unit disc $\mathbb{D}$, then it follows from the Riesz factorization that $H^2 \odot H^2 = H^1$ with equality of norms, and in fact every $h \in H^2 \odot H^2$ can be written as a single product of functions in H^2. Furthermore the paper [ARSW11] contains a number of results about $D \odot D$, where D is the Dirichlet space of the disc consisting of analytic functions f such that $\int_{\mathbb{D}} |f'|^2 \frac{dA}{\pi} < \infty$. However, currently no simple description of which functions belong to $D \odot D$ is available. In particular, it appears to be unknown whether each function in $D \odot D$ can be written as a single product of two Dirichlet functions.

As alluded to earlier the weak product spaces were introduced, because in many cases there is a simple way to express the dual space as another space of analytic functions (see e.g. Theorem 1.3 below). For the definition that will make the connection we need to require another property of $\mathcal{H}$. We write $\mathrm{Hol}(\overline{\Omega})$ for the algebra of all functions f on Ω such that f extends to be analytic in a neighborhood of $\overline{\Omega}$. If $\mathcal{H} \subseteq \mathrm{Hol}(\Omega)$ is a reproducing kernel Hilbert space such that $\mathrm{Hol}(\overline{\Omega})$ is densely contained in $\mathcal{H}$, then we define

$$\mathcal{X}(\mathcal{H}) = \{b \in \mathcal{H} : \exists C > 0 \ |\langle \varphi\psi, b\rangle| \leq C\|\varphi\|\|\psi\| \ \forall \ \varphi, \psi \in \mathrm{Hol}(\overline{\Omega})\}$$

and for $b \in \mathcal{X}(\mathcal{H})$ write $\|b\|_{\mathcal{X}}$ for the infimum of all $C > 0$ such that $|\langle \varphi\psi, b\rangle| \leq C\|\varphi\|\|\psi\|$ for all $\varphi, \psi \in \mathrm{Hol}(\overline{\Omega})$. It is easy to see that $(\mathcal{X}(\mathcal{H}), \|\cdot\|_{\mathcal{X}})$ is a Banach space. Again we have provided some details in Section 2.

It is clear that for each $b \in \mathcal{X}(\mathcal{H})$ the rule $H_b(\varphi, \psi) = \langle \varphi\psi, b\rangle, \varphi, \psi \in \mathrm{Hol}(\overline{\Omega})$ extends to be a bounded bilinear form on $\mathcal{H} \oplus \mathcal{H}$. This bilinear form defines a bounded operator H_b, the Hankel operator with symbol b. Thus, $\mathcal{X}(\mathcal{H})$ is the space of symbols for the Hankel operators on $\mathcal{H}$.

It was a basic observation of Coifman, Rochberg, and Weiss ([CRW76]) that in many cases one can identify $\mathcal{X}(\mathcal{H})$ with the dual space of $\mathcal{H} \odot \mathcal{H}$. For a vector subspace $\mathcal{L} \subseteq \mathcal{H}$ let

$$\mathcal{L}\widehat{\odot}\mathcal{L} = \{\sum_{i=1}^{n} f_i g_i, n \in \mathbb{N}, f_1, ..., f_n, g_1, ..., g_n \in \mathcal{L}\}$$

be the set of finite sums of products of elements in $\mathcal{L}$ and define a norm on $\mathcal{L}\widehat{\odot}\mathcal{L}$ by

$$\|h\|_{\bullet\mathcal{L}} = \inf\{\sum_{i=1}^{n} \|f_i\|\|g_i\| : h = \sum_{i=1}^{n} f_i g_i, f_i, g_i \in \mathcal{L}\}.$$

Clearly $\mathcal{L}\widehat{\odot}\mathcal{L} \subseteq \mathcal{H}\widehat{\odot}\mathcal{H} \subseteq \mathcal{H} \odot \mathcal{H}$ and $\|h\|_* \leq \|h\|_{\bullet\mathcal{H}} \leq \|h\|_{\bullet\mathcal{L}}$ for every $h \in \mathcal{L}\widehat{\odot}\mathcal{L}$. It is easy to see that $\mathcal{L}\widehat{\odot}\mathcal{L}$ is dense in $\mathcal{H} \odot \mathcal{H}$ with respect to the norm $\|\cdot\|_*$ (see Lemma 2.3). We use $(\mathcal{L}\widehat{\odot}\mathcal{L})_\bullet$ to denote the completion of $\mathcal{L}\widehat{\odot}\mathcal{L}$ with respect to the norm $\|\cdot\|_{\bullet\mathcal{L}}$. The above inequality implies that the inclusion of $\mathcal{L}\widehat{\odot}\mathcal{L}$ into $\mathcal{H}\odot\mathcal{H}$ extends to be a contraction $U_{\mathcal{L}} : (\mathcal{L}\widehat{\odot}\mathcal{L})_\bullet \to \mathcal{H}\odot\mathcal{H}$.

Theorem 1.1. *Let $\mathcal{L}\subseteq\mathcal{H}$ be a dense vector subspace. Then $U_{\mathcal{L}}$ is onto and the induced map $\tilde{U}_{\mathcal{L}} : (\mathcal{L}\widehat{\odot}\mathcal{L})_\bullet/\ker U_{\mathcal{L}} \to \mathcal{H}\odot\mathcal{H}$ is isometric.*

It follows that $\|\varphi\|_* = \|\varphi\|_{\bullet\mathcal{L}}$ for all $\varphi\in\mathcal{L}\widehat{\odot}\mathcal{L}$, if and only if $U_{\mathcal{L}}$ is 1-1. Although we know of no examples where the two norms are actually different, we can show that the embedding of $\mathcal{L}\widehat{\odot}\mathcal{L}$ into $\mathcal{H}\odot\mathcal{H}$ is isometric only under some extra hypothesis on $\mathcal{H}$. In fact, in the literature the definition of the space $\mathcal{H}\odot\mathcal{H}$ is often taken to be what we call $(\mathcal{H}\widehat{\odot}\mathcal{H})_\bullet$ (thus following the definition of the projective tensor product). It is not clear that this space is injectively embedded in the holomorphic functions.

We will address this issue in further detail in Section 2, and it turns out that for the Dirichlet space and many other spaces on the unit disc or unit ball of $\mathbb{C}^d$ the spaces $(\mathcal{H}\widehat{\odot}\mathcal{H})_\bullet$ and $\mathcal{H}\odot\mathcal{H}$ can be identified.

For $d\ge 1$ let $\mathbb{B}_d$ denote the open unit ball in $\mathbb{C}^d$, and for $0\le r<1$, $f\in\mathcal{H}$, and $z\in\mathbb{B}_d$ let $f_r(z)=f(rz)$.

Theorem 1.2. *Let $\mathcal{H}\subseteq Hol(\mathbb{B}_d)$ be a reproducing kernel Hilbert space such that*

(1.1) *$\mathcal{H}$ contains $Hol(\overline{\mathbb{B}}_d)$,*
(1.2) *if $f_n, f\in Hol(\overline{\mathbb{B}}_d)$ such that $f_n\to f$ uniformly in some open neighborhood of $\overline{\mathbb{B}}_d$, then $f_n\to f$ in $\mathcal{H}$,*
(1.3) *there is $c>0$ such that, if $0<r<1$ and if $f\in\mathcal{H}$, then $f_r\in\mathcal{H}$ and $\|f_r\|\le c\|f\|$.*

Then $Hol(\overline{\mathbb{B}}_d)$ is dense in $\mathcal{H}$ and

$$\|\varphi\|_* = \|\varphi\|_{\bullet\mathcal{H}} = \|\varphi\|_{\bullet\mathcal{L}} = \inf\{\sum_{i=1}^n \|\varphi_i\|\|\psi_i\| : \varphi=\sum_{i=1}^n \varphi_i\psi_i, \varphi_i,\psi_i\in\mathcal{L}\}$$

for all $\varphi\in\mathcal{L}= Hol(\overline{\mathbb{B}}_d)$.

We note that (1.2) actually follows from (1.1) and the closed graph theorem, we included it for clarity. One of the reasons we are interested in the previous theorem is because it leads to the identification of $\mathcal{X}(\mathcal{H})$ with the the dual of $\mathcal{H}\odot\mathcal{H}$.

Theorem 1.3. *Let $Hol(\overline{\Omega})$ be dense in $\mathcal{H}$, and suppose there is a linear subspace $\mathcal{L}\subseteq Hol(\overline{\Omega})$ which is dense in $\mathcal{H}$ and which satisfies $\|\varphi\|_* = \|\varphi\|_{\bullet\mathcal{L}}$ for all $\varphi\in\mathcal{L}\widehat{\odot}\mathcal{L}$. Then $(\mathcal{H}\odot\mathcal{H})^* = \mathcal{X}(\mathcal{H})$.*

This means if for $b\in\mathcal{X}(\mathcal{H})$ we define L_b on $\mathcal{H}$ by

$$L_b(h)=\langle h,b\rangle,$$

then L_b extends to be bounded on $\mathcal{H} \odot \mathcal{H}$, and the map $b \to L_b$ is a conjugate linear isometric isomorphism of $\mathcal{X}(\mathcal{H})$ onto $(\mathcal{H} \odot \mathcal{H})^*$.

In [ARSW10], Corollary 1.2, the isomorphism between $\mathcal{X}(D)$ and $(D\widehat{\odot}D)_\bullet$ is shown, and their argument easily also establishes Theorem 1.3. We have repeated the short proof in Section 2, because of the distinction between the two norms.

Of course, the representation for the dual space (respectively, the space of Hankel symbols) as in Theorem 1.3 is most useful, if there is a more explicit characterization of the space $\mathcal{X}(\mathcal{H})$. In many cases such a result is known, in particular for $\mathcal{H} = H^2$, $\mathcal{X}(\mathcal{H})$ consists of the BMOA-functions [Fef71] and for $\mathcal{H} = L^2_a$, $\mathcal{X}(\mathcal{H})$ is the Bloch space [CRW76]. It is notable that in both cases $\mathcal{X}(\mathcal{H})$ can be described by a Carleson measure condition. Furthermore, the main result of [ARSW10] is that for $\mathcal{H} = D$, the Dirichlet space of the disc, one has

$$\mathcal{X}(D) = \{b \in D : |b'|^2 dA \text{ is a Carleson measure for } D\}.$$

Recall that a positive Borel measure μ on Ω is a Carleson measure for the space $\mathcal{H}$ of holomorphic functions on Ω, if

$$\int_\Omega |f|^2 d\mu \le C\|f\|^2 \text{ for all } f \in \mathcal{H}.$$

We also mention that the paper [ARSW11] contains a number of further results about $D \odot D$ and $\mathcal{X}(D)$.

In Section 3 we will prove the following theorem.

Theorem 1.4. *If $\mathcal{H} = \mathcal{H}(k) \subseteq$ Hol(Ω) has reproducing kernel k, then*

$$\mathcal{H} \odot \mathcal{H} \subseteq \mathcal{H}(k^2)$$

with $\|h\|_{\mathcal{H}(k^2)} \le \|h\|_*$ for all $h \in \mathcal{H} \odot \mathcal{H}$.

For $\mathcal{H} = H^2$ the inclusion $H^1 \subseteq L^2_a$ was known, but we obtain the best constant for the inclusion. The result appears new for $\mathcal{H} = D$. We refer to Section 3 for detailed comments.

Starting with Section 4 we will restrict our attention to weighted Dirichlet spaces. For a non-negative superharmonic function w on the unit disc $\mathbb{D}$ we consider spaces of the form

$$\mathcal{H} = \{f \in \text{ Hol}(\mathbb{D}) : \int_{\mathbb{D}} |f'(z)|^2 w(z) dA(z) < \infty\}. \tag{1.4}$$

By the representation theorem for superharmonic functions (see [Lan72], page 109) such non-negative superharmonic weights w can be represented by use of some measure μ on the closed unit disc, and one can represent the norm on the space directly in terms of the measure μ.

One obtains $\|f\|^2 = \|f\|^2_{L^2(\mathbb{T})} + \int_{|z|\le 1} D_z(f)d\mu(z)$, where $D_z(f)$ denotes the local Dirichlet integral of f (see Section 4 or [Ale93]). Thus we will write $\mathcal{H} = D(\mu)$ (see [Ale93]).

