HANKEL OPERATORS AND INVARIANT SUBSPACES OF THE DIRICHLET SPACE

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ABSTRACT. The Dirichlet space D is the space of all analytic functions f on the open unit disc $\mathbb{D}$ such that f' is square integrable with respect to two-dimensional Lebesgue measure. In this paper we prove that the invariant subspaces of the Dirichlet shift are in 1-1 correspondence with the kernels of the Dirichlet-Hankel operators. We then apply this result to obtain information about the invariant subspace lattice of the weak product $D \odot D$ and to some questions about approximation of invariant subspaces of D.

Our main results hold in the context of superharmonically weighted Dirichlet spaces.

1. INTRODUCTION

If $f \in \mathrm{Hol}(\mathbb{D})$ denotes an analytic function on the open unit disc $\mathbb{D}$, then we use $\hat{f}(n)$ for its nth Taylor coefficient. We write H^2 for the Hardy space of the unit disc, it has norm given by $\|f\|_{H^2}^2 = \int_{|z|=1} |f(z)|^2 \frac{|dz|}{2\pi} = \sum_{n=0}^{\infty} |\hat{f}(n)|^2$.

In this paper we will consider weighted Dirichlet spaces of the form

$$\mathcal{H} = \{f \in \mathrm{Hol}(\mathbb{D}) : \int_{\mathbb{D}} |f'(z)|^2 U(z) dA(z) < \infty\},$$

where U is a non-negative superharmonic function on $\mathbb{D}$ and dA denotes 2-dimensional Lebesgue measure. Particular examples of such weights are $U(z) = (1-|z|^2)^{1-\alpha}$ for $0 < \alpha \leq 1$. By the representation theorem for superharmonic functions (see [12], page 109) such weights can be represented by use of a finite Borel measure μ on the closed unit disc. We write

$$U_\mu(z) = \int_{|w|<1} \log \left| \frac{1-\overline{w}z}{z-w} \right| \frac{d\mu(w)}{1-|w|^2} + \int_{|w|=1} \frac{1-|z|^2}{|1-\overline{w}z|^2} d\mu(w), \quad z \in \mathbb{D},$$

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then the correspondence $\mu \to U_\mu$ is a bijection of the collection of finite positive measures on $\overline{\mathbb{D}}$ onto the set of positive superharmonic functions on $\mathbb{D}$, and it is a theorem of Aleman, [1], Theorem IV.1.9 that for all $f \in H^2$ we have

$$\int_{\mathbb{D}} |f'(z)|^2 U_\mu(z) dA(z) = \int_{\overline{\mathbb{D}}} D_w(f) d\mu(w), \tag{1.1}$$

where $D_w(f) = \int_{|z|=1} \frac{|f(z)-f(w)|^2}{|z-w|^2} \frac{|dz|}{2\pi}$ is called the local Dirichlet integral of f at $w \in \overline{\mathbb{D}}$. Here we follow the convention that $D_w(f) = \infty$, if f does not have a nontangential limit at $w \in \partial\mathbb{D}$.

We define

$$D(\mu) = \{f \in H^2(\mathbb{D}) : \int_{\overline{\mathbb{D}}} D_w(f) d\mu(w) < \infty\}$$

with norm $\|f\|^2 = \|f\|^2_{H^2} + \int_{\overline{\mathbb{D}}} D_w(f) d\mu(w)$. In the special case where $\mu = m$ is normalized linear Lebesgue measure on the unit circle we obtain $U_m \equiv 1$ and $D(m) = D$, the classical Dirichlet space. In this case we have

$$\|f\|^2 = \int_{\partial\mathbb{D}} |f|^2 dm + \int_{\mathbb{D}} |f'|^2 \frac{dA}{\pi} = \sum_{n=0}^{\infty} (n+1)|\hat{f}(n)|^2,$$

Note also that $D(\mu) = H^2$ when $\mu = 0$. For measures μ that are supported in $\partial\mathbb{D}$ these spaces arose in [16]. The general case was considered in [1], [22], and [19], where all the basic results about $D(\mu)$ can be found.

For any space of analytic functions $\mathcal{B}$ on the open unit disc $\mathbb{D}$ we denote by $(M_z, \mathcal{B})$ the linear transformation defined by $(M_z f)(z) = zf(z)$ and we use $\mathrm{Lat}(M_z, \mathcal{B})$ to denote the collection of invariant subspaces of $(M_z, \mathcal{B})$. We write $\mathcal{M}(\mathcal{B})$ for the set of multipliers of $\mathcal{B}$,

$$\mathcal{M}(\mathcal{B}) = \{\varphi \in \mathrm{Hol}(\mathbb{D}) : \varphi f \in \mathcal{B} \text{ for all } f \in \mathcal{B}\}$$

and for $\varphi \in \mathcal{M}(\mathcal{B})$ we use $M_\varphi \in B(\mathcal{B})$ for the corresponding multiplication operator, $f \to \varphi f$. Furthermore for $\varphi \in \mathcal{M}(\mathcal{B})$ we write $\|\varphi\|_{\mathcal{M}} = \|M_\varphi\|$ for the multiplier norm.

Let $\mathcal{H} = H^2, D(\mu)$, or $L^2_a = L^2(dA) \cap \mathrm{Hol}(\mathbb{D})$, the Bergman space. By Beurling's theorem we have a precise knowledge of $\mathrm{Lat}(M_z, H^2)$, the Beurling lattice. Indeed, if $(0) \neq \mathcal{M} \in \mathrm{Lat}(M_z, H^2)$, then $\mathcal{M} = \varphi H^2$ for some inner function φ, i.e. φ is in the unit ball of H^∞ and satisfies $|\varphi(e^{it})| = 1$ a.e., [5]. Less is known about the invariant subspaces of the Dirichlet and Bergman shifts, but it is well-established that the Bergman lattice differs in structure substantially from the Beurling lattice, [2], while there are a number of similarities between $\mathrm{Lat}(M_z, H^2)$ and $\mathrm{Lat}(M_z, D(\mu))$, [15, 18, 1]. We refer the reader to [9] for a nice overview of what is currently known about the Dirichlet lattice $\mathrm{Lat}(M_z, D)$. The analogues of inner functions play an important role in the invariant subspace theory of both the Bergman and Dirichlet spaces. For $(0) \neq \mathcal{M} \in \mathrm{Lat}(M_z, \mathcal{H})$ we let $n = \inf\{k : \exists f \in \mathcal{M} \text{ with } f^{(k)}(0) \neq 0\}$. Then the extremal problem $\sup\{\mathrm{Re} f^{(n)}(0) : f \in \mathcal{M}, \|f\| \leq 1\}$ has a unique solution, which will be called the extremal function for $\mathcal{M}$. In the case $\mathcal{H} = H^2$ the extremal function for $\mathcal{M}$ is the inner function satisfying $\mathcal{M} = \varphi H^2$. It is easy to check that up to a multiplicative constant of modulus one extremal functions φ in $\mathcal{H}$ are characterized by $\langle z^n \varphi, \varphi\rangle = \delta_{0n}$, where $\delta_{0n} = 1$ for $n = 0$ and $= 0$ for $n > 0$.

The main result that we want to bring to the attention of the reader is a new analogy between the Beurling and Dirichlet lattices that is based on the theory of Hankel operators $\mathcal{H} \to \overline{\mathcal{H}}$. Here we have written $\overline{\mathcal{H}} = \{\overline{f} : f \in \mathcal{H}\}$ for the space of complex conjugates of $\mathcal{H}$. This is a Hilbert space with inner product $\langle \overline{f}, \overline{g}\rangle_{\overline{\mathcal{H}}} = \langle g, f\rangle_{\mathcal{H}}$, $f, g \in \mathcal{H}$.

As in [4] or [19] we define

$$\mathcal{X}(\mathcal{H}) = \{b \in \mathcal{H} : \ \exists\ C > 0\ |\langle \varphi\psi, b\rangle| \leq C\|\varphi\|\|\psi\|, \forall \varphi, \psi \in \mathrm{Hol}(\overline{\mathbb{D}})\}.$$

Note that for every $b \in \mathcal{X}(\mathcal{H})$ the map $(\varphi, \overline{\psi}) \to \langle \varphi\psi, b\rangle$ extends to be a bounded sesquilinear form on $\mathcal{H} \times \overline{\mathcal{H}}$. Thus with each $b \in \mathcal{X}(\mathcal{H})$ we may associate the Hankel operator $H_b \in B(\mathcal{H}, \overline{\mathcal{H}})$,

$$\langle H_b \varphi, \overline{\psi}\rangle_{\overline{\mathcal{H}}} = \langle \varphi\psi, b\rangle_{\mathcal{H}}, \quad \varphi, \psi \in \mathrm{Hol}(\overline{\mathbb{D}}).$$

If $\mathcal{H} = H^2$ our definition of Hankel operator differs by a rank 1 operator from the common definition as operator $H^2 \to H^{2\perp} \subseteq L^2(\partial\mathbb{D})$. For $\mathcal{H} = L^2_a$ our definition coincides with what is typically referred to as "little Hankel operator".

