JOINT EXTENSIONS IN FAMILIES OF CONTRACTIVE COMMUTING OPERATOR TUPLES.

STEFAN RICHTER AND CARL SUNDBERG

Abstract. In this paper we systematically study extension questions in families of commuting operator tuples that are associated with the unit ball in $\mathbb{C}^d$.

Contents

1. Introduction

It is fair to say that the Sz. Nagy dilation theorem is of central importance for the theory of contraction operators on Hilbert spaces. One version of this theorem states that every contraction on a Hilbert space can be extended to a co-isometric operator acting on a larger Hilbert space. Because of the known structure of the co-isometric operators, this means that one can use the function theory of the Hardy space of the unit disc to study arbitrary contractions.

Partial extensions of Sz. Nagy's theorem are available for the study of tuples of operators. The best known result is Ando's theorem which

2000 *Mathematics Subject Classification.* Primary 47A13, 47A20; Secondary 47A45.

Work of the authors was supported by the National Science Foundation, grant DMS-0556051.

says that for any pair of commuting contraction operators S and T acting on a Hilbert space $\mathcal{H}$, there is a pair U, V of commuting co-isometric operators acting on a larger space $\mathcal{K} \supseteq \mathcal{H}$ such that U extends S and V extends T, [2]. It is also known that a direct analogue of Ando's theorem fails for three or more commuting contractions. Ando's theorem relates the study of commuting contractions to function theory on the bidisc, while it remains an open problem to find an effective model for three or more commuting contractions. The spherical contractions and the row contractions are collections of operator tuples which have been studied recently and which can be associated with function theory in the unit ball of $\mathbb{C}^d$. A convenient way to approach many such theorems is through J. Agler's model theory (see [1]). In this note we will present some examples of this model theory for the multivariable context.

The following definition is from [1]. We will assume that all our Hilbert spaces are separable.

Definition 1.1. *Let $d \geq 1$. A family is a collection $\mathcal{F}$ of d-tuples $T = (T_1, .., T_d)$ of Hilbert space operators, $T_i \in \mathcal{B}(\mathcal{H})$, such that:*

(a) $\mathcal{F}$ is bounded, i.e. there exists $c > 0$ such that for all $T = (T_1, .., T_d) \in \mathcal{F}$ we have $\|T_i\| \leq c$ for all $i = 1, ..., d$,

(b) $\mathcal{F}$ is preserved under restrictions to invariant subspaces, i.e. whenever $T \in \mathcal{F}$ and $\mathcal{M} \subseteq \mathcal{H}$ such that $T_i\mathcal{M} \subseteq \mathcal{M}$ for all i, then $T|\mathcal{M} \in \mathcal{F}$,

(c) $\mathcal{F}$ is preserved under direct sums, i.e. whenever $T_n \in \mathcal{F}$ is a sequence of tuples, then $\oplus_n T_n \in \mathcal{F}$,

*(d) $\mathcal{F}$ is preserved under unital *-representations, i.e. if $\pi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{K})$ is a *-homomorphism with $\pi(I) = I$ and if $T = (T_1, .., T_d) \in \mathcal{F}$, then $\pi(T) = (\pi(T_1), .., \pi(T_d)) \in \mathcal{F}$.*

For $d = 1$ some examples are given by the families of contractions, isometries, subnormal contractions, and hyponormal contractions. For $d > 1$ we will be interested only in families which consist of commuting tuples of operators. The family of commuting contractions has already been mentioned. A tuple $(T_1, ..., T_d)$ is called a spherical isometry if $\sum_{i=1}^d \|T_i x\|^2 = \|x\|^2$ for every $x \in \mathcal{H}$. It is immediately clear that the collection of spherical isometries satisfies (a), (b) and (c) of Definition 1.1. Furthermore (d) follows as well, because $(T_1, ..., T_d)$ is a spherical isometry if and only if $\sum_{i=1}^d T_i^* T_i = I$. We will write $\mathcal{F}_{si}$ to denote the family of commuting spherical isometries.

The spherical contractions $\mathcal{F}_{sc}$ are those commuting d-tuples $T = (T_1, .., T_d)$ of Hilbert space operators satisfying $\sum_{j=1}^d T_j^* T_j \leq I$. The collection of adjoint tuples $\mathcal{F}_{sc}^*$ consists of the row contractions $\mathcal{F}_{rc}$.

They satisfy

$$\|\sum_{j=1}^{d} T_j x_j\|^2 \leq \sum_{j=1}^{d} \|x_j\|^2 \text{ for all } x_1, ..., x_d$$

in the Hilbert space. As for the spherical isometries it is easy to check that both $\mathcal{F}_{sc}$ and $\mathcal{F}_{rc} = \mathcal{F}_{sc}^*$ form a family.