The results of Sections 2 and 3 apply to this setting. Based on the above mentioned results for Hardy-, Bergman-, and unweighted Dirichlet spaces one expects that the Carleson measures for these spaces will play a role in the theory. Furthermore, some of the known characterizations of Carleson measures involve capacities which can be defined by considering the corresponding spaces of harmonic functions $D_h(\mu) = \{f + \overline{g} : f, g \in D(\mu), g(0) = 0\}$, $\|f + \overline{g}\|^2 = \|f\|^2 + \|g\|^2$, see [Cha11].

Theorem 1.5. *If μ is a measure on the closed unit disc.*

Then every real-valued function $h \in D_h(\mu)\odot D_h(\mu)$ is a single product of the type $h = uv$, where $u, v \in D_h(\mu)$ are real-valued functions with $u \ge 0$.

In fact, if $h \in L^1(\mathbb{T})$ is real-valued, then $h \in D_h(\mu) \odot D_h(\mu)$, if and only if $h^+, h^- \in D_h(\mu) \odot D_h(\mu)$, where $h^+(e^{it}) = \max(h(e^{it}), 0)$ and $h^-(e^{it}) = -\min(h(e^{it}), 0)$.

Furthermore, if $D_h(\mu)$ is compactly contained in $L^2(\partial\mathbb{D})$, then for every real-valued function $h \in D_h(\mu) \odot D_h(\mu)$ there is a pair $u_0, v_0 \in D_h(\mu)$ with $h = u_0 v_0$, $u_0 \ge 0$, $|v_0| \le u_0$, $\|u_0\| = \|v_0\|$, and

$$\|u_0\|\|v_0\| = \inf\{\sum_{i\ge 1} \|f_i\|\|g_i\| : f_i, g_i \in D_h(\mu), h = \sum_{i\ge 1} f_i g_i\}.$$

See Theorem 4.3 for more information about the norm minimizing pair u_0, v_0. As a corollary we obtain that the capacity (and hence the class of exceptional sets) for the space $D_h(\mu)\odot D_h(\mu)$ equal the capacity for the space $D_h(\mu)$. We will give the details in Section 5.

Of course, Theorem 1.5 implies that every $h \in D(\mu) \odot D(\mu)$ is of the form $h = u_1v_1 + iu_2v_2$ for some real-valued $u_j, v_j \in D_h(\mu)$, $j = 1, 2$. This raises the question what exactly the real parts of $D(\mu)\odot D(\mu)$-functions are. For example, if $\mu = 0$, then $D(\mu) = H^2$ and $D_h(\mu) = L^2(\mathbb{T})$, $D_h(\mu) \odot D_h(\mu) = L^1(\mathbb{T})$, $D(\mu) \odot D(\mu) = H^1$, and it is well-known that not every real-valued L^1-function is the real part of an H^1-function. In contrast to this we will show that for the Dirichlet space D no further restriction is necessary. That follows from the following theorem.

Theorem 1.6. *The Cauchy projection is a bounded linear operator from $D_h \odot D_h$ onto $D \odot D$.*

From this it easily follows that the real and imaginary parts of $D\odot D$-functions are exactly the single products of functions in the harmonic Dirichlet space D_h. We will prove this result in Section 6.

In Section 7 we establish a precise connection between the results of [ARSW10] and [MV02] and thus give an answer to a question posed in [ARSW10].

Section 8 contains some examples that illustrate the difficulties in deciding whether every $D \odot D$-function is a single product of two functions in D.

In various arguments of this paper we will use the notion of a multiplier of a space of functions. Thus recall that a function φ is a multiplier for $\mathcal{H}$ if $\varphi f \in \mathcal{H}$ whenever $f \in \mathcal{H}$. We write $M(\mathcal{H})$ for the set of all multipliers of $\mathcal{H}$ and $\|\varphi\|_M = \|M_\varphi\|_{\mathcal{B}(\mathcal{H})}$ for the multiplier norm.

We would like to thank Alexandru Aleman for help with the proof of Lemma 2.5.

2. The general theory

Lemma 2.1. *If $\mathcal{H} \subseteq Hol(\Omega)$ is a reproducing kernel Hilbert space, then $(\mathcal{H} \odot \mathcal{H}, \|\cdot\|_*)$ is a Banach space of analytic functions such that point evaluations at points of Ω are continuous.*

Proof. We only show the completeness, everything else is elementary. In particular, we note that $|h(z)| \leq \|h\|_* \|k_z\|^2$ for all $h \in \mathcal{H} \odot \mathcal{H}$, hence convergence in $\mathcal{H} \odot \mathcal{H}$ implies local uniform convergence in Ω.

Let $\{h_n\}$ be a Cauchy sequence in $\mathcal{H} \odot \mathcal{H}$.

We may choose a subsequence h_{n_k} such that $\|h_{n_k} - h_{n_{k+1}}\|_* \leq 2^{-k}$ for each $k \geq 1$. Thus for each k there are sequences $\{f_{k,i}\}, \{g_{k,i}\} \subseteq \mathcal{H}$ such that $h_{n_k} - h_{n_{k+1}} = \sum_{i=1}^\infty f_{k,i} g_{k,i}$ and $\sum_{i=1}^\infty \|f_{k,i}\| \|g_{k,i}\| \leq 2 \cdot 2^{-k}$. Then the function

$$h := h_{n_1} - \sum_{k\geq 1} (h_{n_k} - h_{n_{k+1}}) = h_{n_1} - \sum_{k \geq 1} \sum_{i=1}^\infty f_{k,i} g_{k,i} \in \mathcal{H} \odot \mathcal{H}$$

For $j \geq 1$ we have $h - h_{n_j} = -\sum_{k \geq j} \sum_{i=1}^\infty f_{k,i} g_{k,i}$ and hence

$$\|h - h_{n_j}\|_* \leq \sum_{k \geq j} \sum_{i=1}^\infty \|f_{k,i}\| \|g_{k,i}\| \leq 4 \cdot 2^{-j}.$$

This implies

$$\|h - h_n\|_* \leq \|h - h_{n_j}\|_* + \|h_{n_j} - h_n\|_* \to 0$$

since $\{h_n\}$ is a Cauchy sequence. ■

Lemma 2.2. *If $\mathcal{H} \subseteq Hol(\Omega)$ is a reproducing kernel Hilbert space, and if $Hol(\overline{\Omega})$ is a dense subset of $\mathcal{H}$, then $\mathcal{X}(\mathcal{H})$ is a Banach space of analytic functions on Ω such that*

$$|b(z)| \leq \|1\| \|k_z\| \|b\|_{\mathcal{X}} \quad \textit{for all } b \in \mathcal{X}(\mathcal{H}), z \in \Omega.$$

Proof. The inequality is clear since $\mathrm{Hol}(\overline{\Omega})$ is dense in $\mathcal{H}$ and $1 \in \mathrm{Hol}(\overline{\Omega}) \subseteq \mathcal{H}$. In order to show the completeness we first note that $\|b\|^2 = \langle 1 \cdot b, b\rangle \leq \|1\|\|b\|\|b\|_{\mathcal{X}}$, hence $\|b\| \leq \|1\|\|b\|_{\mathcal{X}}$. Let $\{b_n\}$ be a Cauchy sequence in $\mathcal{X}(\mathcal{H})$. Then $\{b_n\}$ is a Cauchy sequence in $\mathcal{H}$ and converges to some $b \in \mathcal{H}$. As in the previous proof we may choose a subsequence b_{n_k} such that $\|b_{n_k} - b_{n_{k+1}}\|_{\mathcal{X}} \leq 2^{-k}$ for each $k \geq 1$. Then for each $k \geq 1$ we have $b - b_{n_k} = \sum_{j=k}^{\infty} b_{n_{j+1}} - b_{n_j}$, where the sum converges in $\mathcal{H}$. Thus for $\varphi, \psi \in \mathrm{Hol}(\overline{\Omega})$ we have

$$\begin{aligned}|\langle \varphi\psi, b - b_{n_k}\rangle| &\leq \sum_{j=k}^{\infty} |\langle \varphi\psi, b_{n_{j+1}} - b_{n_j}\rangle| \\ &\leq \sum_{j=k}^{\infty} \|\varphi\|\|\psi\|\|b_{n_{j+1}} - b_{n_j}\|_{\mathcal{X}} \\ &\leq 2 \cdot 2^{-k}\|\varphi\|\|\psi\|.\end{aligned}$$

Thus $b \in \mathcal{X}(\mathcal{H})$ and $\|b - b_{n_k}\|_{\mathcal{X}} \leq 2 \cdot 2^{-k}$. It now follows easily that $b_n \to b$ in $\mathcal{X}(\mathcal{H})$. ■

Lemma 2.3. *If $\mathcal{L} \subseteq \mathcal{H}$ is a dense vector subspace, if $h \in \mathcal{H} \odot \mathcal{H}$, and if $\varepsilon > 0$, then there are $f_i, g_i \in \mathcal{L}$ such that $\sum_{i=1}^{\infty} \|f_i\|\|g_i\| < \|h\|_* + \varepsilon$ and $h = \sum_{i=1}^{\infty} f_i g_i$.*

In particular, $\mathcal{L}\widehat{\odot}\mathcal{L}$ is dense in $\mathcal{H} \odot \mathcal{H}$ in the $\|\cdot\|_*$-norm and

$$\|h\|_* = \inf\{\sum_{i=1}^{\infty} \|f_i\|\|g_i\| : h = \sum_{i=1}^{\infty} f_i g_i, \quad f_i, g_i \in \mathcal{L}\}.$$

for every $h \in \mathcal{H} \odot \mathcal{H}$.

Proof. First note that if $f \in \mathcal{H}$ and $\delta > 0$, then there is a sequence u_n in $\mathcal{L}$ such that $u_n \to f$ in $\mathcal{H}$ and $\|u_n - u_{n+1}\| < 2^{-n}$. Then for all N we have $f = u_N + \sum_{n=N}^{\infty}(u_{n+1} - u_n)$ where the sum converges in $\mathcal{H}$. For sufficiently large N we have $\|u_N\| + \sum_{n=N}^{\infty} \|u_{n+1} - u_n\| \leq \|f\| + \delta$. Thus, $f = \sum_{j=1}^{\infty} f_j$, $f_j \in \mathcal{L}$ and $\sum_{j=1}^{\infty} \|f_j\| < \|f\| + \delta$.

Now let $h \in \mathcal{H} \odot \mathcal{H}$ and $\varepsilon > 0$. Then there are $f_i, g_i \in \mathcal{H}$ such that $h = \sum_{i=1}^{\infty} f_i g_i$ and $\sum_{i=1}^{\infty} \|f_i\|\|g_i\| < \|h\|_* + \varepsilon/2$. Choose a sequence $\delta_i > 0$ with $\sum_{i=1}^{\infty} \delta_i(\|f_i\| + \|g_i| + \delta_i) < \varepsilon/2$. It is easy to see that this can be done. By the first part of the proof for each i there are sequences f_{ij}, g_{ij} in $\mathcal{L}$ such that $\sum_{j=1}^{\infty} \|f_{ij}\| < \|f_i\|+\delta_i$ and $\sum_{j=1}^{\infty} \|g_{ij}\| < \|g_i\|+\delta_j$.

Then

$$\begin{aligned}\sum_{i=1}^{\infty}\sum_{j,k=1}^{\infty}\|f_{ij}\|\|g_{ik}\| &\le \sum_{i=1}^{\infty}(\|f_i\|+\delta_i)(\|g_i\|+\delta_i)\\ &\le \|h\|_*+\varepsilon\end{aligned}$$

and $h=\sum_{i=1}^{\infty}\sum_{j,k=1}^{\infty}f_{ij}g_{ik}$ is of the required form.

The density of $\mathcal{L}\widehat{\odot}\mathcal{L}$ in $\mathcal{H}\odot\mathcal{H}$ follows, because the partial sums of any representation of h converge in $\|\cdot\|_*$. ■

Proof of Theorem 1.1. Let $h\in\mathcal{H}\odot\mathcal{H}$ and let $\varepsilon>0$.

Then by Lemma 2.3 there are $\{f_i\},\{g_i\}\subseteq\mathcal{L}$ with $\sum_{i=1}^{\infty}\|f_i\|\|g_i\|<\|h\|_*+\varepsilon$ and $h(z)=\sum_{i=1}^{\infty}f_i(z)g_i(z)$ for all $z\in\mathbb{B}_d$.