Carleson measures can be used to describe $\mathcal{X}(\mathcal{H})$ in the cases where $\mathcal{H} = H^2, D$, or L^2_a. Recall that a positive measure μ on $\mathbb{D}$ is called a Carleson measure for $\mathcal{H}$, if there is a $C > 0$ such that $\int_{\mathbb{D}} |p|^2 d\mu \leq C\|p\|^2$ for all polynomials p. Then it is well-known that $\mathcal{X}(H^2) = BMOA = \{b \in H^2 : |b'|^2(1-|z|^2)dA \text{ is a Carleson measure for } H^2\}$, [8]. Similarly, $\mathcal{X}(L^2_a)$ is the Bloch space, and also

$$\mathcal{X}(L^2_a) = \{b \in D : |b'|^2(1-|z|^2)^2 dA \text{ is a Carleson measure for } L^2_a\},$$

see e.g. [24]. Furthermore, in [4] it was shown that

$$\mathcal{X}(D) = \{b \in D : |b'|^2 dA \text{ is a Carleson measure for } D\}.$$

One checks that $\langle H_b(zf), \overline{\psi}\rangle_{\overline{\mathcal{H}}} = \langle H_b f, \overline{z\psi}\rangle_{\overline{\mathcal{H}}}$ for all $f \in \mathcal{H}$ and $\psi \in \mathrm{Hol}(\overline{\mathbb{D}})$. This implies that for each $b \in \mathcal{X}(\mathcal{H})$ we have $\ker H_b \in \mathrm{Lat}(M_z, \mathcal{H})$.

Theorem 1.1. *Let μ be a nonnegative finite Borel measure supported in $\overline{\mathbb{D}}$ and let $\mathcal{M} \in Lat(M_z, D(\mu))$. Then there is a $b \in \mathcal{X}(D(\mu))$ such that $\mathcal{M} = \ker H_b$.*

If $\mathcal{M} \neq (0)$, if φ is the extremal function for $\mathcal{M}$, then $b = M_z^* \varphi \in \mathcal{X}(D(\mu))$ and $\mathcal{M} = \ker H_b$.

For $\mu = 0$ one obtains H^2. Of course, in this case the result is well-known. For the Bergman space the direct analogue of Theorem 1.1 is false, see [23].

We will prove Theorem 1.1 in Section 2. In the later sections of the paper we will apply the theorem to obtain further results about these spaces. If $\mathcal{H} = H^2$ is the Hardy space, or $\mathcal{H} = L^2_a$ is the Bergman space, then it is natural to view $\mathcal{H}$ as part of the family of H^p- or L^p_a-spaces, and investigate how properties of functions and operators on these spaces change as the parameter p changes. However, if $\mathcal{H} = D(\mu)$, or any abstract reproducing kernel Hilbert space, then it is unclear what the most natural class of related spaces should be. In Section 3 of this paper we investigate the weak product $D(\mu) \odot D(\mu)$, which we consider to be a natural analogue of the spaces H^1 and L^1_a in the Hardy and Bergman theories.

The weak product of $\mathcal{H}$ is denoted by $\mathcal{H} \odot \mathcal{H}$ and it is defined to be the collection of all functions $h \in \mathrm{Hol}(\mathbb{D})$ such that there are sequences $\{f_i\}_{i\geq 1}, \{g_i\}_{i\geq 1} \subseteq \mathcal{H}$ with $\sum_{i=1}^\infty \|f_i\|\|g_i\| < \infty$ and $h(z) = \sum_{i=1}^\infty f_i(z) g_i(z)$ for all $z \in \mathbb{D}$. Note that whenever $\sum_{i=1}^\infty \|f_i\|\|g_i\| < \infty$, then

$$\sum_{i=1}^\infty |f_i(z)||g_i(z)| \leq \|k_z\|^2 \sum_{i=1}^\infty \|f_i\|\|g_i\| < \infty,$$

thus the series will converge locally uniformly to the analytic function h.

A norm on $\mathcal{H} \odot \mathcal{H}$ is defined by

$$\|h\|_* = \inf\{\sum_{i=1}^\infty \|f_i\|\|g_i\| : h(z) = \sum_{i=1}^\infty f_i(z) g_i(z) \text{ for all } z \in \mathbb{D}\}.$$

Weak products first appeared in a paper by Coifman, Rochberg and Weiss [7]. It turns out that for spaces $\mathcal{H}$ considered in this paper one can identify the dual of $\mathcal{H} \odot \mathcal{H}$ with $\mathcal{X}(\mathcal{H})$, see [19]. The paper [4] (p. 22-24) contains an excellent motivation for the study of weak products and a summary of results about such spaces, also see [3] and [19]. Here we just mention that if $\mathcal{H}$ is the Hardy space H^2 of the unit disc $\mathbb{D}$, then it follows from the Riesz factorization that $H^2 \odot H^2 = H^1$ with equality of norms, and in fact every $h \in H^2 \odot H^2$ can be written as a single product of functions in H^2. Similarly, it turns out that $L^2_a \odot L^2_a = L^1_a$, [7]. Furthermore, the papers [3], [19], and [13] contain a number of results about $D \odot D$.

It follows from Beurling's theorem and the H^p-theory that every nonzero invariant subspace $\mathcal{M}$ of H^p, $0 < p < \infty$, is of the form $\mathcal{M} = \varphi H^p$ for some inner function φ, [8]. Based on the analogy of $D \odot D$ with H^1 one might expect the structure of $\mathrm{Lat}(M_z, D \odot D)$ to resemble the one of $\mathrm{Lat}(M_z, D)$. Not much is known about this. In [13] the first named author showed that every nonzero $\mathcal{M} \in \mathrm{Lat}(M_z, D \odot D)$ has index 1, i.e. satisfies $\dim \mathcal{M}/z\mathcal{M} = 1$. In this paper we will establish further results along these lines.

We will write $\mathrm{clos}_{\mathcal{B}} K$ for the closure of the set K in the topology of $\mathcal{B}$. Note that two easy corollaries to Beurling's theorem are that

(1.2) for every $\mathcal{M} \in \mathrm{Lat}(M_z, H^2)$ we have $\mathcal{M} = H^2 \cap \mathrm{clos}_{H^1} \mathcal{M}$ and

(1.3) for every $\mathcal{N} \in \mathrm{Lat}(M_z, H^1)$ we have $\mathcal{N} = \mathrm{clos}_{H^1}(\mathcal{N} \cap H^2)$.

We will establish the Dirichlet analogue of (1.2).

Theorem 1.2. *Let μ be a nonnegative finite Borel measure supported in $\overline{\mathbb{D}}$ and let $\mathcal{M} \in Lat(M_z, D(\mu))$, then $\mathcal{M} = D(\mu) \cap \mathrm{clos}_{D(\mu)\odot D(\mu)} \mathcal{M}$.*

This will be Theorem 3.2. Recall that f is called cyclic in $\mathcal{B}$, if the polynomial multiples of f are dense in $\mathcal{B}$. It will follow from Theorem 1.2 that a function $f \in D(\mu)$ is cyclic in $D(\mu)$, if and only if f is cyclic in $D(\mu) \odot D(\mu)$, see Corollary 3.3. Unfortunately we do not know whether the Dirichlet analogue of (1.3) holds.

In Section 4 we will prove the following Theorem as a corollary to Theorem 1.1.

Theorem 1.3. *Let μ be a nonnegative finite Borel measure supported in $\overline{\mathbb{D}}$ and for $j = 1, 2, ...$ let $(0) \neq \mathcal{M}_j, \mathcal{M} \in Lat(M_z, D(\mu))$, let φ_j be the extremal function for $\mathcal{M}_j$, let φ be the extremal function for $\mathcal{M}$, and let P_j, P be the orhogonal projections onto $\mathcal{M}_j, \mathcal{M}$.*

If $\varphi_j(z) \to \varphi(z)$ locally uniformly on $\mathbb{D}$, then $P_j \to P$ in the strong operator topology.

If $\mathcal{H} = H^2$ and if $\mathcal{M} = \varphi H^2$ for some inner function φ, then $P_{\mathcal{M}} = M_\varphi M_\varphi^*$, and it is easily seen (and well-known) that Theorem 1.3 holds for H^2. Furthermore, we note that a Bergman space version of this theorem is also true and that was proved by Shimorin, [21], Theorem 1A.

For a finite or infinite sequence of points $Z = \{z_1, z_2, ...\}$ in $\mathbb{D}$ we write $I(Z)$ for all funtions in $\mathcal{H}$ that are zero at each z_i counting multiplicities. Clearly $I(Z) \in \text{Lat}(M_z, \mathcal{H})$ and we say $I(Z)$ is generated by the sequence Z.

The Caratheodory-Schur Theorem states that every function in the unit ball of H^∞ can be approximated locally uniformly in $\mathbb{D}$ by a sequence of finite Blaschke products, see e.g. [10], Theorem 2.1, or [20], Theorem 5.5.1 for a different proof. In particular, every inner function can be approximated by inner functions that correspond to invariant subspaces that are generated by finite zero sets. Shimorin proved an analogous theorem for the Bergman space, see [21], Theorem 1B. These theorems then can be applied together with the corresponding version of Theorem 1.3 to show that the projections onto all singly generated invariant subspaces can be approximated by projections onto invariant subspaces corresponding to finite zero sets.

Thus we wonder whether all Dirichlet extremal functions can be approximated by Dirichlet extremal functions corresponding to finite zero sets and which functions in D can be approximated by Dirichlet extremal functions. We have two partial results.

Theorem 1.4. *Let μ be a nonnegative finite Borel measure supported in $\overline{\mathbb{D}}$ and let S be an inner function such that $\mathcal{M} = SH^2 \cap D(\mu) \neq (0)$ and let φ be the $D(\mu)$-extremal function for $\mathcal{M}$.*

Then there is a sequence of $D(\mu)$-extremal functions φ_n corresponding to finite zero sets such that $\varphi_n \to \varphi$ locally uniformly in $\mathbb{D}$.