Suppose T is an operator tuple acting on a Hilbert space $\mathcal{H}$ and R is a tuple acting on $\mathcal{K}$. We will write $R \geq T$ if R is an extension of T, i.e. if $\mathcal{H} \subseteq \mathcal{K}$ is a subspace which is invariant for each R_i, and if $T_i = R_i|\mathcal{H}$ for all i. In this case we will call dim $\mathcal{K} \ominus \mathcal{H}$ the rank of the extension. If $R = T \oplus B$ for some operator tuple B, then R is called a trivial extension of T.

Definition 1.2. *Let $\mathcal{F}$ be a family. An operator tuple $T \in \mathcal{F}$ acting on $\mathcal{H}$ is called an extremal for $\mathcal{F}$ if T has only trivial extensions in $\mathcal{F}$, i.e. whenever $R \in \mathcal{F}$ satisfies $R \geq T$, then $\mathcal{H}$ reduces R.*

We shall write $\mathrm{ext}(\mathcal{F})$ for the extremals of the family $\mathcal{F}$.

Theorem. *(J. Agler) If $\mathcal{F}$ is a family and if $T \in \mathcal{F}$, then T can be extended to a tuple $S \in \mathrm{ext}(\mathcal{F})$.*

The Theorem is stated for families of single operators in [1], but it is mentioned there that the result also holds in the multivariable context. For a proof we refer the reader to [14] or the unpublished note [6].

Thus it is an important question to identify the extremals of families of interest. We note that it is easy to see that the extremals for the family of contractions are the co-isometric operators, the extremals for the isometric operators are the unitary operators, and extremals for the subnormal contractions are the normal contractions. It is unknown what the extremals for the hyponormal contractions are (see [13]).

Next we discuss some examples for $d > 1$. Ando's theorem can be used to show that the pairs of two commuting co-isometric operators are extremal for the pairs of commuting contractions. Alternatively, one can use a one-step extension as in the proof of the commutant lifting theorem (see [20], page 65) to identify the extremals. In this case Ando's theorem follows from the above theorem of Agler's. It is an open problem to identify the extremals for the d-tuples of commuting contractions if $d > 2$. On the other hand the extremals for the family of commuting isometries are easily identified as the tuples of commuting unitary operators. The resulting extension theorem is due to Ito [17] and Brehmer [9].

In this paper we will discuss extremals of families that are associated with the unit ball in $\mathbb{C}^d$. A particular emphasis is placed on identifying which spatial and spectral properties of an operator tuple allow nontrivial extensions.

A tuple $U = (U_1, ..., U_d)$ of commuting operators is called spherical unitary if $\sum_{i=1}^d U_i^* U_i = I$ and each U_i is a normal operator. Our first Theorem is the following.

Theorem 1.3. *Let $\mathcal{F}_{si}$ be the family of commuting spherical isometries, then* $\operatorname{ext}(\mathcal{F}_{si})$ *equals the collection of commuting spherical unitaries.*

The resulting extension theorem says that commuting spherical isometries are jointly subnormal and it is due to Athavale [7].

We now turn to spherical and row contractions. An important example of a row contraction is the d-shift $M_z = (M_{z_1}, ..., M_{z_d})$ acting on the Drury-Arveson space H^2_d. H^2_d is the reproducing kernel Hilbert space defined by the kernel

$$k_\lambda(z) = \frac{1}{1 - \langle z, \lambda \rangle} \quad \lambda, z \in \mathbb{B}_d, \quad \langle z, \lambda \rangle = \sum_{i=1}^d z_i \overline{\lambda}_i.$$

H^2_d consists of analytic functions in $\mathbb{B}_d$, and for $d > 1$ it is properly contained in the classical Hardy space $H^2(\partial B_d)$, which has reproducing kernel $\frac{1}{(1-\langle z,\lambda\rangle)^d}$.

Since $M_{z_i}^* k_\lambda = \overline{\lambda}_i k_\lambda$ it follows that

$$\left(\sum_{i=1}^d M_{z_i} M_{z_i}^* k_\lambda\right)(z) = \frac{\langle z, \lambda \rangle}{1 - \langle z, \lambda \rangle} = k_\lambda(z) - 1.$$

This implies that

$$\sum_{i=1}^d M_{z_i} M_{z_i}^* = I - 1 \otimes 1 \le I, \tag{1.1}$$

thus M_z^* is a spherical contraction and M_z is a row contraction.

We say that S is a direct sum of d-shifts if $S = (S_1, ..., S_d)$, $S_i = M_{z_i} \otimes I \in \mathcal{B}(H^2_d \otimes \mathcal{C})$ for some Hilbert space $\mathcal{C}$.

