SOME HILBERT SPACES RELATED WITH THE DIRICHLET SPACE

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ABSTRACT. We study the reproducing kernel Hilbert space with kernel k^d, where d is a positive integer and k is the reproducing kernel of the analytic Dirichlet space.

INTRODUCTION

Consider the Dirichlet space $\mathcal{D}$ on the unit disc $\{z \in \mathbb{C} : \ |z| < 1\}$ of the complex plane. It can be defined as the Reproducing Kernel Hilbert Space (RKHS) having kernel

$$k_z(w) = k(w, z) = \frac{1}{\overline{z}w} \log \frac{1}{1 - \overline{z}w} = \sum_{n=0}^{\infty} \frac{(\overline{z}w)^n}{n+1}.$$

We are interested in the spaces $\mathcal{D}_d$ having kernel k^d, with $d \in \mathbb{N}$. $\mathcal{D}_d$ can be thought of in terms of function spaces on polydiscs, following ideas of Aronszajn [4]. To explain this point of view, note that the tensor d-power $\mathcal{D}^{\otimes d}$ of the Dirichlet space has reproducing kernel $k_d(z_1, \cdots, z_d; w_1, \ldots, w_d) = \Pi_{j=1}^{d} k(z_j, w_j)$. Hence, the space of restrictions of functions in $\mathcal{D}^{\otimes d}$ to the diagonal $z_1 = \cdots = z_d$ has the reproducing kernel k^d, and therefore coincides with $\mathcal{D}_d$.

We will provide several equivalent norms for the spaces $\mathcal{D}_d$ and their dual spaces in Theorem 1. Then we will discuss the properties of these spaces. More precisely, we will investigate:

- $\mathcal{D}_d$ and its dual space HS_d in connection with Hankel operators of Hilbert-Schmidt class on the Dirichlet space $\mathcal{D}$;
- the complete Nevanlinna-Pick property for $\mathcal{D}_d$;
- the Carleson measures for these spaces.

Concerning the first item, the connection with Hilbert-Schmidt Hankel operators served as our original motivation for studying the spaces $\mathcal{D}_d$.

Note that the spaces $\mathcal{D}_d$ live infinitely close to $\mathcal{D}$ in the scale of weighted Dirichlet spaces $\tilde{\mathcal{D}}_s$, defined by the norms

$$\|\varphi\|^2_{\tilde{\mathcal{D}}_s} = \int_{-\pi}^{+\pi} \left|\varphi(e^{it})\right|^2 \frac{dt}{2\pi} + \iint_{|z|<1} |f'(z)|^2 (1 - |z|^2)^s \frac{dA(z)}{\pi}, \ 0 \leq s < 1,$$

where $\frac{dA(z)}{\pi}$ is normalized area measure on the unit disc.

Notation: We use multiindex notation. If $n = (n_1, \ldots, n_d)$ belongs to $\mathbb{N}^d$, then $|n| = n_1 + \cdots + n_d$. We write $A \approx B$ if A and B are quantities that depend on a certain family of variables, and there exist independent constants $0 < c < C$ such that $cA \leq B \leq CA$.

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Theorem 1. *Let d be a positive integer and let*

$$a_d(k) = \sum_{|n|=k} \frac{1}{(n_1+1)\dots(n_d+1)}.$$

Then the norm of a function $\varphi(z) = \sum_{k=0}^{\infty} \widehat{\varphi}(k) z^k$ in $\mathcal{D}_d$ is

$$\|\varphi\|_{\mathcal{D}_d} = \left(\sum_{k=0}^{\infty} a_d(k)^{-1} \left|\widehat{\varphi}(k)\right|^2 \right)^{1/2} \approx [\varphi]_d, \tag{1}$$

where

$$[\varphi]_d = \left(\sum_{k=0}^{\infty} \frac{k+1}{\log^{d-1}(k+2)} \left|\widehat{\varphi}(k)\right|^2 \right)^{1/2}. \tag{2}$$

An equivalent Hilbert norm $|[\varphi]|_d \approx [\varphi]_d$ for φ in terms of the values of φ is given by

$$|[\varphi]|_d = |\varphi(0)|^2 + \left(\iint_{\mathbb{D}} |\varphi'(z)|^2 \frac{1}{\log^{d-1}\left(\frac{1}{1-|z|^2}\right)} \frac{dA(z)}{\pi} \right)^{1/2}. \tag{3}$$