For a function $f \in \mathcal{H}$ we define a harmonic function u_f on $\mathbb{D}$ by

$$u_f(\lambda) = \text{ Re}\langle \frac{1 + \overline{\lambda} z}{1 - \overline{\lambda} z} f, f\rangle.$$

If $\mathcal{H} = H^2$, then $u_f = P[|f|^2]$, the Poisson integral of $|f|^2$. Thus, in this case $u_f \leq 1$ in $\mathbb{D}$ if and only if f is in the unit ball of H^∞. For $\mathcal{H} = L^2_a$ one obtains $u_f(\lambda) = \int_{\mathbb{D}} \frac{1-|\lambda z|^2}{|1-\bar{\lambda} z|^2} |f(z)|^2 \frac{dA}{\pi}$, and notes that the unit ball of H^∞ is properly contained in $\{f \in L^2_a : u_f \leq 1 \text{ in } \mathbb{D}\}$. Shimorin calls functions $f \in L^2_a$ with $u_f \leq 1$ in $\mathbb{D}$ subextremal functions, and he proves in [21] that the subextremal functions form the set of functions that can be approximated by Bergman extremal functions for invariant subspaces corresponding to finite zero sets. Thus in particular, for both spaces $\mathcal{H} = H^2$ and $\mathcal{H} = L^2_a$ it is true that a function $f \in \mathcal{H}$ can be approximated locally uniformly in $\mathbb{D}$ by $\mathcal{H}$-extremal functions, if and only if $u_f \leq 1$ in $\mathbb{D}$.

For the $D(\mu)$-spaces it is known that extremal functions are contractive multipliers, and hence it is easy to see that any limit of extremal functions also must be a contractive multiplier. Additionally we will prove the following.

Theorem 1.5. *Let μ be a nonnegative finite Borel measure supported in $\overline{\mathbb{D}}$. If φ_n is a sequence of $D(\mu)$-extremal functions and if $f \in D(\mu)$ such that $\varphi_n \to f$ locally uniformly in $\mathbb{D}$, then $u_f \leq 1$ in $\mathbb{D}$.*

For the classical Dirichlet space D we will show that the collection $\{f \in D : u_f \leq 1 \text{ in } \mathbb{D}\}$ is a proper subset of the unit ball of the multiplier algebra $\mathcal{M}(D)$. However, even in this restricted setting we do not know whether every function $f \in D$ with $u_f \leq 1$ in $\mathbb{D}$ can be approximated by Dirichlet extremal functions.

2. Invariant subspaces of $D(\mu)$

We start with two simple lemmas that hold for all reproducing kernel Hilbert spaces $\mathcal{H} \subseteq \mathrm{Hol}(\mathbb{D})$ such that

(2.1) $\mathrm{Hol}(\overline{\mathbb{D}}) \subseteq \mathcal{M}(\mathcal{H}) \subseteq \mathcal{H}$ and
(2.2) $\mathrm{Hol}(\overline{\mathbb{D}})$ is dense in $\mathcal{H}$.

Lemma 2.1. *Suppose $\mathcal{H}$ satisfies conditions (2.1) and (2.2).*
If $b \in \mathcal{X}(\mathcal{H})$ and $f \in \mathcal{H}$, then for every $\varphi \in \mathcal{M}(\mathcal{H})$ we have

$$\langle H_b f, \overline{\varphi}\rangle_{\overline{\mathcal{H}}} = \langle \varphi f, b\rangle_{\mathcal{H}} = \langle f, M_\varphi^* b\rangle_{\mathcal{H}}. \tag{2.3}$$

Proof. It is clear that (2.3) holds for $f, \varphi \in \mathrm{Hol}(\overline{\mathbb{D}})$. By a simple approximation we obtain $\langle H_b\psi, \overline{\varphi}\rangle_{\overline{\mathcal{H}}} = \langle \varphi\psi, b\rangle_{\mathcal{H}}$ for all $\psi \in \mathrm{Hol}(\overline{\mathbb{D}})$ and $\varphi \in \mathcal{M}(\mathcal{H})$.

Let $\varphi \in \mathcal{M}(\mathcal{H}), f \in \mathcal{H}$, and $\psi_n \in \mathrm{Hol}(\overline{\mathbb{D}})$ with $\psi_n \to f$ in $\mathcal{H}$, then

$$\begin{aligned}\langle H_b f, \overline{\varphi}\rangle_{\overline{\mathcal{H}}} &= \lim_{n\to\infty} \langle H_b \psi_n, \overline{\varphi}\rangle_{\overline{\mathcal{H}}} \\ &= \lim_{n\to\infty} \langle \varphi\psi_n, b\rangle \\ &= \langle \varphi f, b\rangle \\ &= \langle f, M_\varphi^* b\rangle.\end{aligned}$$

■

Lemma 2.2. *Assume $\mathcal{H}$ satisfies conditions (2.1) and (2.2). If $b \in \mathcal{X}(\mathcal{H})$ and $\varphi \in \mathcal{M}(\mathcal{H})$, then*

(a) $\ker H_b = [b]_*^\perp$, *where $[b]_*$ denotes the smallest subspace that contains b and is invariant under M_ψ^* for every $\psi \in \mathcal{M}(\mathcal{H})$,*

(b) $H_b^* \overline{\varphi} = M_\varphi^* b$ and

(c) $b_1 = M_\varphi^* b \in \mathcal{X}(\mathcal{H})$ and $H_{b_1} = H_b M_\varphi$.

Proof. (a) and (b) follow immediately from Lemma 2.1. In order to see that (c) holds we apply equation (2.3) with $f = uv$ for $u, v \in \mathrm{Hol}(\overline{\mathbb{D}})$ to obtain

$$|\langle uv, M_\varphi^* b\rangle| = |\langle \varphi uv, b\rangle| \leq \|H_b\| \|\varphi u\| \|v\| \leq \|H_b\| \|M_\varphi\| \|u\| \|v\|.$$

This implies $b_1 = M_\varphi^* b \in \mathcal{X}(\mathcal{H})$ and one now easily verifies $H_{b_1} = H_b M_\varphi$. ■

It is shown in [1] that the polynomials are dense in $D(\mu)$. In particular, it follows easily that $D(\mu)$ satisfies (2.1) and (2.2). If $f \in \mathcal{H}$, then we write $[f] = \mathrm{clos}_\mathcal{H}\{pf : p \text{ is a polynomial}\}$ for the smallest M_z-invariant subspace containing f.

We will need the following result, see Lemma IV.4.8 and Theorem IV.4.9 of [1]. Recall that for any extremal function φ of a nonzero invariant subspace $\mathcal{M}$ we have $\varphi \in \mathcal{M} \ominus z\mathcal{M}$.

Theorem 2.3. *If* $(0) \neq \mathcal{M} \in Lat(M_z, D(\mu))$, *then* $\dim \mathcal{M} \ominus z\mathcal{M} = 1$. *Furthermore, if* $\varphi \in \mathcal{M} \ominus z\mathcal{M}$, $\|\varphi\| = 1$, *then* $\varphi \in \mathcal{M}(D(\mu))$ *with* $\|\varphi f\| \leq \|f\|$ *for all* $f \in D(\mu)$, *and* $\mathcal{M} = \overline{\mathrm{ran}\, M_\varphi} = [\varphi]$.

Lemma 2.4. *For any nonnegative finite measure* μ *with support in* $\overline{\mathbb{D}}$ *we have* $\mathcal{M}(D(\mu)) \subseteq \mathcal{X}(D(\mu))$.

Proof. It is well-known and easy to verify from (1.1) that $b \in \mathcal{M}(D(\mu))$, if and only if $b \in H^\infty$ and $|b'|^2 U_\mu dA$ is a Carleson measure for $D(\mu)$. For $b \in \mathcal{M}(D(\mu))$ and $\varphi, \psi \in \mathrm{Hol}(\overline{\mathbb{D}})$ we have

$$\langle \varphi\psi, b\rangle = \int_{\partial\mathbb{D}} \varphi\psi\bar{b}dm + \int_{\mathbb{D}} (\varphi'\psi + \varphi\psi')\overline{b'} U_\mu dA.$$

The first summand is easily bounded by $\|b\|_\infty \|\varphi\|_{H^2} \|\psi\|_{H^2} \leq \|b\|_\mathcal{M} \|\varphi\| \|\psi\|$, while the second summand can be seen to be bounded after applications of the Cauchy-Schwarz inequality and the Carleson measure property of $|b'|^2 U_\mu dA$. The lemma follows. ■

We are now ready to prove our main theorem.

Theorem 2.5. *Let* $\mathcal{M} \in Lat(M_z, D(\mu))$. *Then there is a* $b \in \mathcal{X}(D(\mu))$ *such that* $\mathcal{M} = \ker H_b$.

If $\mathcal{M} \neq (0)$, *if* φ *is the extremal function for* $\mathcal{M}$, *then* $b = M_z^* \varphi \in \mathcal{X}(D(\mu))$ *and* $\mathcal{M} = \ker H_b$.