Set $h_n=\sum_{i=1}^{n}f_ig_i$, then $\{h_n\}\subseteq\mathcal{L}\widehat{\odot}\mathcal{L}$ is a Cauchy sequence in $\|\cdot\|_{\bullet\mathcal{L}}$ and hence there is a $h_0\in(\mathcal{L}\widehat{\odot}\mathcal{L})_\bullet$ such that $\|h_n-h_0\|_{\bullet\mathcal{L}}\to 0$. Then $U_{\mathcal{L}}h_n\to U_{\mathcal{L}}h_0$ in $\mathcal{H}\odot\mathcal{H}$ and $h_n(z)=U_{\mathcal{L}}h_n(z)\to U_{\mathcal{L}}h_0(z)$ for all $z\in\Omega$. This implies $U_{\mathcal{L}}h_0=h$, so that $U_{\mathcal{L}}$ is onto.

We also see that

$$\begin{aligned}\|h\|_* = \|U_{\mathcal{L}}h_0\|_* &\le \|h_0\|_{\bullet\mathcal{L}}\\ &= \lim_{n\to\infty}\|h_n\|_{\bullet\mathcal{L}}\\ &\le \lim_{n\to\infty}\sum_{i=1}^{n}\|f_i\|\|g_i\|\\ &\le \|h\|_*+\varepsilon.\end{aligned}$$

Since $\varepsilon>0$ was arbitrary, this implies that the induced map $\tilde{U}_{\mathcal{L}}:(\mathcal{L}\widehat{\odot}\mathcal{L})_\bullet/\ker U_{\mathcal{L}}\to\mathcal{H}\odot\mathcal{H}$ is isometric. ■

Next we turn to the proof of Theorem 1.2. The following lemma is elementary.

Lemma 2.4. *If $\mathcal{H}$ satisfies conditions (1.1)-(1.3) of Theorem 1.2, then Hol($\overline{\mathbb{B}}_d$) is dense in $\mathcal{H}$ and for every $f\in\mathcal{H}$ we have $\|f_r-f\|\to 0$ as $r\to 1$.*

Proof. Condition (1.3) easily implies that for all $f\in\mathcal{H}$ we have $f_r\to f$ at least weakly. Thus Hol($\overline{\mathbb{B}}_d$) is dense in $\mathcal{H}$.

Now let $f\in\mathcal{H}$ and $\varepsilon>0$. By the above we may choose $g\in$ Hol($\overline{\mathbb{B}}_d$) such that $\|f-g\|<\varepsilon$. Then

$$\begin{aligned}\|f_r-f\| &\le \|f_r-g_r\|+\|g_r-g\|+\|g-f\|\\ &\le (1+c)\varepsilon+\|g_r-g\|\\ &\le (2+c)\varepsilon\end{aligned}$$

for all sufficiently large $r < 1$ (by condition (1.2)). ∎

The following lemma is the key for the proof of Theorem 1.2.

Lemma 2.5. *Let $\mathcal{L} = \ Hol(\overline{\mathbb{B}}_d)$ and suppose that $\mathcal{H}$ satisfies (1.1)-(1.3). If $\{h_n\} \subseteq \mathcal{L}\widehat{\odot}\mathcal{L}$ is a Cauchy sequence in the norm $\|\cdot\|_{\bullet\mathcal{L}}$, and if $h_n(z) \to 0$ for all $z \in \mathbb{B}_d$, then $\|h_n\|_{\bullet\mathcal{L}} \to 0$.*

Proof. For $0 \le r < 1$ define T_r on $\mathcal{L}\widehat{\odot}\mathcal{L}$ by $T_r h = h_r$, $h_r(z) = h(rz)$. Then the definition and hypothesis implies that for $h = \sum_{i=1}^n f_i g_i$, $f_i, g_i \in \mathcal{L}$, we have

$$\|T_r h\|_{\bullet\mathcal{L}} \le \sum_{i=1}^n \|f_{i,r}\| \|g_{i,r}\| \le c^2 \sum_{i=1}^n \|f_i\| \|g_i\|.$$

Thus $\|T_r h\|_{\bullet\mathcal{L}} \le c^2 \|h\|_{\bullet\mathcal{L}}$ for every $h \in \mathcal{L}\widehat{\odot}\mathcal{L}$. Hence it follows that for each r, $0 \le r < 1$, T_r extends to be bounded on $(\mathcal{L}\widehat{\odot}\mathcal{L})_\bullet$ with norm $\le c^2$. Lemma 2.4 and the inequality $\|g\|_{\bullet\mathcal{L}} \le \|1\|\|g\|$ for $g \in \mathcal{L} = \mathcal{L}\widehat{\odot}\mathcal{L}$ implies that

$$\|T_r f - f\|_{\bullet\mathcal{L}} \le \|1\| \|T_r f - f\|,$$

hence $T_r f \to f$ on the dense set $\mathcal{L}\widehat{\odot}\mathcal{L} \subseteq (\mathcal{L}\widehat{\odot}\mathcal{L})_\bullet$, hence $T_r h \to h$ in $\|\cdot\|_{\bullet\mathcal{L}}$ for every $h \in (\mathcal{L}\widehat{\odot}\mathcal{L})_\bullet$.

Since $\{h_n\}$ is a Cauchy sequence in $\|\cdot\|_{\bullet\mathcal{L}}$ there is an $h \in (\mathcal{L}\widehat{\odot}\mathcal{L})_\bullet$ such that $\|h_n - h\|_{\bullet\mathcal{L}} \to 0$. We need to show that $h = 0$ as an element of $(\mathcal{L}\widehat{\odot}\mathcal{L})_\bullet$ and we know that $h_n(z) \to 0$ for all $z \in \mathbb{B}_d$. Since $T_r h \to h$ it will be enough to show that $T_r h = 0$ for every $0 \le r < 1$. But for each $r < 1$ and $n \in \mathbb{N}$ we have

$$\begin{aligned} \|T_r h\|_{\bullet\mathcal{L}} &\le \|T_r(h - h_n)\|_{\bullet\mathcal{L}} + \|T_r h_n\|_{\bullet\mathcal{L}} \\ &\le c^2 \|h - h_n\|_{\bullet\mathcal{L}} + \|1\| \|h_{n,r}\|. \end{aligned}$$

The hypothesis implies that h_n converges to 0 locally uniformly in $\mathbb{B}_d$. This implies that $h_{n,r} \to 0$ uniformly in a neighborhood of $\overline{B}_d$. Hence it follows from condition (1.2) that $\|h_{n,r}\| \to 0$. Since we also have $\|h - h_n\|_{\bullet\mathcal{L}} \to 0$ it follows that $T_r h = 0$. ∎

Proof of Theorem 1.2. By Theorem 1.1 we have to show that $U_\mathcal{L}$ is 1-1, and this follows easily from the previous lemma. Indeed, if $h \in (\mathcal{L}\widehat{\odot}\mathcal{L})_\bullet$ with $Uh = 0$, then there is a sequence $h_n \in \mathcal{L}\widehat{\odot}\mathcal{L}$ such that $h_n \to h$ in $\|\cdot\|_{\bullet\mathcal{L}}$. Then $h_n(z) = Uh_n(z) \to Uh(z) = 0$ for each $z \in \mathbb{B}_d$, hence Lemma 2.5 implies that $\|h_n\|_{\bullet\mathcal{L}} \to 0$. This means $h = 0$. ∎

Proof of Theorem 1.3. Let $b \in \mathcal{X}(\mathcal{H})$ and let $h \in \mathcal{L}\widehat{\odot}\mathcal{L} \subseteq \mathcal{H}$. Then $h = \sum_{i=1}^{n} f_i g_i$ for some $f_1, ..., f_n, g_1, ..., g_n \in \mathcal{L}$ and

$$\begin{aligned} |\text{L}_b(h)| &= |\langle \sum_{i=1}^{n} f_i g_i, b\rangle| \\ &\leq \sum_{i=1}^{n} |\langle f_i g_i, b\rangle| \\ &\leq \sum_{i=1}^{n} \|b\|_{\mathcal{X}} \|f_i\| \|g_i\|. \end{aligned}$$

This implies $|L_b(h)| \leq \|b\|_{\mathcal{X}} \|h\|_{\bullet\mathcal{L}} = \|b\|_{\mathcal{X}} \|h\|_*$ for every $h \in \mathcal{L}\widehat{\odot}\mathcal{L}$. By Lemma 2.3 it follows that L_b extends to be bounded on $\mathcal{H} \odot \mathcal{H}$ with $\|L_b\| \leq \|b\|_{\mathcal{X}}$.

If $L \in (\mathcal{H} \odot \mathcal{H})^*$, then for $f \in \mathcal{H}$ we have

$$|L(f)| \leq \|L\| \|f\|_* \leq \|L\| \|f\| \|1\|$$

since $f = 1 \cdot f$. Thus there is a $b \in \mathcal{H}$ such that $L(f) = \langle f, b\rangle = L_b(f)$ for all $f \in \mathcal{H}$. Furthermore, if $\varphi, \psi \in \operatorname{Hol}(\overline{\Omega})$, then

$$|\langle \varphi\psi, b\rangle| = |L(\varphi\psi)| \leq \|L\| \|\varphi\psi\|_* \leq \|L\| \|\varphi\| \|\psi\|.$$

Hence $\|b\|_{\mathcal{X}} \leq \|L\| = \|L_b\|$. ■

3. Inclusion of $\mathcal{H} \odot \mathcal{H}$ into a Hilbert space

Theorem 3.1. *If $\mathcal{H} = \mathcal{H}(k) \subseteq \operatorname{Hol}(\Omega)$ has reproducing kernel k, then*

$$\mathcal{H} \odot \mathcal{H} \subseteq \mathcal{H}(k^2)$$

with $\|h\|_{\mathcal{H}(k^2)} \leq \|h\|_*$ for all $h \in \mathcal{H} \odot \mathcal{H}$.

Notice that for each $z \in \Omega$ we have $\|k_z^2\|_{\mathcal{H}(k^2)} = k_z(z) = \|k_z\|^2 \geq \|k_z^2\|_* \geq \|k_z^2\|_{\mathcal{H}(k^2)}$ and hence it follows that the norm of the inclusion equals one.

Proof. Let $f, g \in \mathcal{H}$. We will begin by showing that $fg \in \mathcal{H}(k^2)$ and $\|fg\|_{\mathcal{H}(k^2)} \leq \|f\| \|g\|$. The statement is clearly true if either $f = 0$ or $g = 0$, thus we may assume that $\|f\| = \|g\| = 1$.

It is well-known that if $u_\lambda(z)$ and $v_\lambda(z)$ are reproducing kernels for Hilbert spaces $\mathcal{H}(u)$ and $\mathcal{H}(v)$, then $\mathcal{H}(u) \subseteq \mathcal{H}(v)$ with $\|h\|_v \leq \|h\|_u$ for all $h \in \mathcal{H}(u)$ if and only if $v_\lambda(z) - u_\lambda(z)$ is a positive definite kernel (see [Aro50]). If we apply this with $u_\lambda(z) = \overline{h(\lambda)} h(z)$ we see that $h \in \mathcal{H}(v)$ with $\|h\|_v \leq 1$ if and only if $v_\lambda(z) - \overline{h(\lambda)} h(z)$ is positive definite. Thus our hypothesis implies that $k_\lambda(z) - \overline{f(\lambda)}f(z)$ and $k_\lambda(z) - \overline{g(\lambda)}g(z)$ are positive definite and we must show that

$$\begin{aligned} k_\lambda^2(z) - &\overline{f(\lambda)g(\lambda)}f(z)g(z) \\ &= k_\lambda(z)\left(k_\lambda(z) - \overline{f(\lambda)}f(z)\right) + \overline{f(\lambda)}f(z)\left(k_\lambda(z) - \overline{g(\lambda)}g(z)\right) \end{aligned}$$

is positive definite. This follows immediately from the Schur product Theorem and the obvious fact that the sum of positive definite kernels is positive definite.

If $h \in \mathcal{H}\widehat{\odot}\mathcal{H}$, then $h = \sum_{i=1}^n f_i g_i$ for some $f_1, .., f_n, g_1, .., g_n \in \mathcal{H}$. By the above and the fact that $\mathcal{H}(k^2)$ is a normed vector space we have $h \in \mathcal{H}(k^2)$ with

$$\|h\|_{\mathcal{H}(k^2)} \leq \sum_{i=1}^n \|f_i g_i\|_{\mathcal{H}(k^2)} \leq \sum_{i=1}^n \|f_i\|\|g_i\|.$$

This implies $\|h\|_{\mathcal{H}(k^2)} \leq \|h\|_{\bullet\mathcal{H}}$ for all $h \in \mathcal{H}\widehat{\odot}\mathcal{H}$.