Define now the holomorphic space HS_d by the norm:

$$\|\psi\|_{HS_d} = \left(\sum_{k=0}^{\infty} (k+1)^2 a_d(k) \left|\widehat{\psi}(k)\right|^2 \right)^{1/2}. \tag{4}$$

Then, $HS_d \equiv (\mathcal{D}_d)^*$ is the dual space of $\mathcal{D}_d$ under the duality pairing of $\mathcal{D}$. Moreover,

$$\begin{aligned} \|\psi\|_{HS_d} \approx [\psi]_{HS_d} &:= \left(\sum_{k=0}^{\infty} (k+1) \log^{d-1}(k+2) \left|\widehat{\psi}(k)\right|^2 \right)^{1/2} \approx \\ |[\psi]|_{HS_d} &:= \left(|\psi(0)|^2 + \iint_{\mathbb{D}} |\psi'(z)|^2 \log^{d-1}\left(\frac{1}{1-|z|^2}\right) \frac{dA(z)}{\pi} \right)^{1/2}. \end{aligned} \tag{5}$$

Furthermore, the norm can be written as

$$\|\psi\|_{HS_d}^2 = \sum_{(n_1,\dots,n_d)} |\langle e_{n_1} \dots e_{n_d}, \psi \rangle_{\mathcal{D}}|^2, \tag{6}$$

where $\{e_n\}_{n=0}^{\infty}$ is the canonical orthonormal basis of $\mathcal{D}$, $e_n(z) = \frac{z^n}{\sqrt{n+1}}$.

The remainder of this section is devoted to the proof of Theorem 1. The expression for $\|\varphi\|_{\mathcal{D}_d}$ in (1) follows by expanding $(k_z)^d$ as a power series. The equivalence $\|\varphi\|_{\mathcal{D}_d} \approx [\varphi]_d$, as well as $\|\varphi\|_{HS_d} \approx [\varphi]_{HS_d}$, are consequences of the following lemma.

Lemma 1. *For each $d \in \mathbb{N}$ there are constants $c, C > 0$ such that for all $k \geq 0$ we have*

$$c a_d(k) \leq \frac{\log^{d-1}(k+2)}{k+1} \leq C a_d(k).$$

Consequently, if $t \in [0,1)$*, then*

$$c\left(\frac{1}{t}\log\frac{1}{1-t}\right)^d \le \sum_{k=0}^{\infty} \frac{\log^{d-1}(k+2)}{k+1} t^k \le c\left(\frac{1}{t}\log\frac{1}{1-t}\right)^d.$$

Proof of Lemma 1. We will prove the Lemma by induction on $d \in \mathbb{N}$. It is obvious for $d = 1$. Thus let $d \ge 2$ and suppose the lemma is true for $d - 1$. Also we observe that there is a constant $c > 0$ such that for all $k \ge 0$ and $0 \le n \le k$ we have

$$c\log^{d-2}(k+2) \le \log^{d-2}(n+2) + \log^{d-2}(k-n+2) \le 2\log^{d-2}(k+2).$$

Then for $k \ge 0$

$$\begin{aligned}
a_d(k) &= \sum_{n_1+\cdots+n_d=k} \frac{1}{(n_1+1)\dots(n_d+1)} \\
&= \sum_{n=0}^{k} \frac{1}{n+1} \sum_{n_2+\cdots+n_d=k-n} \frac{1}{(n_2+1)\dots(n_d+1)} \\
&\approx \sum_{n=0} \frac{1}{n+1} \frac{\log^{d-2}(k-n+2)}{k-n+1} \quad \text{by the inductive assumption} \\
&= \frac{1}{2}\sum_{n=0}^{k} \frac{\log^{d-2}(n+2) + \log^{d-2}(k-n+2)}{(n+1)(k-n+1)} \\
&\approx \log^{d-2}(k+2) \sum_{n=0}^{k} \frac{1}{(n+1)(k-n+1)} \quad \text{by the earlier observation} \\
&= \frac{\log^{d-2}(k+2)}{k+2} \sum_{n=0}^{k} \frac{1}{n+1} + \frac{1}{k-n+1} \\
&\approx \frac{\log^{d-1}(k+2)}{k+1}.
\end{aligned}$$

□

Next, we prove the equivalence $[\varphi]_{HS_d} \approx |[\varphi]|_{HS_d}$ which appears in (5).

