A REMARK ON THE MULTIPLIERS ON SPACES OF WEAK PRODUCTS OF FUNCTIONS

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Abstract. If $\mathcal{H}$ denotes a Hilbert space of analytic functions on a region $\Omega \subseteq \mathbb{C}^d$, then the weak product is defined by

$$\mathcal{H} \odot \mathcal{H} = \left\{ h = \sum_{n=1}^{\infty} f_n g_n : \sum_{n=1}^{\infty} \|f_n\|_{\mathcal{H}} \|g_n\|_{\mathcal{H}} < \infty \right\}.$$

We prove that if $\mathcal{H}$ is a first order holomorphic Besov Hilbert space on the unit ball of $\mathbb{C}^d$, then the multiplier algebras of $\mathcal{H}$ and of $\mathcal{H} \odot \mathcal{H}$ coincide.

1. Introduction

Let d be a positive integer and let $R = \sum_{i=1}^{d} z_i \frac{\partial}{\partial z_i}$ denote the radial derivative operator. For $s \in \mathbb{R}$ the holomorphic Besov space B_s is defined to be the space of holomorphic functions f on the unit ball $\mathbb{B}_d$ of $\mathbb{C}^d$ such that for some nonnegative integer $k > s$

$$\|f\|_{k,s}^2 = \int_{\mathbb{B}_d} |(I+R)^k f(z)|^2 (1-|z|^2)^{2(k-s)-1} dV(z) < \infty.$$

Here dV denotes Lebesgue measure on $\mathbb{B}_d$. It is well-known that for any $f \in \mathrm{Hol}(\mathbb{B}_d)$ and any $s \in \mathbb{R}$ the quantity $\|f\|_{k,s}$ is finite for some nonnegative integer $k > s$ if and only if it is finite for all nonnegative integers $k > s$, and that for each $k > s$ $\|\cdot\|_{k,s}$ defines a norm on B_s, and that all these norms are equivalent to one another, see [2]. For $s < 0$ one can take $k = 0$ and these spaces are weighted Bergman spaces. In particular, $B_{-1/2} = L^2_a(\mathbb{B}_d)$ is the unweighted Bergman space. For $s = 0$ one obtains the Hardy space of $\mathbb{B}_d$ and one has that for each $k \geq 1$ $\|f\|_{k,0}^2$ is equivalent to $\int_{\partial\mathbb{B}_d} |f|^2 d\sigma$, where σ is the rotationally invariant probability measure on $\partial\mathbb{B}_d$. We also note that for $s = (d-1)/2$ we

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have $B_s = H^2_d$, the Drury-Arveson space. If $d = 1$ and $s = 1/2$, then $B_s = D$, the classical Dirichlet space of the unit disc.

Let $\mathcal{H} \subseteq \mathrm{Hol}(\mathbb{B}_d)$ be a reproducing kernel Hilbert space. The weak product of $\mathcal{H}$ is denoted by $\mathcal{H} \odot \mathcal{H}$ and it is defined to be the collection of all functions $h \in \mathrm{Hol}(\mathbb{B}_d)$ such that there are sequences $\{f_i\}_{i\geq 1}, \{g_i\}_{i\geq 1} \subseteq \mathcal{H}$ with $\sum_{i=1}^\infty \|f_i\|_\mathcal{H}\|g_i\|_\mathcal{H} < \infty$ and for all $z \in \mathbb{B}_d$, $h(z) = \sum_{i=1}^\infty f_i(z)g_i(z)$.

We define a norm on $\mathcal{H} \odot \mathcal{H}$ by

$$\|h\|_* = \inf\left\{\sum_{i=1}^\infty \|f_i\|_\mathcal{H}\|g_i\|_\mathcal{H} : h(z) = \sum_{i=1}^\infty f_i(z)g_i(z) \text{ for all } z \in \mathbb{B}_d\right\}.$$

In what appears below we will frequently take $\mathcal{H} = B_s$, and will use the same notation for this weak product.