Theorem 1.4. *Let $\mathcal{F}_{sc}$ be the family of commuting spherical contractions, and let $T = (T_1, .., T_d)$ be a commuting operator tuple.*

Then the following are equivalent:

(i) $T \in \operatorname{ext}(\mathcal{F}_{sc})$

(ii) $T = S^* \oplus U$, *where U is spherical unitary and S is a direct sum of d-shifts,*

(iii) (a) $\sum_{i=1}^d T_i^*T_i = P =$ *a projection,*
(b) $\sum_{i=1}^d T_iT_i^* \geq I$,
(c) *If* $x_1, .., x_d \in \mathcal{H}$ *with* $T_ix_j = T_jx_i$*, then there is an* $x \in \mathcal{H}$ *with* $x_i = T_ix$ *for all* i*.*

When we write $T = S^* \oplus U$, we want to include the possibility that one of the summands is absent. Note that (iii)(c) says that the Koszul complex for T is exact at $\Lambda^1(\mathcal{H})$ (Section 3 contains a short summary of elementary facts about the Koszul complex).

The resulting extension theorem (i.e. that any $R \in \mathcal{F}_{sc}$ has an extension T of the type as in (ii)) had been known and is due to Müller-Vasilescu [18] and to Arveson [4]. Arveson also proved that the adjoint of the d-shift is an extremal spherical contraction (see [4] pages 205/206). Among other things his proofs are based on his earlier results ([3]) and an analysis of the C^*-algebra generated by the d-shift and the identity operator. We note that the extremality of S^* also follows directly from Agler's Theorem once the implication (i) $\Rightarrow$ (ii) of Theorem 1.4 has been established.

Indeed, by Agler's Theorem the zero tuple $0 = (0, ..., 0)$ acting on a nonzero space extends to an extremal spherical contraction. By (i) $\Rightarrow$ (ii) there must be an extremal of the type $S^* \oplus U$ such that $0 = S^* \oplus U|\mathcal{M}$ where S is a direct sum of d-shifts, U is a spherical unitary tuple, and $\mathcal{M}$ is invariant for $S^* \oplus U$. If the direct summand S^* were absent, then $0 = U|\mathcal{M}$ would have to be a spherical isometry, which is absurd. Thus $S^* \oplus U$ is extremal and definitely has a d-shift as a direct summand. Now it is easy to verify that if $X \oplus Y$ is extremal for a family $\mathcal{F}$, then both X and Y have to be extremal for $\mathcal{F}$ also. Hence the adjoint of the d-shift must be extremal for $\mathcal{F}_{sc}$. For this paper we decided to present yet another proof of the extremality of S^*, this one based on spatial properties of $S = M_z$ as it acts on H_d^2 (Section 3).

When we apply Theorem 1.4 to the tuple of adjoints we obtain the following Corollary.

Corollary 1.5. *(Wold-decomposition) A d-tuple of commuting operators is of the form* $T = S \oplus U$ *if and only if*
(a) $\sum_{i=1}^d T_iT_i^*$ *is a projection,*
(b) $\sum_{i=1}^d T_i^*T_i \geq I$*, and*
(c) whenever $x_1, ..., x_d \in \mathcal{H}$ *with* $\sum_{i=1}^d T_ix_i = 0$*, then there is an antisymmetric matrix* $\{y_{ij}\}_{1 \leq i,j \leq d}$ *with entries* $y_{ij} \in \mathcal{H}$ *such that* $x_i = \sum_{j=1}^d T_jy_{ij}$ *for each* i *(i.e. the Koszul complex for* T *is exact at* $\Lambda^{d-1}(\mathcal{H})$*).*

We will give a short proof at the end of Section 6. Note that when $d = 1$ and condition (a) is satisfied, then T is a partial isometry. In this case each of the conditions (b) or (c) implies that T is 1-1 and thus T must be an isometry. For $d > 1$ neither conditions (a) and (b) nor conditions (a) and (c) alone will imply that $T = S \oplus U$. In fact if $T = (M_z^*, H^2(\partial\mathbb{B}_d))$ then T^* is a spherical isometry, thus it satisfies (a). Furthermore, it is well-known that the Koszul complex for $(M_z^*, H^2(\partial\mathbb{B}_d))$ is exact at all stages except at the first one, so (c) is satisfied provided $d > 1$ (see e.g. Section 2 of [15]), but (b) is not satisfied.