Now let $h \in \mathcal{H} \odot \mathcal{H}$, and let $f_i, g_i \in \mathcal{H}$ such that $h = \sum_{i=1}^\infty f_i g_i$. We set $h_n = \sum_{i=1}^n f_i g_i$, then $h_n \to h$ in $\mathcal{H} \odot \mathcal{H}$ and h_n is a Cauchy sequence in $\|\cdot\|_{\bullet\mathcal{H}}$. By what we have shown it follows that h_n is a Cauchy sequence in $\mathcal{H}(k^2)$. Since convergence in norm implies local uniform convergence in both $\mathcal{H} \odot \mathcal{H}$ and $\mathcal{H}(k^2)$ we see that $h_n \to h$ in $\mathcal{H}(k^2)$, and

$$\|h\|_{\mathcal{H}(k^2)} = \lim_{n\to\infty} \|h_n\|_{\mathcal{H}(k^2)} \leq \limsup_{n\to\infty} \|h_n\|_{\bullet\mathcal{H}} \leq \sum_{i=1}^\infty \|f_i\|\|g_i\|.$$

The result follows by taking the infimum of the right hand side over such representations of h. ∎

By taking $\mathcal{H} = H^2$, $k_\lambda(z) = \frac{1}{1-\overline{\lambda}z}$ we recover the known fact that $H^1 \subseteq L^2_a$. We learned this fact from [BS84], p. 275, where it is attributed to Harold Shapiro, and where the proof uses the Hardy-Littlewood-Fejer inequality and only gives the inequality $\|f\|_{L^2_a} \leq \sqrt{\pi}\|f\|_{H^1}$.

If $\mathcal{H} = D$, the Dirichlet space, then $k_\lambda(z) = \sum_{n=0}^\infty \frac{1}{n+1}\overline{\lambda}^n z^n$. One calculates $k_\lambda^2(z) = \sum_{n=0}^\infty \frac{a_n}{n+2}\overline{\lambda}^n z^n$, where

$$\begin{aligned} a_n &= \sum_{k=0}^n \frac{n+2}{(k+1)(n-k+1)} = \sum_{k=0}^n \frac{1}{k+1} + \frac{1}{n-k+1} \\ &= \sum_{k=0}^n \frac{2}{k+1} \sim \log(n+2). \end{aligned}$$

It follows that there is a $c > 0$ such that if $h(\lambda) = \sum_{n=0}^{\infty} \hat{h}(n)\lambda^n \in D \odot D$, then

$$\sum_{n=0}^{\infty} \frac{n+1}{\log(n+2)} |\hat{h}(n)|^2 \leq c\|h\|_*^2. \tag{3.1}$$

In particular, this implies that $D \odot D \subseteq D_\alpha$ for all $\alpha < 1$. Shuaibing Luo has pointed out to us that the inequality (3.1) can be seen to be equivalent to a result of Brown and Shields, [BS84], p. 299. In fact, the Brown-Shields result is dual to (3.1). They prove that if $b(z) = \sum_n \hat{b}(n) z^n$ and if $\sum_n (n+1) \log(n+2) |\hat{b}(n)|^2 < \infty$, then $|b'|^2 dA$ is a Carleson measure for D, and hence $b \in \mathcal{X}(D)$. Also see [ARSW11], Theorem 4, condition 2(b).

4. Weighted Harmonic Dirichlet spaces

In this section and the next we will discuss weighted Dirichlet spaces with superharmonic weights, i.e. we will be interested in the case where

$$\mathcal{H} = \{f \in \text{ Hol}(\mathbb{D}) : \int_D |f'(z)|^2 w(z) dA(z) < \infty\}$$

for some non-negative superharmonic function w. By the work of [RS91] and [Ale93] one knows that every such space is of the form $\mathcal{H} = D(\mu)$, where μ is a non-negative measure on the closed unit disc and

$$D(\mu) = \{f \in H^2(\mathbb{D}) : \int_{\overline{D}} D_\lambda(f) d\mu(\lambda) < \infty\},$$

with $D_\lambda(f)$ = the local Dirichlet integral of f at $\lambda \in \overline{\mathbb{D}}$ which will be defined below. For the study of boundary behaviour and Carleson measures it is useful to also consider the corresponding spaces of harmonic functions $D_h(\mu)$ which consist of functions of the form $f + \overline{g}$ with $f, g \in D(\mu)$.

If $f \in L^1(\partial\mathbb{D})$ and $\lambda \in \mathbb{D}$ we write $f(\lambda) = \int_{\partial\mathbb{D}} \frac{1-|\lambda|^2}{|w-\lambda|^2} f(w) \frac{|dw|}{2\pi}$ for the Poisson integral of f and define

$$D_\lambda(f) = \int_{w \in \partial\mathbb{D}} \left| \frac{f(w) - f(\lambda)}{w - \lambda} \right|^2 \frac{|dw|}{2\pi}.$$

Clearly $D_\lambda(f) < \infty$ if and only if $f \in L^2(\partial\mathbb{D})$.

If $\lambda = z \in \partial\mathbb{D}$ we need a little preparation to define $D_z(f)$. Let $\alpha \in \mathbb{C}$, then for $w \in \partial\mathbb{D}$ we have

$$|f(w)|^2 \leq 2(|f(w) - \alpha|^2 + |\alpha|^2) \leq 8 \left| \frac{f(w) - \alpha}{w - z} \right|^2 + 2|\alpha|^2.$$

Thus, if $\int_{|w|=1} \left|\frac{f(w)-\alpha}{w-z}\right|^2 \frac{|dw|}{2\pi} < \infty$, then $f \in L^2(\partial\mathbb{D})$ and one checks that the Poisson extension of f has nontangential limit $f(z) = \alpha$ at z, see [RS91]. Thus we can define the local Dirichlet intgral of f at z by

$$D_z(f) = \int_{|w|=1} \left|\frac{f(w)-f(z)}{w-z}\right|^2 \frac{|dw|}{2\pi},$$

if f has a nontangential limit $f(z)$ at z and $D_z(f) = \infty$ otherwise. We remark for clarity that we will always assume that for any $f \in L^1(\partial\mathbb{D})$ the quantity $f(z)$ denotes the nontangential limit of f at z for any point $z \in \partial\mathbb{D}$ where this limit exists.

Lemma 4.1. *If $f, g \in H^2$ and $|\lambda| \leq 1$, then $D_\lambda(f+\overline{g}) = D_\lambda(f)+D_\lambda(g)$.*

Proof. We will first do the case $\lambda = 1$. Set $h = f + \overline{g}$.

If $D_1(f) < \infty$ and $D_1(g) < \infty$, then both f and g have nontangential limits at 1 and so does h with $h(1) = f(1)+\overline{g(1)}$. Furthermore, $\frac{f(w)-f(1)}{w-1}$ and $\frac{g(w)-g(1)}{w-1}$ are in H^2. Thus the identity easily follows from

$$\frac{h(w)-h(1)}{w-1} = \frac{f(w)-f(1)}{w-1} - \overline{w\frac{g(w)-g(1)}{w-1}}. \tag{4.1}$$

Now suppose the L^2-function h has nontangential limit $h(1)$ at 1 and

$$D_1(h) = \int_{|w|=1} \left|\frac{h(w)-h(1)}{w-1}\right|^2 \frac{|dw|}{2\pi} < \infty.$$

Then $\frac{h(w)-h(1)}{w-1} = u(w) + \overline{v(w)}$ for some $u, v \in H^2$. This implies

$$h(w) = h(1) + (w-1)(u(w) + \overline{v(0)}) + \overline{(1-w)\frac{v(w)-v(0)}{w}}.$$

Next let $\lambda \in \mathbb{D}$, substitute $h = f + \overline{g}$, and take the L^2 inner product with $(1-\overline{\lambda}w)^{-1}$. Thus $f(\lambda)+\overline{g(0)} = h(1)+(\lambda-1)(u(\lambda)+\overline{v(0)})+\overline{v'(0)}$, and it follows that f has nontangential limit $f(1) = h(1)+\overline{v'(0)}-\overline{g(0)}$ at 1 and $D_1(f) = \|u\|^2_{H^2} < \infty$.

Similarly $D_1(g) < \infty$, and the result follows from the first part of the proof.

For general $|\lambda| = 1$ the lemma now follows by a rotation, and for $|\lambda| < 1$ the result easily follows from an identity analogous to (4.1). ∎

This enables us to use many of the results from [RS92] and [RS91]. For example, we have $D_z(f) = \text{nt-}\lim_{\lambda\to z} D_\lambda(f)$ for all $f \in L^2(\partial\mathbb{D})$ and all $z \in \partial\mathbb{D}$. For later reference we also note that $D_\lambda(f) = D_\lambda(u) + D_\lambda(v)$, whenever $f = u + iv$ for real-valued functions u and v. This is elementary since $|f(z)-f(\lambda)|^2 = |u(z)-u(\lambda)|^2 + |v(z)-v(\lambda)|^2$.

Lemma 4.2. *Let* $\{f_i\} \subseteq L^2(\partial\mathbb{D})$ *with* $\sum_{i\geq 1} \|f_i\|^2_{L^2} < \infty$ *and define*

$$f = \sqrt{\sum_i |f_i|^2}.$$

Then $f \in L^2(\partial\mathbb{D})$ *and for every* $\lambda \in \overline{\mathbb{D}}$ *we have* $D_\lambda(f) \leq \sum_i D_\lambda(f_i)$.

Note that it is a consequence of this that for any $f \in L^2$ we have $D_\lambda(|f|) \leq D_\lambda(f)$.

Proof. It is clear that $f \in L^2(\partial\mathbb{D})$, thus we only need to prove the inequality. We assume that $\sum_i D_\lambda(f_i) < \infty$.

Note that

$$\int_{\partial\mathbb{D}} |f(w)|^2 \frac{|dw|}{2\pi} = \int_{\partial\mathbb{D}} \sum_i |f_i(w)|^2 \frac{|dw|}{2\pi} = \sum_i \|f_i\|^2_{L^2} < \infty.$$

Hence the hypothesis implies that $S = \{w \in \partial\mathbb{D} : \sum_{i\geq 1} |f_i(w)|^2 < \infty\}$ has full measure in $\partial\mathbb{D}$. For $w \in S$ define the sequence $F(w) = \{f_i(w)\}_{i\geq 1}$ so that $f(w) = \|F(w)\|_{l^2} < \infty$ for all $w \in S$.

We first do the case $\lambda = z \in \partial\mathbb{D}$. For any $g \in L^2$ with $D_z(g) < \infty$ we have

$$|g(z)|^2 \leq 2(|g(z) - g(w)|^2 + |g(w)|^2) \leq 2(4D_z(g) + \|g\|^2_{L^2}).$$

Thus $\sum_i D_z(f_i) < \infty$ implies that $z \in S$.

For $w \in S$ the reverse triangle inequality

$$|\|F(z)\|_{l^2} - \|F(w)\|_{l^2}| \leq \|F(z) - F(w)\|_{l^2}$$

implies

$$\left|\frac{f(z) - f(w)}{z - w}\right|^2 \leq \sum_i \left|\frac{f_i(z) - f_i(w)}{z - w}\right|^2.$$

Thus the result follows in the case $\lambda = z \in \partial\mathbb{D}$.

Now let $|\lambda| < 1$. A short calculation shows that for any $g \in L^2$

$$D_\lambda(g) = \frac{1}{1 - |\lambda|^2} \left(\int P_\lambda(w) |g(w)|^2 \frac{|dw|}{2\pi} - |g(\lambda)|^2 \right).$$

Note that

$$\| \int P_\lambda(w) F(w) \frac{|dw|}{2\pi} \|_{l^2} \leq \int P_\lambda(w) \|F(w)\|_{l^2} \frac{|dw|}{2\pi}$$

and this implies that

$$\sum_{i\geq 1} |f_i(\lambda)|^2 \leq |f(\lambda)|^2.$$

Thus

$$\int P_\lambda(w)|f(w)|^2\frac{|dw|}{2\pi}-|f(\lambda)|^2 \le \sum_{i\ge 1}\left(\int P_\lambda(w)|f_i(w)|^2\frac{|dw|}{2\pi}-|f_i(\lambda)|^2\right).$$

The result follows. In fact, the boundary result follows from the $|\lambda|<1$ result by taking nontangential limits. ■

If μ is a finite nonnegative measure supported in $\overline{\mathbb{D}}$, then we see that

$$D_h(\mu)=\{f\in L^2 : \int_{\overline{\mathbb{D}}} D_\lambda(f)d\mu<\infty\}.$$

The norm on $D_h(\mu)$ is

$$\|f\|_\mu^2=\|f\|_{L^2}^2+\int_{\overline{\mathbb{D}}} D_\lambda(f)d\mu.$$

Generally we will think of $f\in D_h(\mu)\subseteq L^2(\partial\mathbb{D})$ as a function defined on $\partial\mathbb{D}$. Such a function f has a harmonic extension inside $\mathbb{D}$ given by the Poisson integral $P[f](\lambda)$. When we write expressions like fg for $f,g\in D_h(\mu)$, then we are multiplying boundary values, which of course generally may satisfy $P[fg](\lambda)\neq P[f](\lambda)P[g](\lambda)$. Later we will explain that in the case of the classical Dirichlet space this difference is not significant for our purposes (see the remark after Theorem 7.2).