Weak products have their origin in the work of Coifman, Rochberg, and Weiss [5]. In the frame work of the Hilbert space $\mathcal{H}$ one may consider the weak product to be an analogue of the Hardy H^1-space. For example, one has $H^2(\partial\mathbb{B}_d) \odot H^2(\partial\mathbb{B}_d) = H^1(\partial\mathbb{B}_d)$ and $L^2_a(\mathbb{B}_d) \odot L^2_a(\mathbb{B}_d) = L^1_a(\mathbb{B}_d)$, see [5]. For the Dirichlet space D the weak product $D \odot D$ has recently been considered in [1], [4], [9], [6], and [7]. The space $H^2_d \odot H^2_d$ was used in [10]. For further motivation and general background on weak products we refer the reader to [1] and [9].

If $\mathcal{B}$ is a Banach space of analytic functions, then we use $M(\mathcal{B})$ to denote the multiplier algebra of $\mathcal{B}$,

$$M(\mathcal{B}) = \{\varphi : \varphi f \in \mathcal{B} \text{ for all } f \in \mathcal{B}\}.$$

The multiplier norm $\|\varphi\|_M$ is defined to be the norm of the associated multiplication operator $M_\varphi : \mathcal{B} \to \mathcal{B}$. It is easy to check and is well-known that for any space of analytic functions $M(\mathcal{B}) \subseteq H^\infty(\mathbb{B}_d)$, and that for $s \leq 0$ we have $M(B_s) = H^\infty(\mathbb{B}_d)$. For $s > d/2$ the space B_s is an algebra [2], hence $B_s = M(B_s)$, but for $0 < s \leq d/2$ one has $M(B_s) \subsetneq B_s \cap H^\infty(\partial\mathbb{B}_d)$. For those cases $M(B_s)$ has been described by a certain Carleson measure condition, see [3, 8].

It is easy to see that $M(\mathcal{H}) \subseteq M(\mathcal{H} \odot \mathcal{H}) \subseteq H^\infty$ (see Proposition 3.1). Thus, if $s \leq 0$, then $M(B_s) = M(B_s \odot B_s) = H^\infty$. Furthermore, if $s > d/2$, then $B_s = B_s \odot B_s = M(B_s)$ since B_s is an algebra. This raises the question whether $M(B_s)$ and $M(B_s \odot B_s)$ always agree. We prove the following:

Theorem 1.1. *Let $s \in \mathbb{R}$ and $d \in \mathbb{N}$. If $s \leq 1$ or $d \leq 2$, then $M(B_s) = M(B_s \odot B_s)$.*

Note that when $d \leq 2$, then B_s is an algebra for all $s > 1$. Thus for each $d \in \mathbb{N}$ the nontrivial range of the Theorem is $0 < s \leq 1$. If

$d = 1$ then the theorem applies to the classical Dirichlet space of the unit disc and for $d \leq 3$ it applies to the Drury-Arveson space.

2. Preliminaries

For $z = (z_1, ..., z_d) \in \mathbb{C}^d$ and $t \in \mathbb{R}$ we write $e^{it}z = (e^{it}z_1, ..., e^{it}z_d)$ and we write $\langle z, w\rangle$ for the inner product in $\mathbb{C}^d$. Furthermore, if h is a function on $\mathbb{B}_d$, then we define $T_t f$ by $(T_t f)(z) = f(e^{it}z)$. We say that a space $\mathcal{H} \subseteq \mathrm{Hol}(\mathbb{B}_d)$ is radially symmetric, if each T_t acts isometrically on $\mathcal{H}$ and if for all $t_0 \in \mathbb{R}$, $T_t \to T_{t_0}$ in the strong operator topology as $t \to t_0$, i.e. if $\|T_t f\|_{\mathcal{H}} = \|f\|_{\mathcal{H}}$ and $\|T_t f - T_{t_0} f\|_{\mathcal{H}} \to 0$ for all $f \in \mathcal{H}$. For example, for each $s \in \mathbb{R}$ the holomorphic Besov space B_s is radially symmetric when equipped with any of the norms $\|\cdot\|_{k,s}$, $k > s$.