Proof. We first consider the case $\mathcal{M} = (0)$. By Lemma 2.2 we need to show the existence of $b \in \mathcal{X}(D(\mu))$ such that $[b]_* = D(\mu)$, i.e. b is cyclic for M_z^*. Cyclic vectors $b \in \mathcal{H}$ for M_z^* exist in abundance for all Hilbert spaces $\mathcal{H}$ satisfying (2.1) and (2.2), see e.g. [6]. We follow the ideas in [6] to construct a M_z^*-cyclic vector in $\mathcal{M}(D(\mu)) \subseteq \mathcal{X}(D(\mu))$. Let $k_\lambda(z)$ denote the reproducing kernel for $D(\mu)$. Lemma 2.2 (a) of [11] implies that for each $\lambda \in \mathbb{D}$ the function $k_\lambda \in \mathcal{M}(D(\mu))$. Indeed, while it is true that $k_\lambda(z)$ is a complete Nevanlinna Pick kernel ([22]), we note that one can also just follow the proof of Lemma 2.2 (a) of [11] and use Theorem 2.3. Now let $\lambda_j \subseteq \mathbb{D}$ be a sequence of distinct points with $\lambda_j \to 0$ and choose $c_j > 0$ such that $\sum_{j=1}^\infty c_j \|k_{\lambda_j}\|_{\mathcal{M}} < \infty$ and $\sum_{j=1}^\infty c_j \|k_{\lambda_j}\|^2 < \infty$. Define a measure $\sigma = \sum_{j=1}^\infty c_j \delta_{\lambda_j}$, and set $b = \sum_{j=1}^\infty c_j k_{\lambda_j}$, then $b \in \mathcal{M}(D(\mu))$.

Also note that for $g \in D(\mu)$ we have $\int |g|^2 d\sigma = \sum_{j=1}^\infty c_j |g(\lambda_j|^2 \leq \|g\|^2 \sum_{j=1}^\infty c_j \|k_{\lambda_j}\|^2 < \infty$, hence $g \in L^2(\sigma)$.

Now suppose that $g \in D(\mu)$ such that $g \perp [b]_*$. Then for every polynomial p we have

$$0 = \langle g, M_p^* b\rangle = \sum_{j=1}^\infty c_j p(\lambda_j) g(\lambda_j) = \int pg \, d\sigma.$$

It is clear that the polynomials are dense in $L^2(\sigma)$, thus $g(\lambda_j) = 0$ for all j. The analyticity of g implies that $g = 0$. Hence b is M_z^*-cyclic.

Next we suppose $\mathcal{M} \neq (0)$. Let φ be the extremal function for $\mathcal{M}$. Then by Theorem 2.3 $\mathcal{M} = [\varphi]$ and $\varphi \in \mathcal{M}(D(\mu)$, hence by Lemmas 2.4 and 2.2 we have $b = M_z^* \varphi \in \mathcal{X}(D(\mu))$.

Let $\mathcal{N} = [b]_*^\perp$. Then by Lemma 2.2 (a) we have $\mathcal{N} = \ker H_b$, and we have to show that $\mathcal{M} = \mathcal{N}$. By Theorem 2.3 it suffices to prove that $\varphi \in \mathcal{N} \ominus z\mathcal{N}$. For $n \geq 0$ we have

$$\langle H_b \varphi, \overline{z}^n \rangle = \langle z^n \varphi, M_z^* \varphi\rangle = \langle z^{n+1}\varphi, \varphi\rangle = 0,$$

thus $\varphi \in \ker H_b = \mathcal{N}$. Furthermore, if $f \in \mathcal{N}$, then

$$\langle \varphi, zf\rangle = \langle b, f\rangle = \langle 1, H_b f\rangle = 0$$

so $\varphi \perp z\mathcal{N}$ and hence $\varphi \in \mathcal{N} \ominus z\mathcal{N}$.

■

Remark 2.6. *If the extremal function φ for $\mathcal{M}$ satisfies $\varphi(0) \neq 0$, then one can also take $\tilde{b} = 1 - \overline{\varphi(0)}\varphi$ and obtain* $\ker H_{\tilde{b}} = \mathcal{M}$.

Indeed, under that hypothesis one checks that $\tilde{b} = P_{\mathcal{M}^\perp} 1 \perp [\varphi] = \mathcal{M}$, so that $[\tilde{b}]_* \subseteq \mathcal{M}^\perp = [M_z^*\varphi]_*$ (as established in the proof of the previous theorem). Also $M_z^*\varphi = -\frac{1}{\overline{\varphi(0)}} M_z^* \tilde{b} \in [\tilde{b}]_*$. Thus $[\tilde{b}]_* = [M_z^*\varphi]_*$ and this implies $\ker H_{\tilde{b}} = \ker H_{M_z^*\varphi} = \mathcal{M}$.

Remark 2.7. *Similarly, if* $\lambda \in \mathbb{D}$ *such that there is* $f \in \mathcal{M}$ *with* $f(\lambda) \neq 0$*, then it turns out that* $b = P_{\mathcal{M}^\perp} k_\lambda \in \mathcal{M}(D(\mu)) \subseteq \mathcal{X}(D(\mu))$ *and* $\mathcal{M} = \ker H_b$.

See Theorem 2.2.10 of [14]. We will not use this result here.

3. Invariant subspaces of $D(\mu) \odot D(\mu)$

It was a basic observation of Coifman, Rochberg, and Weiss ([7]) that in many cases one can identify $\mathcal{X}(\mathcal{H})$ with the dual space of $\mathcal{H} \odot \mathcal{H}$.

A proof that this duality holds for the spaces $\mathcal{H} = D(\mu)$ can be found in [19], see Theorems 1.2 and 1.3. The needed estimate $\|f_r\| \leq \frac{5}{2}\|f\|$ for all $f \in D(\mu)$, $0 \leq r < 1$, $f_r(z) = f(rz)$ was established in [1].

Theorem 3.1. $(D(\mu) \odot D(\mu))^* = \mathcal{X}(D(\mu))$*, i.e. if for* $b \in \mathcal{X}(D(\mu))$ *we define* L_b *on* $D(\mu)$ *by*

$$L_b(h) = \langle h, b \rangle,$$

then L_b *extends to be bounded on* $D(\mu) \odot D(\mu)$*, and the map* $b \to L_b$ *is a conjugate linear isometric isomorphism of* $\mathcal{X}(D(\mu))$ *onto* $(D(\mu) \odot D(\mu))^*$.

Corollary 3.2. *Let* μ *be a finite positive measure in* $\overline{\mathbb{D}}$*, and let* $\mathcal{M} \in Lat(M_z, D(\mu))$*. Then*

$$\mathcal{M} = D(\mu) \cap \operatorname{clos}_{D(\mu)\odot D(\mu)} \mathcal{M}.$$

Proof. The corollary is clearly true for $\mathcal{M} = (0)$. Thus we suppose that $\mathcal{M} \neq (0)$ and we set $\mathcal{N} = D(\mu) \cap \operatorname{clos}_{D(\mu)\odot D(\mu)} \mathcal{M}$. Clearly $\mathcal{M} \subseteq \mathcal{N} \in \mathrm{Lat}(M_z, D(\mu))$.

Suppose that there is an $f \in \mathcal{N}$ such that $f \notin \mathcal{M}$. By use of Theorem 2.5 we pick $b \in \mathcal{X}(D(\mu))$ such that $\ker H_b = \mathcal{M}$. Then the functional that b defines in the dual of $D(\mu) \odot D(\mu)$ annihilates $\mathcal{M}$ and hence it annihilates $\operatorname{clos}_{D(\mu)\odot D(\mu)} \mathcal{M}$. However, since $f \in \mathcal{N} \setminus \ker H_b$ we have $H_b f \neq 0$. Then there is a multiplier φ such that $\langle \varphi f, b \rangle = \langle H_b f, \overline{\varphi} \rangle_{\overline{D(\mu)}} \neq 0$. But $\varphi f \in \mathcal{N} \subseteq \operatorname{clos}_{D(\mu)\odot D(\mu)} \mathcal{M}$, and hence b does not annihilate all of $\operatorname{clos}_{D(\mu)\odot D(\mu)} \mathcal{M}$. This contradiction proves the corollary. ∎

Corollary 3.3. *Let* $f \in D(\mu)$*. Then* f *is cyclic in* $D(\mu)$ *if and only if* f *is cyclic in* $D(\mu) \odot D(\mu)$.

Proof. Since $\|pf - 1\|_* \leq \|pf - 1\|$ is is clear that cyclic vectors in $D(\mu)$ are cyclic in $D(\mu) \odot D(\mu)$. If f is not cyclic in $D(\mu)$, then we can take $\mathcal{M} = [f] \in \mathrm{Lat}(M_z, D(\mu))$ and apply the previous Corollary to conclude that $\operatorname{clos}_{D(\mu)\odot D(\mu)}[f] \neq D(\mu) \odot D(\mu)$, i.e. f is not cyclic in $D(\mu) \odot D(\mu)$. ∎

4. Approximation by $D(\mu)$-extremal functions

If E_n is a sequence of subspaces of a Banach space E, then define

$$\underline{\lim} E_n = \{x \in E : \exists\ x_n \in E_n \text{ with } x_n \to x\}.$$

It is easy to see that $\underline{\lim} E_n$ is always closed subspace. See [21].

Lemma 4.1. *For any E_n, E in a Hilbert space we have*

$$\underline{\lim} E_n \cap \underline{\lim} E_n^\perp = (0).$$

Consequently, if $E \subseteq \underline{\lim} E_n$ and $E^\perp \subseteq \underline{\lim} E_n^\perp$, then $E = \underline{\lim} E_n$ and $E^\perp = \underline{\lim} E_n^\perp$.