Lemma 2. *Let* $d \in \mathbb{N}$*. Then*

$$\int_0^1 t^k \left(\frac{1}{t}\log\frac{1}{1-t}\right)^{d-1} dt \approx \frac{\log^{d-1}(k+2)}{k+1}, \qquad k \ge d.$$

Given the Lemma, we expand

$$\begin{aligned}
|[\psi]|^2_{HS_d} &= |\widehat{\psi}(0)|^2 + \iint_{\mathbb{D}} \left|\sum_{k=1}^{\infty} \widehat{\psi}(k) k z^{k-1}\right|^2 \log^{d-1}\frac{1}{1-|z|^2}\frac{dA(z)}{\pi} \\
&= |\widehat{\psi}(0)|^2 + \sum_{k=1}^{\infty} k^2 \left|\widehat{\psi}(k)\right|^2 \int_0^1 \log^{d-1}\frac{1}{1-t} t^{k-1} dt \\
&\approx |\widehat{\psi}(0)|^2 + \sum_{k=1}^{\infty} k^2 \left|\widehat{\psi}(k)\right|^2 \frac{\log^{d-1}(k+2)}{k+1} \\
&\approx [\psi]^2_{HS_d},
\end{aligned}$$

obtaining the desired conclusion.

Proof of Lemma 2. The case $d = 1$ is obvious, leaving us to consider $d \geq 2$. We will also assume that $k \geq 2$. Then by Lemma 1 we have

$$\begin{aligned}\int_0^1 t^k \left(\frac{1}{t}\log\frac{1}{1-t}\right)^{d-1} dt \;&\approx\; \int_0^1 t^k \sum_{n=0}^{\infty} \frac{\log^{d-2}(n+2)}{n+1} t^n dt \\ &=\; \sum_{n=0}^{\infty} \frac{\log^{d-2}(n+2)}{(n+1)(n+k+1)} = S_1 + S_2,\end{aligned}$$

where

$$\begin{aligned}S_1 &= \sum_{n=0}^{k-1} \frac{\log^{d-2}(n+2)}{(n+1)(n+k+1)} \approx \frac{1}{k+1}\sum_{n=0}^{k-1}\frac{\log^{d-2}(n+2)}{n+1} \approx \frac{1}{k+1}\int_1^{k+2}\frac{\log^{d-2}(t)}{t}dt \\ &= \frac{1}{d-1}\,\frac{\log^{d-1}(k+2)}{k+1}\end{aligned}$$

and

$$\begin{aligned}S_2 &= \sum_{n=k}^{\infty} \frac{\log^{d-2}(n+2)}{(n+1)(n+k+1)} \leq \sum_{n=k+1}^{\infty}\frac{\log^{d-2}(n+1)}{n^2} \leq \sum_{j=1}^{\infty}\sum_{n=k^j}^{k^{j+1}-1}\frac{\log^{d-2}(n+1)}{n^2} \\ &\leq \sum_{j=1}^{\infty}(j+1)^{d-2}\log^{d-2}k \sum_{n=k^j}^{k^{j+1}-1}\frac{1}{n^2} \leq \log^{d-2}(k+2)\sum_{j=1}^{\infty}(j+1)^{d-2}\int_{k^j-1}^{\infty}\frac{1}{x^2}dx \\ &= \frac{\log^{d-2}(k+2)}{k+1}\sum_{j=1}^{\infty}(j+1)^{d-2}\frac{k+1}{k^j-1} \leq \frac{\log^{d-2}(k+2)}{k+1}\sum_{j=1}^{\infty}(j+1)^{d-2}\frac{k+1}{(k-1)k^{j-1}} \\ &\leq \frac{\log^{d-2}(k+2)}{k+1}\sum_{j=1}^{\infty}(j+1)^{d-2}\frac{3}{2^{j-1}} \;= o\left(\frac{\log^{d-1}(k+2)}{k+1}\right).\end{aligned}$$ □

Now, the duality between $\mathcal{D}_d$ and HS_d under the duality pairing given by the inner product of $\mathcal{D}$ is easily seen by considering $[\cdot]_d$ and $[\cdot]_{HS_d}$. They are weighted ℓ^2 norms and duality is established by means of the Cauchy-Schwarz inequality.