It is elementary to verify the following lemma.

Lemma 2.1. *If $\mathcal{H} \subseteq Hol(\mathbb{B}_d)$ is radially symmetric, then so is $\mathcal{H} \odot \mathcal{H}$.*

Note that if h and φ are functions on $\mathbb{B}_d$, then for every $t \in \mathbb{R}$ we have $(T_t\varphi)h = T_t(\varphi T_{-t}h)$, hence if a space is radially symmetric, then T_t acts isometrically on the multiplier algebra. For $0 < r < 1$ we write $f_r(z) = f(rz)$.

Lemma 2.2. *If $\mathcal{H} \subseteq Hol(\mathbb{B}_d)$ is radially symmetric, and if $\varphi \in M(\mathcal{H} \odot \mathcal{H})$, then for all $0 < r < 1$ we have $\|\varphi_r\|_{M(\mathcal{H}\odot\mathcal{H})} \leq \|\varphi\|_{M(\mathcal{H}\odot\mathcal{H})}$.*

Proof. Let $\varphi \in M(\mathcal{H} \odot \mathcal{H})$ and $h \in \mathcal{H} \odot \mathcal{H}$, then for $0 < r < 1$ we have

$$\varphi_r h = \int_{-\pi}^{\pi} \frac{1 - r^2}{|1 - re^{it}|^2} (T_t\varphi)h \frac{dt}{2\pi}.$$

This implies

$$\|\varphi_r h\|_* \leq \int_{-\pi}^{\pi} \frac{1 - r^2}{|1 - re^{it}|^2} \|(T_t\varphi)h\|_* \frac{dt}{2\pi} \leq \|\varphi\|_{M(\mathcal{H}\odot\mathcal{H})} \|h\|_*.$$

Thus, $\|\varphi_r\|_{M(\mathcal{H}\odot\mathcal{H})} \leq \|\varphi\|_{M(\mathcal{H}\odot\mathcal{H})}$. ■

3. Multipliers

The following Proposition is elementary.

Proposition 3.1. *We have $M(\mathcal{H}) \subseteq M(\mathcal{H} \odot \mathcal{H}) \subseteq H^\infty$ and if $\varphi \in M(\mathcal{H})$, $\|\varphi\|_{M(\mathcal{H}\odot\mathcal{H})} \leq \|\varphi\|_{M(\mathcal{H})}$.*

As explained in the Introduction, the following will establish Theorem 1.1.

Theorem 3.2. *Let $0 < s \leq 1$. Then $M(B_s) = M(B_s \odot B_s)$ and there is a $C_s > 0$ such that*

$$\|\varphi\|_{M(B_s \odot B_s)} \leq \|\varphi\|_{M(B_s)} \leq C_s \|\varphi\|_{M(B_s \odot B_s)}$$

for all $\varphi \in M(B_s)$.

Here for each s we have the norm on B_s to be $\|\cdot\|_{k,s}$, where k is the smallest natural number $> s$.

Proof. We first do the case $0 < s < 1$. Then $k = 1$, and $\|f\|^2_{B_s} = \int_{\mathbb{B}_d} |(I+R)f(z)|^2 dV_s(z)$, where $dV_s(z) = (1-|z|^2)^{1-2s} dV(z)$. For later reference we note that a short calculation shows that $\int_{\mathbb{B}_d} |Rf|^2 dV_s \leq \|f\|^2_{B_s}$.