Proof. Suppose $x \in \underline{\lim} E_n \cap \underline{\lim} E_n^\perp$, then there are sequences $x_n \in E_n$ and $y_n \in E_n^\perp$ such that $x_n \to x$ and $y_n \to x$, hence $\|x_n\|^2 + \|y_n\|^2 = \|x_n - y_n\|^2 \to 0$, hence $x = 0$. ∎

Lemma 4.2. *Let $P, P_1, P_2, ...$ be projections with ranges $E, E_1, E_2, ...$ Then the following are equivalent:*
(a) $P_n \to P$ in the weak operator topology,
(b) $P_n \to P$ in the strong operator topology,
(c) $E = \underline{\lim} E_n$ and $E^\perp = \underline{\lim} E_n^\perp$,
(d) $E \subseteq \underline{\lim} E_n$ and $E^\perp \subseteq \underline{\lim} E_n^\perp$.

Proof. The equivalence of (a) and (b) is elementary and well-known, the equivalence of (b) and (c) is Lemma 2 of [21], (c) $\Rightarrow$(d) is trivial and (d) $\Rightarrow$ (c) follows from Lemma 4.1. ∎

Proposition 4.3. *Assume that $\mathcal{H}$ satisfies (2.1) and (2.2).*
(a) If $u_n, u \in \mathcal{M}(\mathcal{H})$ with $u_n \to u$ in $\mathcal{H}$, then

$$[u] \subseteq \underline{\lim} [u_n].$$

(b) If $b_n, b \in \mathcal{X}(\mathcal{H})$ with $b_n \to b$ in $\mathcal{H}$, then

$$\ker H_b^\perp \subseteq \underline{\lim} \ker H_{b_n}^\perp.$$

Proof. (a) The set $\{pu : p \text{ a polynomial}\}$ is dense in $[u]$. (a) follows, because by (2.1) each polynomial p is a multiplier, hence we have $pu_n \in [u_n]$ and $pu_n \to pu$.

(b) Set $E_n = \ker H_{b_n}^\perp$. Note that the set $\{H_b^* \overline{u} : u \in \mathcal{M}(\mathcal{H})\}$ is dense in $\operatorname{clos\,ran} H_b^* = \ker H_b^\perp$. Thus by Lemma 2.2 (b) it will suffice to show that $H_b^* \overline{u} = M_u^* b \in \underline{\lim} E_n$ for every $u \in \mathcal{M}(\mathcal{H})$. Let $u \in \mathcal{M}(\mathcal{H})$, then $H_{b_n}^* \overline{u} \in E_n$ for each n and $H_{b_n}^* \overline{u} = M_u^* b_n \to M_u^* b \in \underline{\lim} E_n$. ∎

Example 4.4. Let $Z = \{\lambda_1, ..., \lambda_m\} \subseteq \mathbb{D}$ with $0 \notin Z$ and let $\lambda \in \mathbb{D} \setminus Z$, $\lambda \neq 0$, and let $\mathcal{M} = I(Z)$, the zero-based invariant subspace with zeros Z. If P is the projection onto $\mathcal{M}^\perp$ we let $b = Pk_0$. Then b is a finite linear combination of reproducing kernels and hence H_b is bounded and $\mathcal{M} = \ker H_b$.

If we let $b_n = b + \frac{1}{n}k_\lambda$, then $b_n \to b$ in $\mathcal{H}$ (or even in $\mathcal{M}(\mathcal{H})$). Also, if k_λ is not a linear combination of $k_{\lambda_1}, ..., k_{\lambda_m}$, then for each n we have $\ker H_{b_n} = I(Z \cup \{\lambda\}) \neq \ker H_b$ and $E_n =$ closed linear span of $\{k_z : z \in Z \cup \{\lambda\}\} \neq \ker H_b^\perp$. Thus we don't always get equality in Proposition 4.3 (b).

Corollary 4.5. *Let μ be a finite positive measure in $\overline{\mathbb{D}}$. For $j = 1, 2, ...$ let $(0) \neq \mathcal{M}_j, \mathcal{M} \in Lat(M_z, D(\mu))$, let φ_j be the extremal function for $\mathcal{M}_j$, let φ be the extremal function for $\mathcal{M}$, and let P_j, P be the orthogonal projections onto $\mathcal{M}_j, \mathcal{M}$.*

If $\varphi_j(z) \to \varphi(z)$ for all $z \in \mathbb{D}$, then $P_j \to P$ in the strong operator topology.

Proof. Since $\varphi_n(z) \to \varphi(z)$ for all $z \in \mathbb{D}$ and $\|\varphi_n\| = \|\varphi\| = 1$ for all n we have $\varphi_n \to \varphi$ weakly and hence $\|\varphi_n - \varphi\|^2 = 2 - 2\ \mathrm{Re}\langle\varphi_n, \varphi\rangle \to 0$ as $n \to \infty$. By Theorem 2.3 φ_n and φ satisfy the hypothesis of Proposition 4.3 (a), hence $\mathcal{M} = [\varphi] \subseteq \underline{\lim}[\varphi_n]$.

Next set $b_n = M_z^*\varphi_n$ and $b = M_z^*\varphi$. Then $b_n \to b$ in $\mathcal{H}$ and by Lemma 2.2 we have $b_n, b \in \mathcal{X}(D(\mu))$. Thus by Theorem 2.5 $\mathcal{M}^\perp = \ker H_b^\perp \subseteq \underline{\lim} \ker H_{b_n}^\perp = \underline{\lim}\mathcal{M}_n^\perp$. The Corollary now follows from Lemma 4.2. ∎

Lemma 4.6. *Let μ be a finite positive measure in $\overline{\mathbb{D}}$, let S be an inner function, and let $f \in H^2$. Then for all $\lambda \in \mathbb{D}$ we have*

$$\frac{1-|\lambda|}{1+|\lambda|}\|Sf\|^2 \leq \|\frac{S-\lambda}{1-\overline{\lambda}S}f\|^2 \leq \frac{1+|\lambda|}{1-|\lambda|}\|Sf\|^2.$$

Proof. Let $|z| < 1$. Then since $\frac{S-S(z)}{w-z} \perp SH^2$ with respect to the H^2 inner product we have

$$\begin{aligned} D_z(\frac{S-\lambda}{1-\overline{\lambda}S}) &= \frac{(1-|\lambda|^2)^2}{|1-\overline{\lambda}S(z)|^2}\int_{|w|=1}\frac{1}{|1-\overline{\lambda}S(w)|^2}\left|\frac{S(z)-S(w)}{z-w}\right|^2\frac{|dw|}{2\pi} \\ &= \frac{(1-|\lambda|^2)^2}{|1-\overline{\lambda}S(z)|^2}\int_{|w|=1}\left|\sum_{n=0}\overline{\lambda}^n S^n(w)\frac{S(z)-S(w)}{z-w}\right|^2\frac{|dw|}{2\pi} \\ &= \frac{(1-|\lambda|^2)}{|1-\overline{\lambda}S(z)|^2}D_z(S). \end{aligned}$$

This easily implies

$$\frac{1-|\lambda|}{1+|\lambda|}D_z(S) \leq D_z(\frac{S-\lambda}{1-\overline{\lambda}S}) \leq \frac{1+|\lambda|}{1-|\lambda|}D_z(S) \tag{4.1}$$

for all $|z| < 1$. If $|z| = 1$, then any local Dirichlet integral $D_z(g)$ is the limit of $D_{z_n}(g)$ whenever $|z_n| < 1$ and $z_n \to z$ nontangentially, see [17], Lemma 3.3. Hence (4.1) holds for all $|z| \le 1$. The lemma now follows from the definition of the norm in $D(\mu)$ and because for any inner function B, any $f \in H^2$, and any $z \in \overline{\mathbb{D}}$ we have $D_z(Bf) = D_z(B)|f(z)|^2 + D_z(f)$, see [17], Lemma 3.4. ■

Theorem 4.7. *Let μ be a finite positive measure in $\overline{\mathbb{D}}$. Let S_0 be an inner function such that $\mathcal{M} = S_0H^2 \cap D(\mu) \ne (0)$ and let φ be the $D(\mu)$-extremal function for $\mathcal{M}$.*

Then there is a sequence $Z_n = \{z_{1n}, z_{2n}, ..., z_{k_n n}\}$ of finite sequences in $\mathbb{D}$ such that the $D(\mu)$-extremal functions φ_n for $I(Z_n)$ converge locally uniformly in $\mathbb{D}$ to φ.

Proof. Write $S_0(z) = z^kS(z)$, where $k \ge 0$ and S is an inner function with $S(0) \ne 0$. Then the extremal function φ for $\mathcal{M}$ is of the form $\varphi = z^kSu$ for some $u \in H^2$. Since inner factors increase the $D(\mu)$-norm (see [17], Lemma 3.4) it is clear that u is an outer function and that $S(0)u(0) > 0$.

By Lemma 4.6 we have $\|z^k\frac{S-\lambda}{1-\bar\lambda S}u\|^2 < \infty$ for all $|\lambda| < 1$ and $\|z^k\frac{S-\lambda}{1-\bar\lambda S}u\| \to \|z^kSu\| = 1$ as $|\lambda| \to 0$. Thus $\mathcal{M}_\lambda := z^k\frac{S-\lambda}{1-\bar\lambda S}H^2 \cap D(\mu) \ne (0)$ for all $|\lambda| < 1$. For $\lambda \in \mathbb{D} \setminus \{0\}$ we set $S_\lambda = z^k\frac{S-\lambda}{1-\bar\lambda S}\Big/\|z^k\frac{S-\lambda}{1-\bar\lambda S}u\|$, then $S_\lambda u \in \mathcal{M}_\lambda$, $\|S_\lambda u\| = 1$, and $S_\lambda(z) \to S_0(z)$ as $|\lambda| \to 0$.