Next we will prove that $[\varphi]_d \approx |[\varphi]|_d$. This is equivalent to proving that the dual space of HS_d, with respect to the Dirichlet inner product $\langle\cdot\,,\,\cdot\rangle_{\mathcal{D}}$, is the Hilbert space with the norm $|[\cdot]|_d$.

Let $d \in \mathbb{N}$ and set, for $0 \leq t < 1$, $w_d(t) = \left(\frac{1}{t}\log\frac{1}{1-t}\right)^d$ and, for $|z| < 1$, $W_d(z) = w_d(|z|^2)$.

Lemma 3. *Let $d \in \mathbb{N}$ and $t \in [0,1)$. Then*

$$\int_{1-\varepsilon}^1 w_d(t)dt \cdot \int_{1-\varepsilon}^1 \frac{1}{w_d(t)}dt \approx \varepsilon^2 \quad \textit{as } \varepsilon \to 0.$$

Proof. Write $\tilde{w}(t) = (\log \frac{1}{1-t})^d$, and note that it suffices to establish the lemma for $\tilde{w}$ in place of w. Let $\varepsilon > 0$. Then $\tilde{w}$ is increasing in $[0,1)$ and $\tilde{w}(1-\varepsilon^{k+1}) = (k+1)^d(\log \frac{1}{\varepsilon})^d$, hence

$$\begin{aligned}
\int_{1-\varepsilon}^{1} \tilde{w}(t)dt &= \sum_{k=1}^{\infty} \int_{1-\varepsilon^k}^{1-\varepsilon^{k+1}} \tilde{w}(t)dt \\
&\le \sum_{k=1}^{\infty} \tilde{w}(1-\varepsilon^{k+1})(\varepsilon^k - \varepsilon^{k+1}) \\
&= \sum_{k=1}^{\infty} (k+1)^d (\log \frac{1}{\varepsilon})^d \varepsilon^k (1-\varepsilon) \\
&\approx \varepsilon (\log \frac{1}{\varepsilon})^d \frac{1}{(1-\varepsilon)^d}
\end{aligned}$$

For $1/\tilde{w}$ we just notice that it is decreasing and hence

$$\int_{1-\varepsilon}^{1} \frac{1}{\tilde{w}(t)} dt \le \frac{1}{\tilde{w}(1-\varepsilon)} \varepsilon = \frac{\varepsilon}{(\log \frac{1}{\varepsilon})^d}$$

Thus as $\varepsilon \to 0$ we have

$$\varepsilon^2 \le \int_{1-\varepsilon}^{1} \tilde{w}(t)dt \int_{1-\varepsilon}^{1} \frac{1}{\tilde{w}(t)} dt = O(\varepsilon^2). \qquad \square$$

For $0 < h < 1$ and $s \in [-\pi, \pi)$, let $S_h(e^{is})$ be the Carleson square at e^{is}, i.e.

$$S_h(e^{is}) = \{re^{it} : 1-h < r < 1, |t-s| < h\}.$$

A positive function W on the unit disc is said to satisfy the Békollé-Bonami condition (B2) if

$$\int_{S_h(e^{is})} W dA \cdot \int_{S_h(e^{is})} \frac{1}{W} dA \le ch^4$$

for every Carleson square $S_h(e^{is})$. If $d \in \mathbb{N}$ and if $W(z) = W_d(z) = w_d(|z|^2)$, then

$$\int_{S_h(e^{is})} W_d dA \cdot \int_{S_h(e^{is})} \frac{1}{W_d} dA = h^2 \int_{1-h}^{1} w_d(t)dt \cdot \int_{1-h}^{1} \frac{1}{w_d(t)} dt \approx h^4$$

by Lemma 3. Thus W_d satisfies the condition (B2). Furthermore, note that if $f(z) = \sum_{k=0}^{\infty} \hat{f}(k) z^k$ is analytic in the open unit disc, then

$$\int_{|z|<1} |f(z)|^2 w_d(|z|^2) \frac{dA(z)}{\pi} = \sum_{k=0}^{\infty} w_k |\hat{f}(k)|^2,$$

where $w_k = \int_0^1 t^k w_d(t)dt \approx \frac{\log^d(k+2)}{k+1}$.