We now define the corresponding weak product space. Since $\|f\|_{L^2}\le \|f\|_\mu$ we note that $h=\sum_{i\ge1} f_ig_i$ converges in L^1, whenever

$$\sum_{i\ge 1}\|f_i\|_\mu\|g_i\|_\mu<\infty.$$

Thus we set

$$D_h(\mu)\odot D_h(\mu)=\{\sum_{i\ge1} f_ig_i : f_i,g_i\in D_h(\mu) \text{ with } \sum_{i\ge1}\|f_i\|_\mu\|g_i\|_\mu<\infty\},$$

with norm

$$\|h\|_{1,\mu}=\inf\{\sum_{i\ge1}\|f_i\|_\mu\|g_i\|_\mu : f_i,g_i\in D_h(\mu), h=\sum_{i\ge1} f_ig_i \text{ a.e.}\}.$$

Exactly as in the proof of Lemma 2.1 one shows that $D_h(\mu)\odot D_h(\mu)$ is a Banach space, and clearly $D_h(\mu)\odot D_h(\mu)\subseteq L^1$ for each μ.

Note that since $\|g\|_\mu=\|\overline{g}\|_\mu$ we may always assume that $h=\sum_{i\ge1} f_i\overline{g}_i$ with $\sum_{i\ge1}\|f_i\|_\mu\|g_i\|_\mu<\infty$. We also observe that $\overline{h}\in D_h(\mu)\odot D_h(\mu)$ with $\|\overline{h}\|_{1,\mu}=\|h\|_{1,\mu}$, whenever $h\in D_h(\mu)\odot D_h(\mu)$. This implies that $h\in D_h(\mu)\odot D_h(\mu)$ if and only if both $\mathrm{Re}h$ and $\mathrm{Im}h\in D_h(\mu)\odot D_h(\mu)$ and $\|h\|_{1,\mu}\le\|\mathrm{Re}h\|_{1,\mu}+\|\mathrm{Im}h\|_{1,\mu}\le 2\|h\|_{1,\mu}$.

We also note that if $h \in D(\mu) \odot D(\mu)$, then $h \in D_h(\mu) \odot D_h(\mu)$ and $\|h\|_{1,\mu} \leq \|h\|_*$. We will see that for the classical Dirichlet space one has equivalence of norms, so that $D \odot D$ is a closed subspace of $D_h \odot D_h$.

Theorem 4.3. *If $h \in D_h(\mu) \odot D_h(\mu)$ is real-valued, then there are real-valued functions $u, v \in D_h(\mu)$, $u \geq 0$, such that $h = uv$ and*

$$\|h\|_{1,\mu} = \inf\{\|u\|_\mu \|v\|_\mu : u, v \in D_h(\mu), u \geq 0, |v| \leq u, h = uv\}.$$

If $D_h(\mu)$ is compactly contained in L^2, then the infimum is attained.

Furthermore, if the infimum is attained, then one can choose an infimizing pair (u, v) with $u \geq 0$, $|v| \leq u$ and $\|u\|_\mu = \|v\|_\mu$. It will satisfy $\langle \varphi u, u\rangle_\mu = \langle \varphi v, v\rangle_\mu$ for all $D_h(\mu)$-multipliers φ.

In Section 8 we will see examples which show that it is possible that even for nonnegative functions h it may happen that any norm minimizing pair satisfies $u \neq v$.

Proof. Let $h \in D_h(\mu) \odot D_h(\mu)$ be real-valued and pick any $f_i, g_i \in D_h(\mu)$ with $\sum_{i\geq 1} \|f_i\|_\mu \|g_i\|_\mu < \infty$ and such that $h = \sum_{i\geq 1} f_i \overline{g}_i$. We may assume that for each i we have $\|f_i\|_\mu = \|g_i\|_\mu$. By the parallelogram law we observe

$$\|f_i\|_\mu \|g_i\|_\mu = \|\frac{f_i + g_i}{2}\|_\mu^2 + \|\frac{f_i - g_i}{2}\|_\mu^2$$

and since h is real-valued, we have

$$h = \sum_{i\geq 1} \mathrm{Re} f_i \overline{g}_i = \sum_{i\geq 1} \left|\frac{f_i + g_i}{2}\right|^2 - \left|\frac{f_i - g_i}{2}\right|^2.$$

Let $f = \sqrt{\sum_{i\geq 1} \left|\frac{f_i+g_i}{2}\right|^2}$ and $g = \sqrt{\sum_{i\geq 1} \left|\frac{f_i-g_i}{2}\right|^2}$, then Lemma 4.2 implies that

$$\|f\|_\mu^2 + \|g\|_\mu^2 \leq \sum_{i\geq 1} \|\frac{f_i + g_i}{2}\|_\mu^2 + \|\frac{f_i - g_i}{2}\|_\mu^2 = \sum_{i\geq 1} \|f_i\|_\mu \|g_i\|_\mu.$$

Note that $h = f^2 - g^2 = (f + g)(f - g)$ and a short calculation shows that $\|f + g\|_\mu \|f - g\|_\mu \leq \|f\|_\mu^2 + \|g\|_\mu^2$. Furthermore, f and g are nonnegative. Thus $u = f + g$ is nonnegative and $v = f - g$ satisfies $|v| \leq u$. Hence

$$\|h\|_{1,\mu} = \inf\{\|u\|_\mu \|v\|_\mu : h = uv, u, v \text{ real-valued }, u \geq 0, |v| \leq u\}.$$

Note that it follows that

$$\|h\|_{1,\mu} = \inf\{\|u\|_\mu \|v\|_\mu : h = uv, u, v \text{ real-valued }, u \geq 0, \|u\|_\mu = \|v\|_\mu\}.$$

Now suppose $D_h(\mu)$ is compactly contained in L^2. Let $u_n, v_n \in D_h(\mu)$ be real-valued, $h = u_n v_n$, $u_n \geq 0$, $\|u_n\|_\mu = \|v_n\|_\mu$ such that

$\|u_n\|_\mu \|v_n\|_\mu \to \|h\|_{1,\mu}$. By possibly dropping to a subsequence we may assume that $u_n \to u$ and $v_n \to v$ weakly in $D_h(\mu)$. The hypothesis then implies that $u_n \to u$, $v_n \to v$ in the L^2-norm, and hence $h = uv$. We clearly have $u \geq 0$ and $\|u\|_\mu \leq \liminf \|u_n\|_\mu$, $\|v\|_\mu \leq \liminf \|v_n\|_\mu$, and $\|u\|_\mu \|v\|_\mu \leq \|h\|_{1,\mu}$. Thus $\|h\|_{1,\mu} = \|u\|_\mu \|v\|_\mu$ and this also implies that $\|u\|_\mu = \|v\|_\mu$.

Next we prove that if $u, v \in D_h(\mu)$ are real-valued with $\|uv\|_{1,\mu} = \|u\|_\mu \|v\|_\mu$, then an infimizing pair with further properties can be chosen. We may assume that $\|u\|_\mu = \|v\|_\mu$. Set

$$u_1 = |\frac{u+v}{2}| + |\frac{u-v}{2}|, \ v_1 = |\frac{u+v}{2}| - |\frac{u-v}{2}|.$$

Then $u_1, v_1 \in D_h(\mu)$ since $\||w|\|_\mu \leq \|w\|_\mu$ as follows from the remark after Lemma 4.2. Furthermore $u_1 v_1 = (\frac{u+v}{2})^2 - (\frac{u-v}{2})^2 = uv$ and by the parallelogram law

$$\begin{aligned} \|u_1\|_\mu \|v_1\|_\mu &\leq \frac{\|u_1\|_\mu^2 + \|v_1\|_\mu^2}{2} \\ &= \|\frac{|u+v|}{2}\|_\mu^2 + \|\frac{|u-v|}{2}\|_\mu^2 \\ &\leq \|\frac{u+v}{2}\|_\mu^2 + \|\frac{u-v}{2}\|_\mu^2 = \|u\|_\mu \|v\|_\mu \end{aligned}$$

Hence we must have equality throughout which implies that $\|u_1\|_\mu = \|v_1\|_\mu$. Thus (u_1, v_1) is a real-valued infimizing pair with $u_1 \geq 0$, $|v_1| \leq u_1$ and $\|u_1\|_\mu = \|v_1\|_\mu$.

Now write $M(D_h(\mu))$ for the set of multipliers for $D_h(\mu)$, and $\|\cdot\|_M$ for the multiplier norm.

Let $f, g \in D_h(\mu)$ be real-valued with $\|f\|_\mu = \|g\|_\mu$ and $\|f\|_\mu \|g\|_\mu \leq \|u\|_\mu \|v\|_\mu$ for all $u, v \in D(\mu)$ with $fg = uv$.

Let $\varphi \in M(D_h(\mu))$ and let $\alpha \in \mathbb{C}$ with $|\alpha| \|\varphi\|_M < 1$. Then $\frac{1}{(1-\alpha\varphi)} = 1 + \alpha\varphi + O(|\alpha|^2) \in M(D_h(\mu))$, and $u = f(1-\alpha\varphi), v = g/(1-\alpha\varphi) \in D_h(\mu)$ satisfy $fg = uv$. Thus

$$\begin{aligned} \|f\|_\mu^2 \|g\|_\mu^2 &\leq \|u\|_\mu^2 \|v\|_\mu^2 \\ &= (\|f\|_\mu^2 - 2\mathrm{Re}\ \alpha\langle\varphi f, f\rangle + O(|\alpha|^2))(\|g\|_\mu^2 + 2\mathrm{Re}\ \alpha\langle\varphi g, g\rangle + O(|\alpha|^2)) \\ &= \|f\|_\mu^2 \|g\|_\mu^2 + \|f\|_\mu^2 (2\mathrm{Re}\ \alpha(\langle\varphi g, g\rangle - \langle\varphi f, f\rangle)) + O(|\alpha|^2), \end{aligned}$$

since $\|f\|_\mu = \|g\|_\mu$. Since this inequality holds for all small enough α the result follows easily. ■

Example 4.4. Let $\mu = \delta_\lambda$ for some $|\lambda| < 1$, take $h(z) = |z-\lambda|^2$, and $u_n(z) = (z-\lambda)z^n$, $v_n = \overline{u}_n$. Then $h = u_n v_n$ for each n, but $u_n, v_n \to 0$ weakly in $D_h(\delta_\lambda)$. This example shows that in the previous proof without any extra argument or hypothesis one could not have concluded that h equals the product of the weak limits of u_n and v_n.

If h is a real-valued function, then write $h^+ = \max\{h, 0\} = \frac{|h|+h}{2}$, $h^- = -\min\{h, 0\} = \frac{|h|-h}{2}$.