For $|\lambda| < |S(0)|$ let u_λ be the function such that $\varphi_\lambda = S_\lambda u_\lambda$ is the extremal function for $\mathcal{M}_\lambda$. We will show that $\varphi_\lambda \to \varphi$ locally uniformly in $\mathbb{D}$ as $|\lambda| \to 0$.

We have $1 = \|S_\lambda u_\lambda\| \ge \|u_\lambda\|$. It suffices to prove that if v is any weak limit of u_λ as $|\lambda| \to 0$, then $u = v$.

Thus let $v \in D(\mu)$ such that $\exists|\lambda_n| \to 0$ with $u_{\lambda_n} \to v$ locally uniformly in $\mathbb{D}$.

The extremality of $\varphi_{\lambda_n} = S_{\lambda_n}u_{\lambda_n}$ in $\mathcal{M}_{\lambda_n}$ and the properties of $S_{\lambda_n}u$ imply that $|u_{\lambda_n}(0)| \ge |u(0)|$ for each n. Thus $|v(0)| \ge |u(0)|$.

But we also have $S_{\lambda_n}(z)u_{\lambda_n}(z) \to S_0(z)v(z)$ in $\mathbb{D}$ and hence $S_{\lambda_n}u_{\lambda_n} \to S_0v$ weakly in $D(\mu)$. Thus $\|S_0v\| \le 1$ and $S_0v \in \mathcal{M}$. Then the extremality of $\varphi = S_0u$ in $\mathcal{M}$ implies $|u(0)| \ge |v(0)|$. The uniqueness of the extremal function $\varphi = S_0u$ implies that there is a $|c| = 1$ such that $S_0u = cS_0v$. From the extremal condition we also have $S(0)u(0) > 0$ and $\frac{S(0)-\lambda_n}{1-\bar\lambda_n S(0)}u_{\lambda_n}(0) > 0$. Taking $n \to \infty$ we conclude $S(0)v(0) > 0$ and hence $u = v$. Thus $\varphi_\lambda = S_\lambda u_\lambda \to \varphi = S_0u$ locally uniformly in $\mathbb{D}$.

By Frostman's theorem we have $B_\lambda = z^k\frac{S-\lambda}{1-\bar\lambda S}$ is a Blaschke product for all $\lambda \in \mathbb{D}\setminus K$, where K is an exceptional set of zero logarithmic capacity. Note that it is clear that if B_λ is a Blaschke product, then $\mathcal{M}_\lambda$ is a zero based invariant subspace. Thus the above shows that φ can be approximated locally uniformly by extremal functions corresponding to zero based invariant subspaces. The Theorem follows, because if Z is any zero sequence for $D(\mu)$, then the extremal function for $I(Z)$ can be approximated locally uniformly on $\mathbb{D}$ by the extremal functions corresponding to the partial subsequences of Z. That can be seen by an argument similar to the one given above. We sketch the details below for the case where $0 \notin Z$.

Suppose $Z = \{z_1, z_2, \ldots\}$, $0 \notin Z$, and let $\varphi \in I(Z)$ be the extremal function. Then for $n \in \mathbb{N}$, let $Z_n = \{z_1, z_2, \ldots, z_n\}$, then $I(Z_n) \supseteq I(Z) = \cap_k I(Z_k)$. Suppose φ_n is the extremal function for $I(Z_n)$, then $\varphi_n(0) \geq \varphi_{n+1}(0) \geq \ldots \geq \varphi(0)$. Let g be any weak limit of φ_n, then $g \in I(Z)$ and $\lim_n \varphi_n(0) = g(0)$, thus $\varphi(0) \leq g(0)$ and so $g = \varphi$ by the uniqueness of the extremal function for $I(Z)$. ■

5. A necessary condition for approximation

For this section again we suppose that μ is a nonnegative finite Borel measure that is supported in $\overline{\mathbb{D}}$. As is easy to check the local Dirichlet integral has the property that

$$D_w(zf) = D_w(f) + |f(w)|^2 \tag{5.1}$$

for all $|w| < 1$ and all $f \in H^2$. If $|w| = 1$ and $D_w(f) < \infty$, then f has finite non-tangential limit $f(w)$ at w and equation (5.1) holds in this case as well ([17]). This implies that $M_z^* M_z - I$ is a positive operator on $D(\mu)$. Let D denote the positive square root of $M_z^* M_z - I$. Then equation (5.1) implies

$$\|Df\|^2 = \|zf\|^2 - \|f\|^2 = \int_{|z|\leq 1} |f|^2 d\mu \tag{5.2}$$

for any $f \in D(\mu)$.

Lemma 5.1. *If $\mu \neq 0$, then the inclusion map $i : D(\mu) \to L^2(\mu)$ is compact.*

Proof. Using the notation from before the lemma we see from (5.2) that $i^* i = D^2$ and hence it suffices to show that D is compact. Since M_z is bounded below and its range has codimension one in $D(\mu)$ (see [1]) it follows that $F_n = (z^n D(\mu))^\perp$ is n-dimensional and $D_n = DP_{F_n}$ is a finite rank operator. Equation (5.1) implies that for $f \in D(\mu)$, $\lambda \in \overline{\mathbb{D}}$, and $n \in \mathbb{N}$ we have $D_\lambda(z^n f) = D_\lambda(f) + \sum_{k=0}^{n-1} |\lambda|^{2k} |f(\lambda)|^2$. Hence

$\|z^n f\|^2 = \|f\|^2 + \int_{\overline{\mathbb{D}}} \sum_{k=0}^{n-1} |\lambda|^{2k} |f(\lambda)|^2 d\mu(\lambda)$ and therefore whenever $f \neq 0$

$$\begin{aligned}\frac{\|Dz^n f\|^2}{\|z^n f\|^2} &= \frac{\|z^{n+1}f\|^2 - \|z^n f\|^2}{\|f\|^2 + \int_{\overline{\mathbb{D}}} \sum_{k=0}^{n-1} |\lambda|^{2k} |f(\lambda)|^2 d\mu(\lambda)} \\ &= \frac{\int_{\overline{\mathbb{D}}} |\lambda|^{2n} |f(\lambda)|^2 d\mu(\lambda)}{\|f\|^2 + \int_{\overline{\mathbb{D}}} \sum_{k=0}^{n-1} |\lambda|^{2k} |f(\lambda)|^2 d\mu(\lambda)} \\ &\leq \frac{\int_{\overline{\mathbb{D}}} |\lambda|^{2n} |f(\lambda)|^2 d\mu(\lambda)}{\|f\|^2 + n \int_{\overline{\mathbb{D}}} |\lambda|^{2n} |f(\lambda)|^2 d\mu(\lambda)} \\ &\leq \frac{1}{n}.\end{aligned}$$

This implies

$$\|D - D_n\|^2 = \sup_{f \neq 0} \frac{\|Dz^n f\|^2}{\|z^n f\|^2} \leq \frac{1}{n} \to 0 \text{ as } n \to \infty,$$

thus D is compact. ■

Recall from the introduction that for $f \in D(\mu)$ and $\lambda \in \mathbb{D}$ we defined

$$u_f(\lambda) = \text{Re } \langle \frac{1 + \overline{\lambda} z}{1 - \overline{\lambda} z} f, f \rangle.$$

Then u_f is harmonic on $\mathbb{D}$ and u_f is identically equal to 1 for any extremal function f. The definition of u_f is motivated by Shimorin's work in [21].

Theorem 5.2. *If φ_n are extremal functions in $D(\mu)$ and if $\varphi_n(z) \to f(z)$ for every $z \in \mathbb{D}$, then $u_f(z) \leq 1$ for all $z \in \mathbb{D}$.*

Proof. The theorem is trivial for $\mu = 0$, thus we will assume that $\mu \neq 0$. Let $\lambda \in \mathbb{D}$ and $g \in D(\mu)$. Then $u_g(\lambda) = \|g\|^2 + 2\text{Re } \langle \frac{\overline{\lambda} z}{1 - \overline{\lambda} z} g, g \rangle$ and hence

$$\begin{aligned}\|\frac{g}{1 - \overline{\lambda} z}\|^2 = \|(1 + \frac{\overline{\lambda} z}{1 - \overline{\lambda} z}) g\|^2 &= u_g(\lambda) + |\lambda|^2 \|z \frac{g}{1 - \overline{\lambda} z}\|^2 \\ &= u_g(\lambda) + |\lambda|^2 \|D \frac{g}{1 - \overline{\lambda} z}\|^2 + |\lambda|^2 \|\frac{g}{1 - \overline{\lambda} z}\|^2.\end{aligned}$$

By use of (5.2) this implies

$$(1 - |\lambda|^2) \|s_\lambda g\|^2 = u_g(\lambda) + |\lambda|^2 \int |s_\lambda(z) g(z)|^2 d\mu(z), \tag{5.3}$$

where $s_\lambda(z) = \frac{1}{1 - \overline{\lambda} z}$.