A special case of Theorem 2.1 of Luecking's paper [7] says that if W satisfies the condition (B2) by Bekollé and Bonami [5], then one has a duality between the spaces $L^2_a(WdA)$ and

$L^2_a(\frac{1}{W}dA)$ with respect to the pairing given by $\int_{|z|<1} f\overline{g}dA$. Thus, we have

$$\int_{|z|<1} |g(z)|^2 \frac{1}{W_d(z)} dA \approx \sup_{f\neq 0} \frac{\left|\int_{|z|<1} g(z)\overline{f(z)}\frac{dA(z)}{\pi}\right|^2}{\int_{|z|<1}|f(z)|^2 W_d(z)dA}$$
$$= \sup_{f\neq 0} \frac{\left|\sum_{k=0}^\infty \frac{\hat{g}(k)}{(k+1)\sqrt{w_k}}\sqrt{w_k}\overline{\hat{f}(k)}\right|^2}{\sum_{k=0}^\infty w_k|\hat{f}(k)|^2}$$
$$= \sum_{k=0}^\infty \frac{1}{(k+1)^2 w_k}|\hat{g}(k)|^2$$

This finishes the proof of (5). It remains to demonstrate (6). We defer its proof to the next section.

By Theorem 1 we have the following chain of inclusions:

$$\ldots \hookrightarrow HS_{d+1} \hookrightarrow HS_d \hookrightarrow \ldots \hookrightarrow HS_2 \hookrightarrow HS_1 = \mathcal{D} = \mathcal{D}_1 \hookrightarrow \mathcal{D}_2 \hookrightarrow \ldots \hookrightarrow \mathcal{D}_d \hookrightarrow \mathcal{D}_{d+1} \hookrightarrow \ldots$$

with duality w.r.t. $\mathcal{D}$ linking spaces with the same index. It might be interesting to compare this sequence with the sequence of Banach spaces related to the Dirichlet spaces studied in [2]. Note that for $d \geq 3$ the reproducing kernel of HS_d is continuous up to the boundary. Hence functions in HS_d extend continuously to the closure of the unit disc, for $d \geq 3$.

Hilbert-Schmidt norms of Hankel-type operators

Let $\{e_n\}$ be the canonical orthonormal basis of $\mathcal{D}$, $e_n(z) = \frac{z^n}{\sqrt{n+1}}$. Equation (6) follows from the computation

$$\sum_{k=0}^\infty \sum_{|n|=k} |\langle e_{n_1}...e_{n_d}, \psi\rangle|^2 = \sum_{k=0}^\infty \sum_{|n|=k} \frac{1}{(n_1+1)\cdot ...\cdot (n_d+1)}|\langle z^{n_1}...z^{n_d}, \psi\rangle|^2$$
$$= \sum_{k=0}^\infty \sum_{|n|=k} \frac{1}{(n_1+1)\cdot ...\cdot (n_d+1)}|\langle z^k, \psi\rangle|^2 = \sum_{k=0}^\infty \sum_{|n|=k} \frac{(k+1)^2}{(n_1+1)\cdot ...\cdot (n_d+1)}|\hat{\psi}(k)|^2$$
$$= \sum_{k=0}^\infty (k+1)a_d(k)|\hat{\psi}(k)|^2 \approx \sum_{k=0}^\infty \frac{\log^{d-1}(k+2)}{k+1}|\hat{\psi}(k)|^2.$$

Polarizing this expression for $\|\cdot\|_{HS_d}$, the inner product of HS_d can be written

$$\langle \psi_1, \psi_2\rangle_{HS_d} = \sum_{(n_1,\ldots,n_d)} \langle \psi_1, e_{n_1}\ldots e_{n_d}\rangle_{\mathcal{D}} \langle e_{n_1}\ldots e_{n_d}, \psi_2\rangle_{\mathcal{D}}.$$

Hence, for any $\lambda, \zeta \in \mathbb{D}$,

$$\langle k_\lambda, k_\zeta\rangle_{HS_d} = \sum_{n\in\mathbb{N}^d} \langle k_\lambda, e_{n_1}\ldots e_{n_d}\rangle_{\mathcal{D}} \langle e_{n_1}\ldots e_{n_d}, k_\zeta\rangle_{\mathcal{D}} = \sum_{n\in\mathbb{N}^d} \overline{e_{n_1}(\lambda)\ldots e_{n_d}(\lambda)} e_{n_1}(\zeta)\ldots e_{n_d}(\zeta)$$
$$= \left(\sum_{i=0}^\infty \overline{e_i(\lambda)}e_i(\zeta)\right)^d = k_\lambda(\zeta)^d = \langle k^d_\lambda, k^d_\zeta\rangle_{\mathcal{D}_d}.$$