Corollary 4.5. *If $h \in L^1(\partial\mathbb{D})$, then $h \in D_h(\mu) \odot D_h(\mu)$ if and only if both Re h, Im $h \in D_h(\mu) \odot D_h(\mu)$. If h is real-valued, then $h \in D_h(\mu) \odot D_h(\mu)$ if and only if both $h^+, h^- \in D_h(\mu) \odot D_h(\mu)$.*

In particular, if $h \in D_h(\mu) \odot D_h(\mu)$ is real-valued, then $|h| \in D_h(\mu) \odot D_h(\mu)$

Proof. We have already noted the statement of the first sentence. If $h \in D_h(\mu) \odot D_h(\mu)$ is real-valued, then by Theorem 4.3 $h = uv$ for $u, v \in D_h(\mu)$ with $u \geq 0$. Thus $h^+ = uv^+$ and $h^- = uv^-$ and $v^+, v^- \in D_h(\mu)$ by the remark following Lemma 4.2. The converse is trivial. ■

Question 4.6. *If $h \in D_h(\mu) \odot D_h(\mu)$ is complex-valued, then is $|h| \in D_h(\mu) \odot D_h(\mu)$?*

5. Capacities for weighted Dirichlet spaces

It is easy to check that for each measure μ on $\overline{\mathbb{D}}$ the real parts of functions in $D_h(\mu) \subseteq L^2(\partial\mathbb{D})$ form a "Dirichlet space" in the sense of Beurling-Deny [BD59]. Each such space is associated with a Markov process and an extensive general potential theory of such spaces has been developed. For the specific case of the spaces $D_h(\mu)$ we recall the definition of capacity. First if $U \subseteq \partial\mathbb{D}$ is open, then

$$\text{cap}U = \inf\{\|f\|_\mu^2 : f \in D_h(\mu), f \geq 1 \text{ a.e. on } U\},$$

and for arbitrary sets $E \subseteq \partial\mathbb{D}$ one sets

$$\text{cap}E = \inf\{\ \text{cap}U : E \subseteq U, U \text{ open }\}.$$

This capacity turns out to be a Choquet capacity ([Cho54]) and as a consequence one has

$$\text{cap}E = \sup\{\ \text{cap}K : K \subseteq E, K \text{ compact }\}$$

for every Borel set $E \subseteq \partial\mathbb{D}$. A property hold quasi-everywhere (q.e.) if it holds except perhaps on a set of capacity 0. We refer the reader to Chapter 2 of [FOT11] for the results mentioned above and further basic results about these capacities and exceptional sets. For measures μ that are supported in $\partial\mathbb{D}$ the paper [Gui12] also contains a nice overview. Furthermore we mention that Chacon [Cha11] showed that the nontangential maximal function is bounded on $D_h(\mu)$ for μ supported in $\partial\mathbb{D}$, and it follows that the harmonic extensions of $D_h(\mu)$-functions have nontangential limits q.e. Chacon went on to use this result to give a capacitary characterization of the Carleson measures for $D(\mu)$.

We also note that the capacity one obtains this way for the Dirichlet space $D_h = D_h(m)$, m normalized Lebesgue measure on $\partial\mathbb{D}$, is comparable to classical logarithmic capacity restricted to $\partial\mathbb{D}$. In this case it is also well-known that $D_h = \{k * f : f \in L^2(\mathbb{T})\}$ for $k(e^{it}) = |1-e^{it}|^{-1/2}$, and that $\|k * f\|_{D_h}$ is comparable to $\|f\|_{L^2}$ (see e.g. [RRS94], Section 2, and the references therein). This leads to another way to define the capacity of open sets U as $\inf\{\|f\|_{L^2}^2 : f \in L^2, f \geq 0, k*f \geq 1$ a.e. on $U\}$. Of course, by the equivalence of norms this quantity is comparable to the quantity that we have defined. This is the approach that is carried out e.g. in [AH96]. It generalizes to spaces other than Hilbert spaces, although it is not clear that the spaces $D_h(\mu)$ could be treated this way for all μ.

The following proposition shows that the capacity concept does not need to be changed for $D_h(\mu) \odot D_h(\mu)$.

Proposition 5.1. *Let $U \subseteq \partial\mathbb{D}$ be open, then*

$$\inf\{\|h\|_{1,\mu} : h \in D_h(\mu) \odot D_h(\mu), h \text{ real-valued}, h \geq 1 \text{ a.e. on } U\} = \text{cap } U.$$

Proof. If $f \in D_h(\mu)$ with $f \geq 1$ a.e. on U, then $f^2 \in D_h(\mu) \odot D_h(\mu)$ and $f^2 \geq 1$ a.e. on U. Thus it is clear that

$$\inf\{\|h\|_{1,\mu} : h \in D_h(\mu) \odot D_h(\mu), h \text{ real-valued}, h \geq 1 \text{ a.e. on } U\} \leq \text{ cap } U.$$

Now let $h \in D_h(\mu) \odot D_h(\mu)$ be real-valued and $h \geq 1$ a.e. on U. Then by Theorem 4.3 we have

$$\|h\|_{1,\mu} = \inf\{\|u\|_\mu \|v\|_\mu : h = uv, u, v \text{ real-valued } u \geq 0\}.$$

Thus let $h = uv$ for real-valued $u, v \in D_h(\mu)$, $u \geq 0$. We may assume $\|u\|_\mu = \|v\|_\mu$. Then $f = \frac{u+v}{2} \geq \sqrt{uv} \geq 1$ a.e. on U and hence

$$\text{cap } U \leq \|f\|_\mu^2 = \|\frac{u+v}{2}\|_\mu^2 \leq \|u\|_\mu \|v\|_\mu.$$

This implies cap $U \leq \|h\|_{1,\mu}$ for any real-valued $h \in D_h(\mu) \odot D_h(\mu)$ with $h \geq 1$ a.e. on U. ∎

6. The projection theorem for D

We now restrict our attention to the classical Dirichlet space. We will write $D = D(m)$, $D_h = D_h(m)$, and $\|\cdot\| = \|\cdot\|_m$ will be the norm of D, $\|\cdot\|_*$ the norm of $D \odot D$, and $\|\cdot\|_1 = \|\cdot\|_{1,m}$ the norm of $D_h \odot D_h$.

We know from the results of Section 4 that for any measure μ and any analytic $h \in D(\mu) \odot D(\mu)$, there are real-valued functions $u_1, u_2, u_3, u_4 \in D_h(\mu)$ such that $h = u_1 u_2 + i u_3 u_4$. We will now see that for D the converse holds, namely that whenever $u_1, u_2, u_3, u_4 \in D_h$ are real-valued such that $h = u_1u_2 + iu_3u_4 \in H^1$, then $h \in D \odot D$. We also note that D satisfies the conditions (1.1), (1.2), (1.3) of Theorem 1.2, thus $(D \odot D)^* = \mathcal{X}(D)$.

In this section we use P denote the Cauchy projection. Note that in other sections P has also been used to denote the Poisson kernel. It should be possible to distinguish these from the context.

Lemma 6.1. *There is a $c > 0$ such that*

$$\|Pf\overline{g}\|_* \leq c\|f\|\|g\|$$

for all $f, g \in D$.

Proof. We start by noting that $\langle p, q\rangle_D = \langle p, (zq)'\rangle_{H^2}$ and $\langle p, q\rangle_{H^2} = \langle (zp)', q\rangle_{L^2_a}$ for all polynomials p and q. We combine these two equalities to obtain

$$|\langle Pf\overline{g}, b\rangle_D| = |\int_{|z|=1} f\overline{g}\overline{(zb)'}\frac{|dz|}{2\pi}| = |\langle (zf)', g(zb)'\rangle_{L^2_a}| \leq \|f\|_D\|g(zb)'\|_{L^2_a}$$

for all polynomials f, g, and b. It follows that the inequality

$$|\langle Pf\overline{g}, b\rangle_D| \leq \|f\|_D\|g(zb)'\|_{L^2_a}$$

holds whenever f and g are polynomials and $b \in D$.

Now recall that we used $\mathcal{X}(D) \subseteq D$ to denote the dual space of $D \odot D$. It is easy to check that $zb \in \mathcal{X}(D)$, whenever $b \in \mathcal{X}(D)$. Furthermore, as we mentioned in Section 1, a theorem of Arcozzi, Rochberg, Sawyer, and Wick characterizes the functions $b \in \mathcal{X}(D)$ as the analytic functions such that the measure $|b'|^2dA$ is a Carleson measure for the Dirichlet space (see [ARSW10], also see [CO12] for an alternate proof). Thus there is a $c > 0$ such that

$$|\langle P(f\overline{g}), b\rangle_D| \leq c\|f\|_D\|g\|_D\|b\|_{\mathcal{X}}.$$

By taking the supremum of this inequality over all unit vectors $b \in \mathcal{X}$ we see that the lemma follows for all polynomials f and g.

Now if $f, g \in D$ and if f_n, g_n are polynomials with $f_n \to f$ and $g_n \to g$ in D, then the inequality easily implies that $\{Pf_n\overline{g}_n\}$ is a Cauchy sequence in $D \odot D$ and hence it converges to some $h \in D \odot D$. Similarly, it is clear that $f_n\overline{g}_n$ converges to $f\overline{g}$ in $L^1(\mathbb{T})$. By comparing Fourier coefficients we see that $h = P(f\overline{g})$. The lemma follows. ∎

Theorem 6.2. *There is a $c > 0$ such that for all $h \in D_h \odot D_h$*

$$\|Ph\|_* \leq c\|h\|_1.$$

In particular, if $h \in D \odot D$ then $h = Ph$ and hence $\|h\|_1 \leq \|h\|_* \leq c\|h\|_1$. Hence the norms $\|\cdot\|_*$ and $\|\cdot\|_1$ are equivalent on $D \odot D$ and $D \odot D$ is a closed subspace of $D_h \odot D_h$.

Proof. Since $\|\mathrm{Re}h\|_1 + \|\mathrm{Im}h\|_1 \leq 2\|h\|_1$ it will be sufficient to establish the theorem for real-valued functions.

Note that if $u \in D_h$ is real-valued, then $u = f + \overline{f}$ for some $f \in D, f(0) \in \mathbb{R}$. Thus, if $h \in D_h \odot D_h$ is real-valued, then it follows from the results of Section 4 that there are f and $g \in D$ such that $f(0), g(0) \in \mathbb{R}$, $h = (f+\overline{f})(g+\overline{g})$, and $\|h\|_1 = \|f+\overline{f}\|\|g+\overline{g}\| \geq \|f\|\|g\|$. Then $Ph = fg + P(f\overline{g}) + P(g\overline{f}) + f(0)g(0)$ and Lemma 6.1 implies

$$\begin{aligned}\|Ph\|_* &\leq \|fg\|_* + \|P(f\overline{g})\|_* + \|P(g\overline{f})\|_* + |f(0)g(0)| \\ &\leq (2+2c)\|f\|\|g\| \leq (2+2c)\|h\|_1.\end{aligned}$$

■

Corollary 6.3. *$D \odot D = H^1 \cap (D_h \odot D_h)$ and if $h \in H^1$, then $h \in D \odot D$ if and only if both Reh and Imh $\in D_h \odot D_h$.*

7. A connection to a paper by Mazya and Verbitsky

In their paper [ARSW10], Sec. 1.2 the authors mention similarities and analogies between their results characterizing the space of analytic functions $\mathcal{X}(D)$ and results of Mazya and Verbitsky, [MV02], which are real variable type results. It turns out that one of the Mazya-Verbitsky results implies a characterization of $\mathcal{X}(D_h)$, and hence is related to topics discussed in this paper. We will now present a precise connection between these two types of results.

Recall that a non-negative measure μ on the open unit disc is called a Carleson measure for the Dirichlet space, if there is a constant $c > 0$ such that

$$\int_{\mathbb{D}} |f|^2 d\mu \leq c\|f\|_D$$

for all $f \in D$. There are several intrinsic characterizations of such Carleson measures available, see e.g. [Ste80] or [KS88]. Since every function $f \in D_h$ is of the form $f = f_1 + \overline{f}_2$ with $f_1, f_2 \in D$ and $f_2(0) = 0$, $\|f\|^2_{D_h} = \|f_1\|^2_D + \|f_2\|^2_D$, it is clear that μ satisfies the above (analytic) Carleson measure condition, if and only if μ is a Carleson measure for D_h, i.e. if

$$\int_{\mathbb{D}} |P_z[f]|^2 d\mu(z) \leq c\|f\|^2_{D_h}$$

for all $f \in D_h$. Here $P_z[f]$ denotes the Poisson integral of f at $z \in \mathbb{D}$. If $b = b_1 + \overline{b}_2 \in D_h$ with $b_1, b_2 \in D$ and $b_2(0) = 0$, then $|\nabla P_z[b]|^2 dA =$

$2(|b_1'|^2+|b_2'|^2)dA$ is a Carleson measure for D, if and only if both $|b_1'|^2dA$ and $|b_2'|^2dA$ are Carleson measures for D.

We start by restating the main result of [ARSW10].

Theorem 7.1. *([ARSW10]) Let $b \in D$. Then the following two conditions are equivalent:*

(a) $b \in \mathcal{X}(D)$, i.e. there is a constant $c > 0$ such that $|\langle \varphi\psi, b\rangle_D| \leq c\|\varphi\|_D\|\psi\|_D$ for all $\varphi, \psi \in$ $Hol(\overline{\mathbb{D}})$.

(b) $|b'|^2dA$ is a Carleson measure for D.