Now let φ_n and f be as in the theorem. Since φ_n is a norm-bounded sequence that converges pointwise we conclude that φ_n converges weakly to $f \in D(\mu)$. $s_\lambda \in \mathcal{M}(D(\mu))$ implies that $s_\lambda \varphi_n$ converges weakly to $s_\lambda f$. Hence $\|s_\lambda f\|^2 \le \liminf_{n\to\infty} \|s_\lambda \varphi_n\|^2$. Furthermore, Lemma 5.1 implies that $s_\lambda \varphi_n \to s_\lambda f$ in $L^2(\mu)$. Thus

$$\int_{|z|\le 1} |s_\lambda(z)\varphi_n(z)|^2 d\mu(z) \to \int_{|z|\le 1} |s_\lambda(z) f(z)|^2 d\mu(z).$$

But now (5.3) implies that $u_f(\lambda) \le \liminf_{n\to\infty} u_{\varphi_n}(\lambda) = 1$. ∎

We have noted in the Introduction that local uniform limits of extremal functions also must be contractive multipliers. We will show now that for the Dirichlet space D the condition $u_f \le 1$ in $\mathbb{D}$ already implies that f is a contractive multiplier. We start with an observation that holds for all $D(\mu)$.

If φ is an extremal function in $D(\mu)$, then it was shown in [16] and [1] that

$$\|p\varphi\|^2 = \int_{|z|=1} |p(z)|^2 \frac{|dz|}{2\pi} + \int_{|z|\le 1} D_z(p)|\varphi(z)|^2 d\mu(z)$$

for every polynomial p. The following lemma is a generalization of this.

Lemma 5.3. *Let $f \in D(\mu)$, then for any polynomial p we have*

$$\|pf\|^2 = \lim_{r\to 1} \int_{|z|=1} |p(z)|^2 u_f(rz) \frac{|dz|}{2\pi} + \int_{|z|\le 1} D_z(p)|f(z)|^2 d\mu(z).$$

Proof. First note that by the definition of u_f we have

$$\lim_{r\to 1} \int_{|z|=1} z^k u_f(rz) \frac{|dz|}{2\pi} = \langle z^k f, f\rangle$$

whenever k is a nonnegative integer. Similarly, for $k < 0$ we have $\lim_{r\to 1} \int_{|z|=1} z^k u_f(rz) \frac{|dz|}{2\pi} = \langle f, z^{|k|} f\rangle$.

Now write

$$D_z(h, g) = \int_{|w|=1} \frac{h(z) - h(w)}{z - w} \frac{\overline{g(z)} - \overline{g(w)}}{\overline{z} - \overline{w}} \frac{|dw|}{2\pi}.$$

Recall from [17], Lemma 3.4 and its proof, that $D_z(Bh) = D_z(B)|h(z)|^2 + D_z(h)$ for all inner functions B, all $h \in H^2$, and all $z \in \overline{\mathbb{D}}$. By polarization this implies $D_z(Bh, Bg) = D_z(B)h(z)\overline{g(z)} + D_z(h, g)$ for $h, g \in H^2$. If $m \ge n$ we first apply this with $B(w) = w^n$, $h(w) = w^{m-n}$, and $g(w) = 1$ and obtain $D_z(w^m, w^n) = D_z(w^n) z^{m-n}$. Next we apply the same formula with $B(w) = w^n$, $h(w) = w^{m-n}f(w)$, and $g(w) = f(w)$ and obtain for all $m \geq n$

$$\begin{aligned} D_z(w^m f, w^n f) &= D_z(w^n)z^{m-n}|f(z)|^2 + D_z(w^{m-n}f, f) \\ &= D_z(w^m, w^n)|f(z)|^2 + D_z(w^{m-n}f, f). \end{aligned}$$

Now the definition of the $D(\mu)$-norm implies that for $m \geq n$ we have

$$\begin{aligned} \langle z^m f, z^n f\rangle &= \int_{|z|\leq 1} D_z(w^m, w^n)|f(z)|^2 d\mu(z) + \langle z^{m-n}f, f\rangle \\ &= \int_{|z|\leq 1} D_z(w^m, w^n)|f(z)|^2 d\mu(z) + \lim_{r\to 1}\int_{|z|=1} z^{m-n}u_f(rz)\frac{|dz|}{2\pi}. \end{aligned}$$

A similar calculation holds for $m < n$ and hence the lemma follows. ■

Note that if $u_f \leq 1$ in $\mathbb{D}$, then $1 - u_f$ is a nonnegative harmonic function in $\mathbb{D}$, hence there is a nonnegative measure σ such that $u_f = 1 - P[\sigma]$, where $P[\sigma]$ denotes the Poisson integral of σ. Then Lemma 5.3 implies that

$$\|pf\|^2 = \|p\|^2_{H^2} + \int_{|z|\leq 1} D_z(p)|f|^2 d\mu - \int_{|z|=1} |p|^2 d\sigma \leq \max(1, \|f\|^2_\infty)\|p\|^2$$

for every polynomial p. Thus the inclusion

$$\{f \in D(\mu) : u_f \leq 1 \text{ in } \mathbb{D} \text{ and } \|f\|_\infty \leq 1\} \subseteq \{f \in \mathcal{M}(D(\mu)) : \|f\|_{\mathcal{M}} \leq 1\}$$

holds for all μ.

Theorem 5.4. *If $f \in D$, then $\|f\|^2_\infty \leq \sup_{\lambda\in\mathbb{D}} u_f(\lambda)$. Consequently,*

$$\{f \in D : u_f \leq 1 \text{ in } \mathbb{D}\} \subseteq \{f \in \mathcal{M}(D) : \|f\|_{\mathcal{M}(D)} \leq 1\}$$

Proof. The reproducing kernel for the Dirichlet space is $k_\lambda(z) = \sum_{n=0}^\infty \frac{\overline{\lambda}^n z^n}{n+1}$, and it is a complete Nevanlinna-Pick kernel with $k_\lambda(0) = 1$, see e.g. [22]. Then Lemma 2.2 (b) of [11] says that for all $f \in D$ and $\lambda \in \mathbb{D}$

$$|f(\lambda)|^2 \leq 2\text{Re}\langle k_\lambda f, f\rangle - \|f\|^2.$$

A short calculation shows that

$$\int_0^1 u_f(s\lambda)ds = 2\text{Re}\langle k_\lambda f, f\rangle - \|f\|^2.$$

Hence the two inequalities together imply $|f(\lambda)|^2 \leq \sup_{z\in\mathbb{D}} u_f(z)$. Taking the supremum over $\lambda \in \mathbb{D}$ gives the desired estimate. ■

Remark 5.5. *Since there are bounded analytic functions in D that are not multipliers, it follows that the boundedness of f does not imply the boundedness of u_f. The following provides a simple explicit example with $\|f\|_{\mathcal{M}} \leq 1$ and $\sup_{z\in\mathbb{D}} u_f(z) > 1$. It is shown in* [11, Lemma 2.2] *that for complete NP kernels one has* $\|k_\lambda\|_{\mathcal{M}} \leq 2\|k_\lambda\|^2$ *and clearly,* $u_{k_\lambda}(z) = \frac{1-|\lambda z|^2}{|1-\overline{\lambda}z|^2}\|k_\lambda\|^2$. *Thus setting* $f_\lambda = k_\lambda/(2\|k_\lambda\|^2)$ *we have* $\|f_\lambda\|_{\mathcal{M}} \leq 1$ *and*

$$u_{f_\lambda}(z) = \frac{1}{4}\frac{1-|\lambda z|^2}{|1-\overline{\lambda}z|^2\|k_\lambda\|^2},$$

and hence $\sup_{z\in\mathbb{D}} u_{f_\lambda}(z) = \frac{1+|\lambda|}{4(1-|\lambda|)\|k_\lambda\|^2} \to \infty$ *as* $|\lambda| \to 1$ *for* k_λ *the reproducing kernel for the Dirichlet space.*

6. Open Questions

Our work raises a number of open questions. The first one is whether an analogue of statement (1.3) holds for $D \odot D$. In particular, we don't even know the answers to the following questions.

Question 6.1. *(a) If* $\mathcal{N} \in$ *Lat*$(M_z, D \odot D)$ *with* $\mathcal{N} \neq (0)$*, then is* $\mathcal{N} \cap D \neq (0)$ *?*

(b) Is every zero sequence for $D \odot D$ *a zero sequence for* D*?*

Of course, the work of this paper on approximation by extremal functions is motivated by the following question.

Question 6.2. *If* $\mathcal{M} \in$ *Lat*(M_z, D), $\mathcal{M} \neq (0)$*, then is there a sequence of finite codimensional subspaces* $\mathcal{M}_j \in$ *Lat*(M_z, D) *such that* $P_{\mathcal{M}_j} \to P_{\mathcal{M}}$ *in the strong operator topology?*

Corollary 4.5 and Theorem 4.7 imply that this is true for subspaces of the type $SH^2 \cap D$ where S is inner. A class of invariant subspaces of D containing outer functions is given by $D_E = \{f \in D : f = 0 \; q.e. \text{ on } E\}$. Here $E \subseteq \mathbb{T}$ is a compact set of positive logarithmic capacity and $f = 0$ q.e. on E means $f(z) = 0$ for all $z \in E \backslash Z$ for some set Z of logarithmic capacity 0. It is known that there are such sets E with $D_E \neq (0)$, see [9].

Question 6.3. *Let* $E \subseteq \mathbb{T}$ *be a compact set with positive logarithmic capacity such that* $D_E \neq (0)$*, and let* φ *be the extremal function for* D_E*.*

Then are there extremal functions φ_j *for zero set based invariant subspaces* $I(Z_j)$ *such that* $\varphi_j \to \varphi$ *locally uniformly in* $\mathbb{D}$*?*

The argument at the end of the proof of Theorem 4.7 shows that if the answer to the question is affirmative, then it can be done with finite zero sets. Hence Question 6.2 would have an affirmative answer for such subspaces.