That is,

Proposition 1. *The map $U : k_\lambda \mapsto k_\lambda^d$ extends to a unitary map $HS_d \to \mathcal{D}_d$.*

When $d = 2$, HS_2 contains those functions b for which the Hankel operator $H_b : \mathcal{D} \to \overline{\mathcal{D}}$, defined by $\langle H_b e_j, \overline{e_k}\rangle_{\overline{\mathcal{D}}} = \langle e_j e_k, b\rangle_{\mathcal{D}}$, belongs to the Hilbert-Schmidt class.

Analogous interpretations can be given for $d \geq 3$, but then function spaces on polydiscs are involved. We consider the case $d = 3$, which is representative. Consider first the operator $T_b : \mathcal{D} \to \overline{\mathcal{D}} \otimes \overline{\mathcal{D}}$ defined by

$$\left\langle T_b f, \overline{g} \otimes \overline{h} \right\rangle_{\overline{\mathcal{D}} \otimes \overline{\mathcal{D}}} = \langle fgh, b\rangle_{\mathcal{D}}.$$

The formula uniquely defines an operator, whose action is

$$\begin{aligned} T_b f(z, w) &= \langle T_b f, \overline{k}_z \overline{k}_w\rangle_{\overline{\mathcal{D}} \otimes \overline{\mathcal{D}}} \\ &= \langle f k_z k_w, b\rangle_{\mathcal{D}} \\ &= \sum_{n,m,j} \hat{f}(j) \frac{\overline{z}^n}{n+1} \frac{\overline{w}^m}{m+1} \langle \zeta^{n+m+j}, b\rangle_{\mathcal{D}} \\ &= \sum_{n,m,j} \hat{f}(j) \overline{\hat{b}(n+m+j)} \frac{n+m+j+1}{(n+1)(m+1)} \overline{z}^n \overline{w}^m \end{aligned}$$

Then, the Hilbert-Schmidt norm of T_b is:

$$\sum_{l,m,n} \left| \langle T_b e_l, e_m e_n\rangle_{\overline{\mathcal{D}} \otimes \overline{\mathcal{D}}} \right|^2 = \sum_{l,m,n} |\langle e_l e_m e_n, b\rangle_{\mathcal{D}}|^2 = \|b\|_{HS_3}^2.$$

Similarly, we can consider $U_b : \mathcal{D} \otimes \mathcal{D} \to \overline{\mathcal{D}}$ defined by

$$\left\langle U_b(f \otimes g), \overline{h} \right\rangle_{\overline{\mathcal{D}}} = \langle fgh, b\rangle_{\mathcal{D}}.$$

The action of this operator is given by

$$U_b(f \otimes g)(\overline{z}) = \sum_{l,m,n=0}^{\infty} \widehat{f}(l) \widehat{g}(m) \frac{(l+m+n+1)\overline{\widehat{b}(l+m+n)}}{n+1} \overline{z}^n.$$

The Hilbert-Schmidt norm of U_b is still $\|b\|_{HS_3}$.

Carleson measures for the spaces $\mathcal{D}_d$ and HS_d

The (B2) condition allows us to characterize the Carleson measures for the spaces $\mathcal{D}_d$ and HS_d. Recall that a nonnegative Borel measure μ on the open unit disc is Carleson for the Hilbert function space H if the inequality

$$\int_{|z|<1} |f|^2 d\mu \leq C(\mu) \|f\|_H^2$$

holds with a constant $C(\mu)$ which is independent of f. The characterization [3] shows that, since the (B2) condition holds, then