Note that the implication (b)⇒(a) follows easily from the product rule and the Cauchy-Schwarz inequality. The reverse implication is harder to prove and it is the main result of [ARSW10]. The analogue of this theorem for the harmonic Dirichlet space is the following.

Theorem 7.2. *Let $b \in D_h$. Then the following two conditions are equivalent:*

(a) $b \in \mathcal{X}(D_h)$, i.e. there is a constant $c > 0$ such that $|\langle \varphi\psi, b\rangle_{D_h}| \leq c\|\varphi\|_{D_h}\|\psi\|_{D_h}$ for all $\varphi, \psi \in C^\infty(\mathbb{T})$.

(b) $|\nabla P_z[b]|^2dA(z)$ is a Carleson measure for D.

Again the implication (b)⇒(a) is elementary, although there is a minor subtlety due to the fact that in general $P_z[\varphi\psi] \neq P_z[\varphi]P_z[\psi]$ for $z \in \mathbb{D}$. However, it follows from Green's first identity $\int_{\mathbb{D}}(\Delta u)v + \nabla u \cdot \nabla v dA = \int_{\partial\mathbb{D}} v\frac{\partial u}{\partial n}ds$ that for all $C^\infty(\mathbb{T})$-functions b, φ, ψ we have

$$\int_{\mathbb{D}} \nabla P_z[\bar{b}] \cdot \nabla \left(P_z[\varphi\psi] - P_z[\varphi]P_z[\psi]\right) dA(z) = 0.$$

By an approximation argument it follows that

$$\int_{\mathbb{D}} \nabla P_z[\bar{b}] \cdot \nabla P_z[\varphi\psi] dA = \int_{\mathbb{D}} \nabla P_z[\bar{b}] \cdot \nabla \left(P_z[\varphi]P_z[\psi]\right) dA(z)$$

for all $b \in D_h$ and $\varphi, \psi \in C^\infty(\mathbb{T})$. Hence

$$\begin{aligned}
\langle \varphi\psi, b\rangle_{D_h} &= \int_{\mathbb{T}} \varphi\psi\bar{b}\frac{|dz|}{2\pi} + \int_{\mathbb{D}} \nabla\left(P_z[\varphi\psi]\right) \cdot \nabla P_z[\bar{b}]\frac{dA(z)}{\pi} \\
&= \int_{\mathbb{T}} \varphi\psi\bar{b}\frac{|dz|}{2\pi} + \int_{\mathbb{D}} \nabla\left(P_z[\varphi]P_z[\psi]\right) \cdot \nabla P_z[\bar{b}]\frac{dA(z)}{\pi} \\
&= \int_{\mathbb{T}} \varphi\psi\bar{b}\frac{|dz|}{2\pi} + \int_{\mathbb{D}} \left(P_z[\varphi]\nabla P_z[\psi] + P_z(\psi)\nabla P_z[\varphi]\right) \cdot \nabla P_z[\bar{b}]\frac{dA(z)}{\pi}
\end{aligned}$$

and now the implication (b)⇒(a) follows as in the analytic case by an application of the Cauchy-Schwarz inequality (the first summand poses no problem in either case).

We also remark that the implication (a)⇒(b) of Theorem 7.2 follows easily from the corresponding implication of Theorem 7.1. Indeed if $b \in D_h$, then $b = b_1 + \overline{b_2}$ for b_1 and $b_2 \in D$. Furthermore, if b satisfies condition (a) of Theorem 7.2, then both b_1 and b_2 satisfy condition (a) of Theorem 7.1. Thus by Theorem 7.1 both $|b_1'|^2 dA$ and $|b_2'|^2 dA$ are Carleson measures for D and we have already noted that this implies (b) of Theorem 7.2.

It appears that Theorem 7.1 is stronger than Theorem 7.2. Indeed, if $b \in D$ satisfies condition (a) of Theorem 7.1, then one sees that it will automatically satisfy (a) of Theorem 7.2 if we also assume the inequality $\|h\|_* \le c\|h\|_1$ for all $h \in D \odot D$. However, we note that we have established this inequality in Theorem 6.2 by use of Theorem 7.1.

Finally, we mention that in the case when $b \in D$, then the implication (a) ⇒(b) of Theorem 7.2 can be shown by elementary means. Indeed, by a calculation similar to the one in the proof of Lemma 6.1 the hypothesis implies that

$$\begin{aligned} \left| \int_{\mathbb{D}} (z\psi)' \overline{\varphi(zb)'} \, \frac{dA}{\pi} \right| &= \langle P\overline{\varphi}\psi, b\rangle_D \\ &= \langle \overline{\varphi}\psi, b\rangle_{D_h} \\ &\le C\|\psi\|_{D_h} \|\overline{\varphi}\|_{D_h} \\ &= C\|\psi\|_D \|\varphi\|_D \end{aligned}$$

for all analytic polynomials φ and ψ. Here P was used to denote the Cauchy projection. Now taking the sup over all analytic polynomials ψ with $\|\psi\|_D^2 = \int_{\mathbb{D}} |(z\psi)'|^2 \frac{dA}{\pi} = 1$ we obtain $\int_{\mathbb{D}} |\varphi(zb)'|^2 \frac{dA}{\pi} \le C^2 \|\varphi\|_D^2$ for all polynomials φ and this implies condition (b) of either of the two theorems.

We will now show that one of the results of [MV02] implies Theorem 7.2. We mention that Cascante and Ortega [CO12] established a different kind of connection between the papers [ARSW10] and [MV02]: They used the proofs of [MV02] as inspiration to reprove the main theorem of [ARSW10].

We will use the notation as in [MV02]. Let $n \in \mathbb{N}$. We write $L_2 = L_2(\mathbb{R}^n) = L^2(\mathbb{R}^n)$, and for $u \in C^\infty(\mathbb{R}^n)$ we use $\|u\|_{L_2^1} = \|\nabla u\|_{L_2}$, and $\|u\|_{W_2^1}^2 = \|u\|_{L_2}^2 + \|\nabla u\|_{L_2}^2$. Let $\Omega \subseteq \mathbb{R}^n$ a bounded region with smooth boundary and write $\mathring{L}_2^1(\Omega)$ for the closure of $C_c^\infty(\Omega)$ in the $\|\cdot\|_{L_2^1}$-norm. Finally we use $G_\Omega(x, y)$ and $G_\Omega[\cdot]$ for the Green's function and Green's operator for Ω. Hence if $u \in C_c^\infty(\Omega)$, then $f(x) = G_\Omega[u](x) = \int_\Omega G_\Omega(x, y)u(y)dy$ satisfies $-\Delta f = u$ in Ω and $f = 0$ on $\partial\Omega$.

For the next Theorem and its application we need to define $G_\Omega[V]$ where V is a compactly supported distribution. In order to do this we consider the Riesz potential $K_n(x) = c_n|x|^{-(n-2)}$ for $n \geq 3$, $K_2(x) = \frac{1}{2\pi}\log\frac{1}{|x|}$, and $K_1(x) = -\frac{|x|}{2}$. If $V \in C_c^\infty(\mathbb{R}^n)'$ has compact support in Ω, then the convolution $K_n * V$ defines a harmonic function off the support of V, see e.g. [Rud91]. Let $h \in C^\infty(\overline{\Omega})$ be the harmonic function on Ω with $h = K_n * V$ on $\partial\Omega$, then we define $G_\Omega[V] = K_n * V - h$. Thus $G_\Omega[V]$ is a distribution on Ω such that $\langle G_\Omega[V], (-\Delta w)\rangle = \langle V, w\rangle$ for all $w \in C_c^\infty(\Omega)$ and such that $G_\Omega[V]$ is a smooth function on $\overline{\Omega}\setminus$ spt V, harmonic in $\Omega\setminus$ spt V, and 0 on $\partial\Omega$.

Similarly, if V is a distribution, and if we say that $G_\Omega[V]$ is a W_2^1-function, then we mean that there is a vectorfield $F = (F_1, ..., F_n)$, $F_i \in L^2_{loc}$ such that $F = \nabla G_\Omega[V]$, i.e.

$$\int_\Omega F \cdot \nabla w dx = \langle G_\Omega[V], (-\Delta w)\rangle = \langle V, w\rangle$$

for all $w \in C_c^\infty(\Omega)$.

Mazya and Verbitsky's paper contains various types of related results that apply in different contexts. Section 2 contains homogeneous results (involving $\mathring{L}_2^1(\Omega)$ and the $\|\cdot\|_{L_2^1}$-norm), and the theorems are valid only for $n \geq 3$, because of the existence in $n = 1$ and $n = 2$ of functions u that are 1 on an arbitrary compact set and such that $\|\nabla u\|_{L_2}$ is arbitrarily small. We will thus use the results of Section 4 of [MV02]. However, those results are all stated only for the operator $(I - \Delta)^{-1}$ rather than for $(-\Delta)^{-1}$ which is needed for our purposes. This is a minor point as we will see. Thus, we will first derive an easy Corollary to the work of [MV02], and then see how it implies Theorem 7.2.

Theorem 7.3. *([MV02]) Suppose $V \in C_c^\infty(\mathbb{R}^n)'$ has compact support in Ω and there is a $c > 0$ such that*

$$|\langle V, uw\rangle| \leq c\|\nabla u\|_{L_2}\|\nabla w\|_{L_2} \quad \forall\ u, w \in C_c^\infty(\Omega).$$

Then $G_\Omega[V]$ is a W_2^1-function and there is a $C > 0$ such that

$$\int_\Omega |\nabla G_\Omega[V]|^2|u|^2 dx \leq C^2\|u\|^2_{W_2^1} \quad \forall u \in C_c^\infty(\Omega).$$

Proof. We'll just work with $n = 2$, $dx = dA(x)$. For $n \geq 3$ one can skip most of the proof and use Theorem I of [MV02] to directly deduce inequality 7.3 below, or else one could follow the whole proof below substituting higher order Riesz and Bessel potentials as appropriate (thus using Theorem 4.4 of [MV02]).

Fix $\eta \in C_c^\infty(\mathbb{R}^2)$ with $0 \leq \eta \leq 1$, spt $\eta \subseteq \Omega$, $\eta = 1$ in a neighborhood of spt V. Then for $u, w \in C_c^\infty(\mathbb{R}^2)$ we have

$$\begin{aligned}
|\langle V, uw\rangle &= |\langle V, \eta^2 uw\rangle| \\
&\leq c\|\nabla(\eta u)\|_{L_2}\|\nabla(\eta w)\|_{L_2} \\
&\leq c(\|u\nabla\eta\|_{L_2} + \|\eta\nabla u\|_{L_2})(\|w\nabla\eta\|_{L_2} + \|\eta\nabla w\|_{L_2}) \\
&\leq C\|u\|_{W_2^1}\|w\|_{W_2^1}.
\end{aligned}$$

Then Theorem 4.4 of [MV02] tells us that $(I-\Delta)^{-1}V$ is a W_2^1-function and

$$\int_{\mathbb{R}^2} |\nabla(I-\Delta)^{-1}V|^2|u|^2 dA \leq C^2\|u\|_{W_2^1}^2 \tag{7.1}$$

for all $u \in C_c^\infty\mathbb{R}^2$, where $(I-\Delta)^{-1}$ is convolution with

$$J(x) = \frac{1}{2}\int_0^\infty e^{-\frac{|x|^2}{4\delta}} e^{-\delta}\frac{d\delta}{\delta}. \tag{7.2}$$

This is a Bessel potential, a fundamental solution for $I-\Delta$ (see [Ste70], V.3).

We want to replace $(I-\Delta)^{-1}$ by G_Ω and in order to do this we will first replace $(I-\Delta)^{-1}$ by$(-\Delta)^{-1}$, which is the convolution with

$$K(x) = K_2(x) = \frac{1}{2\pi}\log\frac{1}{|x|}.$$

K is the corresponding Riesz potential, a fundamental solution for $-\Delta$.