Question 6.4. *Let $LE(D)$ be the set of local uniform limits of functions of the type $c\varphi$, where $|c| = 1$ and φ is a Dirichlet extremal function. Is it true that*

$$LE(D) = \{f \in D : u_f \le 1\}?$$

Theorems 5.2, 5.4, and Remark 5.5 imply that

$$LE(D) \subseteq \{f \in D : u_f \le 1\} \subsetneqq \{f \in \mathcal{M}(D) : \|f\|_{\mathcal{M}} \le 1\}.$$

The following example is due to Carl Sundberg and it shows that there are nonzero functions in $LE(D)$ which are not extremal functions. Note that if $f(z) = az^j$, then $u_f(\lambda) = (j+1)|a|^2$.

Theorem 6.5. *(Sundberg, private communication) If $f(z) = az^j$ for some $j \ge 0$ and $a \ge 0$ with $a^2 \le \frac{1}{j+1}$, then there is a sequence of Dirichlet extremal functions φ_n corresponding to finite zero sets such that $\varphi_n \to f$ locally uniformly in $\mathbb{D}$.*

Proof. Throughout this proof we fix $j \ge 0$. If $a^2 = \frac{1}{j+1}$, then f is an extremal function and there is nothing to prove. If $a = 0$, then we just take $\varphi_n(z) = \frac{z^n}{\sqrt{n+1}}$ and the theorem follows in this case. Thus we will assume that $0 < a < \frac{1}{\sqrt{j+1}}$.

We continue the proof with the observation that if $0 < |b| < 1$ and if $\mathcal{H}$ is a reproducing kernel Hilbert space on $\mathbb{D}$ with reproducing kernel $k_\lambda(z)$ such that k_0 and k_b are linearly independent, then $\|k_0\|^2 - \frac{|k_b(0)|^2}{\|k_b\|^2} \ne 0$ and hence

$$\psi(z) = \frac{k_0(z) - \frac{k_0(b)}{k_b(b)} k_b(z)}{\sqrt{\|k_0\|^2 - \frac{|k_b(0)|^2}{\|k_b\|^2}}}$$

is the extremal function for the subspace $\{g \in \mathcal{H} : g(b) = 0\}$. It satisfies $\psi(0) = \sqrt{\|k_0\|^2 - \frac{|k_b(0)|^2}{\|k_b\|^2}}$.

For $n \in \mathbb{N}$ let

$$\mathcal{H}_n = \{g \in \mathrm{Hol}(\mathbb{D}) : \|g\|^2_{\mathcal{H}_n} = \sum_{k=0}^{\infty} (nk + j + 1)|\hat{g}(k)|^2 < \infty\},$$

then $\mathcal{H}_n$ is a Hilbert space of analytic functions with the property that $\langle f, g\rangle_{\mathcal{H}_n} = \langle z^j(f \circ z^n), z^j(g \circ z^n)\rangle_D$ for all $f, g \in \mathcal{H}_n$. Furthermore, the reproducing kernel for $\mathcal{H}_n$ is $k^n_\lambda(z) = \sum_{k=0}^{\infty} \frac{1}{nk+j+1} \overline{\lambda}^k z^k$.

Note that $k^n_0(z) = \frac{1}{j+1}$ and $k^n_b(b) \to \infty$ as $|b| \to 1$. Thus we may choose $b_n \in \mathbb{D}\backslash\{0\}$ such that $\|k^n_0\|^2 - \frac{|k^n_{b_n}(0)|^2}{\|k^n_{b_n}\|^2} = \frac{1}{j+1}(1 - \frac{1}{(j+1)\|k^n_{b_n}\|^2}) = a^2$.

Now let ψ_n be the $\mathcal{H}_n$-extremal function for $\{g \in \mathcal{H}_n : g(b_n) = 0\}$, and set $\varphi_n(z) = z^j\psi_n(z^n)$. Notice that φ_n is analytic in a neighborhood of $\overline{\mathbb{D}}$, hence it is clear that the invariant subspace of D that is generated by φ_n is a finite zero set based invariant subspace $\mathcal{M}_n$.

Any nonnegative integer is of the form $m = kn + r$ for some integers $k \geq 0$ and $0 \leq r < n$. If $0 < r < n$, then by the form of the power series of the functions in the inner product it is clear that

$$\langle z^m \varphi_n, \varphi_n \rangle_D = \langle z^{kn+r+j}(\psi_n \circ z^n), z^j(\psi_n \circ z^n) \rangle_D = 0.$$

If $r = 0$, then we have

$$\langle z^m \varphi_n, \varphi_n \rangle_D = \langle z^{kn+j}(\psi_n \circ z^n), z^j(\psi_n \circ z^n) \rangle_D = \langle z^k \psi_n, \psi_n \rangle_{\mathcal{H}_n} = \delta_{0k} = \delta_{0m}.$$

Thus since $\varphi_n^{(j)}(0) > 0$ it follows that φ_n is the extremal function for $\mathcal{M}_n$.

Finally we note that $\psi_n(z) = a + z g_n(z)$ for some analytic function g_n. Thus $\varphi_n(z) = f(z) + z^{j+n} g_n(z^n)$ and since $\|\varphi_n\| = 1$ for all n we conclude that $\varphi_n \to f$ weakly in D. This implies $\varphi_n \to f$ locally uniformly in $\mathbb{D}$. ■

References

[1] A. Aleman. *The multiplication operator on Hilbert spaces of analytic functions.* Habilitation. Fernuniversitaet Hagen, 1993.

[2] C. Apostol, H. Bercovici, C. Foias, and C. Pearcy. Invariant subspaces, dilation theory, and the structure of the predual of a dual algebra. I. *J. Funct. Anal.*, 63(3):369–404, 1985.

[3] N. Arcozzi, R. Rochberg, E. Sawyer, and B. D. Wick. Function spaces related to the Dirichlet space. *J. Lond. Math. Soc. (2)*, 83(1):1–18, 2011.

[4] Nicola Arcozzi, Richard Rochberg, Eric Sawyer, and Brett D. Wick. Bilinear forms on the Dirichlet space. *Anal. PDE*, 3(1):21–47, 2010.

[5] Arne Beurling. On two problems concerning linear transformations in Hilbert space. *Acta Math.*, 81:17, 1948.

[6] Paul S. Bourdon and Joel H. Shapiro. Spectral synthesis and common cyclic vectors. *Michigan Math. J.*, 37(1):71–90, 1990.

[7] R. R. Coifman, R. Rochberg, and Guido Weiss. Factorization theorems for Hardy spaces in several variables. *Ann. of Math. (2)*, 103(3):611–635, 1976.

[8] Peter L. Duren. *Theory of H^p spaces.* Pure and Applied Mathematics, Vol. 38. Academic Press, New York-London, 1970.

[9] Omar El-Fallah, Karim Kellay, Javad Mashreghi, and Thomas Ransford. *A primer on the Dirichlet space*, volume 203 of *Cambridge Tracts in Mathematics.* Cambridge University Press, Cambridge, 2014.

[10] John B. Garnett. *Bounded analytic functions*, volume 96 of *Pure and Applied Mathematics.* Academic Press, Inc. [Harcourt Brace Jovanovich, Publishers], New York-London, 1981.

[11] Devin C. V. Greene, Stefan Richter, and Carl Sundberg. The structure of inner multipliers on spaces with complete Nevanlinna-Pick kernels. *J. Funct. Anal.*, 194(2):311–331, 2002.

[12] N. S. Landkof. *Foundations of modern potential theory.* Springer-Verlag, New York, 1972. Translated from the Russian by A. P. Doohovskoy, Die Grundlehren der mathematischen Wissenschaften, Band 180.

[13] Shuaibing Luo. On the index of invariant subspaces in the space of weak products of Dirichlet functions. *Complex Analysis and Operator Theory, to appear.*

[14] Shuaibing Luo. *Some aspects of function theory for Dirichlet-type spaces.* Ph.D. dissertation. The University of Tennessee, 2014.

[15] Stefan Richter. Invariant subspaces of the Dirichlet shift. *J. Reine Angew. Math.*, 386:205–220, 1988.

[16] Stefan Richter. A representation theorem for cyclic analytic two-isometries. *Trans. Amer. Math. Soc.*, 328(1):325–349, 1991.

[17] Stefan Richter and Carl Sundberg. A formula for the local Dirichlet integral. *Michigan Math. J.*, 38(3):355–379, 1991.

[18] Stefan Richter and Carl Sundberg. Multipliers and invariant subspaces in the Dirichlet space. *J. Operator Theory*, 28(1):167–186, 1992.

[19] Stefan Richter and Carl Sundberg. Weak products of Dirichlet functions. *J. Funct. Anal.*, 266(8):5270–5299, 2014.

[20] Walter Rudin. *Function theory in polydiscs.* W. A. Benjamin, Inc., New York-Amsterdam, 1969.

[21] S. M. Shimorin. Approximate spectral synthesis in the Bergman space. *Duke Math. J.*, 101(1):1–39, 2000.

[22] Serguei Shimorin. Complete Nevanlinna-Pick property of Dirichlet-type spaces. *J. Funct. Anal.*, 191(2):276–296, 2002.

[23] Kehe Zhu. Maximal inner spaces and Hankel operators on the Bergman space. *Integral Equations Operator Theory*, 31(3):371–387, 1998.

[24] Kehe Zhu. *Operator theory in function spaces*, volume 138 of *Mathematical Surveys and Monographs.* American Mathematical Society, Providence, RI, second edition, 2007.

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