Theorem 2. *For $d \in \mathbb{N}$, a measure $\mu \geq 0$ on $\{|z| < 1\}$ is Carleson for $\mathcal{D}_d$ if and only if for $|a| < 1$ we have:*

$$\int_{\tilde{S}(a)} \log^{d-1}\left(\frac{1}{1-|z|^2}\right) (1-|z|^2) \mu(S(z) \cap S(a))^2 \frac{dxdy}{(1-|z|^2)^2} \leq C_1(\mu)\mu(S(a)),$$

where $S(a) = \{z : 0 < 1 - |z| < 1 - |a|,\ |\arg(z\overline{a})| < 1 - |a|\}$ *is the Carleson box with vertex* a *and* $\tilde{S}(a) = \{z : 0 < 1 - |z| < 2(1 - |a|),\ |\arg(z\overline{a})| < 2(1 - |a|)\}$ *is its "dilation".*

The characterization extends to HS_2, with the weight $\log^{-1}\left(\frac{1}{1-|z|^2}\right)$. Since functions in HS_d are continuous for $d \geq 3$, all finite measures are Carleson measures for these spaces. Once we know the Carleson measures, we can characterize the multipliers for $\mathcal{D}_d$ in a standard way.

The complete Nevanlinna-Pick property for $\mathcal{D}_d$

Next, we prove that the spaces $\mathcal{D}_d$ have the Complete Nevanlinna-Pick (CNP) Property. Much research has been done on CNP kernels in the past twenty years, following seminal work of Sarason and Agler. See the monograph [1] for a comprehensive and very readable introduction to this topic. We give here a definition which is simple to state but perhaps not the most conceptual. An irreducible kernel $k : X \times X \to \mathbb{C}$ has the CNP if there is a positive definite function $F : X \to \mathbb{D}$ and a nowhere vanishing function $\delta : X \to \mathbb{C}$ such that:

$$k(x, y) = \frac{\overline{\delta}(x)\delta(y)}{1 - F(x, y)}$$

whenever x, y lie in X. CNP is a property of the kernel, not of the Hilbert space itself.

Theorem 3. *There are norms on* $\mathcal{D}_d$ *such that the CNP property holds.*

Proof. A kernel $k : \mathbb{D} \times \mathbb{D} \to \mathbb{C}$ of the form $k(w, z) = \sum_{k=0}^{\infty} a_k(\overline{z}w)^k$ has the CNP property if $a_0 = 1$ and the sequence $\{a_n\}_{n=0}^{\infty}$ is positive and log-convex:

$$\frac{a_{n-1}}{a_n} \leq \frac{a_n}{a_{n+1}}.$$

See [1], Theorem 7.33 and Lemma 7.38. Consider $\eta(x) = \alpha \log\log(x) - \log(x)$, with real α. Then, $\eta''(x) = \frac{\log^2(x) - \alpha\log(x) - \alpha}{x^2\log^2(x)}$, which is positive for $x \geq M_\alpha$, depending on α. Let now

$$a_n = \frac{\log^{d-1}(M_d(n+1))}{\log(M_d) \cdot (n+1)} \approx \frac{1}{n+1} + \frac{\log^{d-1}(n+1)}{n+1} \tag{7}$$

Then, the sequence $\{a_n\}_{n=0}^{\infty}$ provides the coefficients for a CNP kernel for the space $\mathcal{D}_d$. □

The CNP property has a number of consequences. For instance, we have that the space $\mathcal{D}_d$ and its multiplier algebra $M(\mathcal{D}_d)$ have the same interpolating sequences. Recall that a sequence $Z = \{z_n\}_{n=0}^{\infty}$ is *interpolating* for a RKHS H with reproducing kernel k^H if the weighted restriction map $R : \varphi \mapsto \left\{\frac{\varphi(z_n)}{k^H(z_n,z_n)^{1/2}}\right\}_{n=0}^{\infty}$ maps H boundedly onto ℓ^2; while Z is interpolating for the multiplier algebra $M(H)$ if $Q : \psi \mapsto \{\psi(z_n)\}_{n=0}^{\infty}$ maps $M(H)$ boundedly onto ℓ^∞. The reader is referred to [1] and to the second chapter of [8] for more on this topic.

It is a reasonable guess that the *universal interpolating sequences* for $\mathcal{D}_d$ and for its multiplier space $M(\mathcal{D}_d)$ are characterized by a Carleson condition and a separation condition, as described in [8] (see the Conjecture at p. 33). See also [6], which contains the best known result on interpolation in general CNP spaces. Unfortunately we do not have an answer for the spaces $\mathcal{D}_d$.

References

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