It is easy to see from (7.2) that $J \in L^1(\mathbb{R}^2)$, that J is C^∞ away from 0, and J and all its derivatives decay exponentially at ∞. This plus the compactness of spt V is more than enough to justify the following manipulations:

$$\begin{aligned}
\nabla(-\Delta)^{-1}V - &\nabla(I-\Delta)^{-1}V \\
&= \nabla\left[(-\Delta)^{-1}((I-\Delta)-(-\Delta))(I-\Delta)^{-1}\right]V \\
&= \nabla\left[(-\Delta)^{-1}(I-\Delta)^{-1}\right]V \\
&= \nabla[K * (I-\Delta)^{-1}]V \\
&= K * [\nabla(I-\Delta)^{-1}V] \\
&= K_+ * [\nabla(I-\Delta)^{-1}V] - K_- * [\nabla(I-\Delta)^{-1}V]
\end{aligned}$$

The second term is bounded, because of the exponential decay of $\nabla(I-\Delta)^{-1}V$ at ∞. Hence it is clear that

$$\int_{\mathbb{R}^2} |K_- * [\nabla(I-\Delta)^{-1}V]|^2|u|^2 dA \leq \|K_- * [\nabla(I-\Delta)^{-1}V]\|_\infty^2\|u\|_{W_2^1}^2$$

for all $u \in C_c^\infty(\mathbb{R}^2)$. Next we note that since inequality (7.1) holds for all $u \in C_c^\infty(\mathbb{R}^2)$ it holds for all translates of $\nabla(I-\Delta)^{-1}V$. Thus it will hold for all convex combinations of translates of $\nabla(I-\Delta)^{-1}V$ and this easily implies

$$\int_\Omega |K_+ * [\nabla(I-\Delta)^{-1}V]|^2|u|^2 dA \le \|K_+\|^2_{L_1(\mathbb{R}^2)} C^2 \|u\|^2_{W_2^1} \quad \forall u \in C_c^\infty(\mathbb{R}^2).$$

Hence $(-\Delta)^{-1}V$ is a W_2^1-function and there is a constant $C > 0$ such that

$$\int_{\mathbb{R}^2} |\nabla(-\Delta)^{-1}V|^2|u|^2 dA \le C^2\|u\|^2_{W_2^1} \tag{7.3}$$

for all $u \in C_c^\infty(\mathbb{R}^2)$.

Now recall that $(-\Delta)^{-1}V = K * V$ and $G_\Omega[V] = K * V - h$, where h is smooth on $\overline{\Omega}$. The result follows. ■

In the proof of (a)⇒(b) of Theorem 7.2 by use of Theorem 7.3 we will replace the disc $\mathbb{D}$ by the annulus

$$\mathcal{A} = \{z \in \mathbb{C} : 1/2 < |z| < 2\}.$$

For $u \in L^2(\mathbb{T})$ define $\hat{u}$ in Ω by $\hat{u}$ is harmonic in $\mathcal{A}\setminus\mathbb{T}$, $\hat{u} = u$ on $\mathbb{T}$, and $\hat{u} = 0$ on $\partial\mathcal{A}$. Of course this will imply that $\hat{u}(z) = \hat{u}(\frac{1}{\bar{z}})$ for $z \in \mathcal{A}$. Then it is not difficult to see that $\|u\|^2_{D_h}$ is equivalent to

$$\|u\|^2_{\mathcal{A}} = \int_{\mathcal{A}} |\nabla\hat{u}|^2 dA,$$

but we will need the following stronger statement.

Lemma 7.4. *If $w, u \in D_h$, then*

$$\left|\int_{\mathcal{A}} \nabla\hat{w}\cdot\nabla\overline{\hat{u}}\frac{dA}{4\pi} - \int_{\mathbb{D}} \nabla P[w]\cdot\nabla\overline{P[u]}\frac{dA}{2\pi}\right| \le C\|w\|_{L^1(\mathbb{T})}\|u\|_{L^1(\mathbb{T})}.$$

Proof. For $n \in \mathbb{Z}$ let $u_n(e^{it}) = e^{int}$. Then $P_{re^{it}}[u_n] = r^{|n|}e^{int}$, $|\nabla P_{re^{it}}[u_n]|^2 = 2n^2r^{2|n|-2}$ and hence

$$\int_{\mathbb{D}} |\nabla P[u_n]|^2 dA = 2\pi|n|. \tag{7.4}$$

We can also find $\hat{u}_n$ explicitly. In $1/2 \le r \le 1$ we have

$$\hat{u}_n(re^{it}) = \left(\frac{4^{|n|}}{4^{|n|}-1}r^{|n|} - \frac{1}{4^{|n|}-1}r^{-|n|}\right)e^{int} \text{ for } n \ne 0$$

and $\hat{u}_0(e^{it}) = \frac{\log 2r}{\log 2}$. Thus for $n \ne 0$

$$\left|\nabla\hat{u}_n(re^{it})\right|^2 = 2n^2\left[\left(\frac{4^{|n|}}{4^{|n|}-1}\right)^2 r^{2|n|-2} + \frac{1}{(4^{|n|}-1)^2}r^{-2|n|-2}\right]$$

and $|\nabla \hat{u}_0(re^{it})|^2 = \frac{1}{(\log 2)^2}\frac{1}{r^2}$. Thus,

$$\int_{1/2<|z|<1} |\nabla \hat{u}_n|^2\, dA = 2\pi |n| \frac{4^{|n|}+1}{4^{|n|}-1}, \text{ if } n \neq 0$$

and $\int_{1/2<|z|<1} |\nabla \hat{u}_0|^2\, dA = \frac{2\pi}{\log 2}$.

In $1 \leq r \leq 2$ we have $\hat{u}_n(re^{it}) = \hat{u}_n(\frac{1}{r}e^{it})$ and $\int_{1<|z|<2} |\nabla \hat{u}_n|^2 dA = \int_{1/2<|z|<1} |\nabla \hat{u}_n|^2\, dA$. Hence

$$\int_{\mathcal{A}} |\nabla \hat{u}_n|^2\, dA = 4\pi |n| \frac{4^{|n|}+1}{4^{|n|}-1}, \text{ if } n \neq 0 \tag{7.5}$$

and $\int_{\mathcal{A}} |\nabla \hat{u}_0|^2\, dA = \frac{4\pi}{\log 2}$.

Also note that

$$\int_{\mathcal{A}} \nabla \hat{u}_n \cdot \nabla \overline{\hat{u}_m} dA = \int_{\mathbb{D}} \nabla P[u_n] \cdot \nabla \overline{P[u_m]} dA = 0 \text{ for all } n \neq m. \tag{7.6}$$

Now let $w(e^{it}) = \sum_{-\infty}^{\infty} a_n e^{int}$ and $u(e^{it}) = \sum_{-\infty}^{\infty} b_n e^{int}$, then by (7.4),(7.5), and (7.6) we have

$$\begin{aligned}
\left| \int_{\mathcal{A}} \nabla \hat{w} \cdot \nabla \bar{\hat{u}} \frac{dA}{4\pi} - \int_{\mathbb{D}} \nabla P[w] \cdot \nabla \overline{P[u]} \frac{dA}{2\pi} \right| &= \left| \frac{a_0 \bar{b}_0}{\log 2} + \sum_{n\neq 0} (|n| \frac{4^{|n|}+1}{4^{|n|}-1} - |n|) a_n \bar{b}_n \right| \\
&\leq \frac{|a_0||b_0|}{\log 2} + \sum_{n \neq 0} \frac{2|n|}{4^{|n|}-1} |a_n||b_n| \\
&\leq C \|\{a_n\}\|_\infty \|\{b_n\}\|_\infty \\
&\leq C \|w\|_{L^1(\mathbb{T})} \|u\|_{L^1(\mathbb{T})}
\end{aligned}$$

■

Proof of (a)⇒(b) of Theorem 7.2. Lemma 7.4 shows that the hypothesis (a) of Theorem 7.2 implies that

$$\int_{\mathcal{A}} \nabla \hat{w} \cdot \nabla \widehat{u\bar{v}} dA \leq C \|u\|_{D_h} \|v\|_{D_h} \text{ for all } u, v \in C^\infty(\mathbb{T}). \tag{7.7}$$

Furthermore, it is clear from the definitions that for any $v \in L^2(\mathbb{T})$ the function $P_z[v] - \hat{v}(z)$ is harmonic in $\mathbb{D} \cap \mathcal{A}$, 0 on $|z| = 1$ and C^∞ on $|z| = 1/2$. Hence there is a $C > 0$ (dependent on w) such that for all z with $1/2 < |z| < 1$ we have

$$|\, |\nabla \hat{w}(z)| - |\nabla P_z[w]|\, | \leq C$$

and $|\hat{u}(z) - P_z[u]| \leq 3\|u\|_{L^1(\mathbb{T})}$ by the maximum principle. Thus it suffices to verify that

$$\int_{\mathcal{A}} |\hat{u}\nabla\hat{w}|^2 dA \leq C^2\|u\|^2_{D_h} \text{ for all } u \in C^\infty(\mathbb{T}). \tag{7.8}$$

Now let $U_1, U_2 \in C_c^\infty(\mathcal{A})$ and define $u_j = U_j|\mathbb{T}$ for $j = 1, 2$. Recall Green's first identity says that

$$\int_\Omega [\varphi\Delta\psi + \nabla\varphi \cdot \nabla\psi] dA = \int_{\partial\Omega} \varphi \frac{\partial\psi}{\partial n} ds$$

for sufficiently smooth Ω and φ, ψ. We apply this in $\mathcal{A} \cap \{z : |z| < 1\}$ and $\mathcal{A} \cap \{z : |z| > 1\}$ to obtain

$$\int_{\mathcal{A}} \nabla\hat{w} \cdot \nabla(U_1U_2 - \widehat{u_1u_2}) dA = 0.$$

This would first be true for smooth w and then by an approximation for all $w \in D_h$. Thus inequality (7.7) and the Dirichlet principle imply

$$\begin{aligned} |\int_{\mathcal{A}} \nabla\hat{w} \cdot \nabla U_1U_2 dA|^2 &\leq C^2 \int_{\mathcal{A}} |\nabla\hat{u}_1|^2 dA \int_{\mathcal{A}} |\nabla\hat{u}_2|^2 dA \\ &\leq C^2 \int_{\mathcal{A}} |\nabla U_1|^2 dA \int_{\mathcal{A}} |\nabla U_2|^2 dA. \end{aligned}$$

We now think of $V = -\Delta\hat{w}$ as a distribution, then

$$\langle V, U_1U_2\rangle = -\langle\Delta\hat{w}, U_1U_2\rangle = \int_{\mathcal{A}} \nabla\hat{w} \cdot \nabla(U_1U_2) dA,$$

so that $|\langle V, U_1U_2\rangle| \leq C\|\nabla U_1\|_{L_2}\|\nabla U_2\|_{L_2}$ for all $U_1, U_2 \in C_c^\infty(\mathcal{A})$. Furthermore, spt $V = \mathbb{T}$ and $\hat{w} = G_{\mathcal{A}}[V]$. Hence (7.8) follows from Theorem 7.3. ■

8. Examples

The following examples are meant to illustrate some of the problems we encountered as we were trying to determine conditions for norm-minimizers in the weak product spaces.

Example 8.1. If $n \in \mathbb{N}$ then $\|z^n\|^2_D = n + 1$. For $k = 0, ..., n$ set $f_k(z) = (\frac{n-k+1}{k+1})^{1/4} z^k$ and $g_k(z) = (\frac{k+1}{n-k+1})^{1/4} z^{n-k}$, then for each k we get $z^n = f_k g_k$ and $\langle\varphi f_k, f_k\rangle_D = \langle\varphi g_k, g_k\rangle_D$ for every multiplier φ, i.e. for each k the pair (f_k, g_k) satisfies the necessary condition for a norm-minimizer of Theorem 4.3. Yet it is clear that (f_k, g_k) is not a norm minimizer for $\|z^n\|_*$, whenever $0 < k < n$. In this case it appears to be best to write $z^n = 1 \cdot z^n$.

If $z \in \Omega$ and if k_z denotes the reproducing kernel for the space $\mathcal{H}$ of analytic functions on Ω, then, of course, $k_z^2 \in \mathcal{H} \odot \mathcal{H}$ and it is easy to see that $\|k_z^2\|_* = \|k_z\|^2$. For outer or positive functions h it is tempting to think that $h = \sqrt{h}\sqrt{h}$ would be the best decomposition for $\|h\|_*$. That is not always the case. The following example was suggested to us by Paul Bourdon.

Example 8.2. (a) The analytic Dirichlet space D. Let $h(z) = (1+z^3)^2$. Then $1 \cdot \|h\|_D = \sqrt{24} < 5 = \|\sqrt{h}\|_D^2$.

(b) The harmonic Dirichlet space D_h. The function $h(e^{it}) = |1+e^{5it}|^4$ is a non-negative function in D_h and we have $1 \cdot \|h\|_{D_h} = \sqrt{250} < 16 = \|\sqrt{h}\|_{D_h}^2$.

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