We write $\|R\varphi\|_{Ca(B_s)}$ for the Carleson measure norm of $|R\varphi|^2$, i.e.

$$\|R\varphi\|^2_{Ca(B_s)} = \inf\left\{C > 0 : \int_{\mathbb{B}_d} |f|^2 |R\varphi|^2 dV_s \leq C\|f\|^2_{B_s} \text{ for all } f \in B_s\right\}.$$

Since $\|\varphi f\|^2_{B_s} = \int_{\mathbb{B}_d} |\varphi(z)(I+R)f(z) + f(z)R\varphi(z)|^2 dV_s(z)$ it is clear that $\|\varphi\|_{M(B_s)}$ is equivalent to $\|\varphi\|_\infty + \|R\varphi\|_{Ca(B_s)}$. Thus, it suffices to show that there is a $c > 0$ such that $\|R\varphi\|_{Ca(B_s)} \leq c\|\varphi\|_{M(B_s \odot B_s)}$ for all $\varphi \in M(B_s \odot B_s)$.

First we note that if b is holomorphic in a neighborhood of $\overline{\mathbb{B}_d}$ and $h = \sum_{i=1}^\infty f_i g_i \in B_s \odot B_s$, then

$$\begin{aligned}\int_{\mathbb{B}_d} |(Rh)Rb| dV_s &\leq \sum_{i=1}^\infty \int_{\mathbb{B}_d} |(Rf_i)g_i Rb| dV_s + \int_{\mathbb{B}_d} |(Rg_i)f_i Rb| dV_s \\ &\leq \sum_{i=1}^\infty \|f_i\|_{B_s} \left(\int_{\mathbb{B}_d} |g_i Rb|^2 dV_s\right)^{1/2} + \|g_i\|_{B_s} \left(\int_{\mathbb{B}_d} |f_i Rb|^2 dV_s\right)^{1/2} \\ &\leq 2\sum_{i=1}^\infty \|f_i\|_{B_s} \|g_i\|_{B_s} \|Rb\|_{Ca(B_s)}.\end{aligned}$$

Hence

$$\int_{\mathbb{B}_d} |(Rh)Rb| dV_s \leq 2\|h\|_* \|Rb\|_{Ca(B_s)},$$

where we have continued to write $\|\cdot\|_*$ for $\|\cdot\|_{B_s \odot B_s}$.

Let $\varphi \in M(B_s \odot B_s)$ and let $0 < r < 1$. Then for all $f \in B_s$ we have $f^2, \varphi_r f^2 \in B_s \odot B_s$, hence

$$\begin{aligned}\int_{\mathbb{B}_d} |f|^2 |R\varphi_r|^2 dV_s &= \int_{\mathbb{B}_d} |R(\varphi_r f^2) - \varphi_r R(f^2)|\ |R\varphi_r| dV_s \\ &\leq 2(\|\varphi_r f^2\|_* + \|\varphi\|_\infty \|f^2\|_*)\|R\varphi_r\|_{Ca(B_s)} \\ &\leq 2(\|\varphi\|_{M(B_s\odot B_s)}\|f^2\|_* + \|\varphi\|_\infty \|f^2\|_*)\|R\varphi_r\|_{Ca(B_s)} \\ &\leq 4\|\varphi\|_{M(B_s\odot B_s)}\|f\|^2_{B_s}\|R\varphi_r\|_{Ca(B_s)}.\end{aligned}$$

Next we take the sup of the left hand side of this expression over all f with $\|f\|_{B_s} = 1$ and we obtain $\|R\varphi_r\|^2_{Ca(B_s)} \leq 4\|\varphi\|_{M(B_s\odot B_s)}\|R\varphi_r\|_{Ca(B_s)}$ which implies that $\|R\varphi_r\|_{Ca(B_s)} \leq 4\|\varphi\|_{M(B_s\odot B_s)}$ holds for all $0 < r < 1$. Thus, for $0 < s < 1$ the result follows from Fatou's lemma as $r \to 1$.

If $s = 1$, then $\|f\|^2_{2,1} \sim \int_{\partial\mathbb{B}_d} |(I + R)f(z)|^2 d\sigma(z)$ and the argument proceeds as above. ■

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