In order to exhibit an example which satisfies (a) and (b) but not (c) we let $\mathcal{M}_0 = \{f \in H^2_d : f(0) = 0\}$ and $T = S|\mathcal{M}_0$, where S is the d-shift acting on H^2_d. It is clear that T satisfies (b), because S does. Furthermore, $\sum_{i=1}^d S_i S_i^* = P = I - 1 \otimes 1$ is the projection from H^2_d onto $\mathcal{M}_0$ (see (1.1)). Then $\sum_{i=1}^d T_i T_i^* = \sum_{i=1}^d P S_i P S_i^* P = P\left(\sum_{i=1}^d S_i S_i^* - S_i(1 \otimes 1) S_i^*\right) P = P - \sum_{i=1}^d z_i \otimes z_i$ and this is a projection, because $\|z_i\| = 1$ for all i and $z_i \perp z_j$ for all $i \neq j$ (see equation (1.2) below). Thus, T satisfies (a). We will now show that for $d > 1$ T does not satisfy (c). Let $f_1(z) = z_2, f_2(z) = -z_1$, and $f_3 = ... = f_d = 0$. Then $f_i \in \mathcal{M}_0$ and $\sum_{i=1}^d T_i f_i = 0$. If $z_2 = f_1(z) = \sum_{j=1}^d z_j g_{1j}$ for some $g_{1j} \in \mathcal{M}_0$, then we take a partial derivative with respect to z_2 and obtain $1 = g_{12}(z) + \sum_{j=1}^d z_j \frac{\partial}{\partial z_2} g_{1j}(z)$. Evaluating at $z = 0$ we conclude $1 = g_{12}(0)$, but this contradicts $g_{12} \in \mathcal{M}_0$.

For the family of row contractions we have partial results.

Theorem 1.6. *Let $\mathcal{F}_{rc}$ be the family of commuting row contractions. Let $T \in \mathcal{F}_{rc}$ and write $D_* = (I - \sum_{i=1}^d T_i T_i^*)^{1/2}$.*

(i) *If $D_* = 0$, then $T \in \text{ext}(\mathcal{F}_{rc})$.*
(ii) *If D_* is onto, then $T \notin \text{ext}(\mathcal{F}_{rc})$.*
(iii) *If D_* is a projection, then $T \notin \text{ext}(\mathcal{F}_{rc})$ if and only if there are $x_1, .., x_d \in \bigcap_{i=1}^d \ker T_i^*$ with $\sum_{i=1}^d \|x_i\|^2 > 0$ and $T_i x_j = T_j x_i$ for all i, j.*
(iv) *If D_* has rank one, i.e. if $D_* = u \otimes u$ for some $u \neq 0$, then $T \in \text{ext}(\mathcal{F}_{rc})$ if and only if dim span $\{u, T_1 u, .., T_d u\} \geq 3$.*

If $d = 1$, then part (i) of Theorem 1.6 describes all extremals (the co-isometric operators). For $d > 1$ the d-shift is an example of an extremal with $D_* \neq 0$. For the d-shift one verifies that D_* is a projection of rank 1 (see equation (1.1)), so its extremality can be derived either from part (iii) or part (iv) of Theorem 1.6. We will see that in all of the above cases, when T is not extremal, then T actually has a nontrivial rank 1 extension in $\mathcal{F}_{rc}$.

If $S = (M_z, H^2_d)$ is the d-shift, and if $\mathcal{M} \subsetneqq H^2_d$ is invariant for S, then $T = P_{\mathcal{M}^\perp} S|\mathcal{M}^\perp \in \mathcal{F}_{rc}$ and D_* has rank 1. Because of this one can use Theorem 1.6 to verify the following Corollary (see Corollary 8.4).

Corollary 1.7. *Let $\mathcal{F}_{rc}$ be the family of commuting row contractions. If $\mathcal{M} \neq H^2_d$ is an invariant subspace for the d-shift $S = (M_z, H^2_d)$, and if*

$$\mathcal{L} = \{a + \sum_{i=1}^{d} b_i z_i : a, b_1, .., b_d \in \mathbb{C}\}$$

denotes the collection of polynomials of degree less than or equal to one, then $T = P_{\mathcal{M}^\perp} S|\mathcal{M}^\perp \in \mathrm{ext}(\mathcal{F}_{rc})$ if and only if dim $\mathcal{M} \cap \mathcal{L} < d - 1$.

Under the hypothesis of the Corollary one easily checks that $D^2_* = \varphi \otimes \varphi$, where $\varphi = P_{\mathcal{M}^\perp} 1$ (see the proof of Corollary 8.4). Thus D_* is a projection if and only if $1 \in \mathcal{M}^\perp$, and the Corollary can be used to exhibit many examples of extremal row contractions whose defect operators are not projections. For example, if $d = 2$ and $\lambda_i = (\lambda_{i1}, \lambda_{i2})$, $i = 1, 2, 3$ are three distinct points in $\mathbb{B}_2$, then we can let

$$\mathcal{M} = \{f \in H^2_2 : f(\lambda_1) = f(\lambda_2) = f(\lambda_3) = 0\}.$$

In this case $T = (T_1, T_2)$ acts on the 3-dimensional space

$$\mathcal{M}^\perp = \text{ span}\{k_{\lambda_1}, k_{\lambda_2}, k_{\lambda_3}\},$$

and by the Corollary T is extremal if and only if $\mathcal{M} \cap \mathcal{L} = \{0\}$. From this one deduces with a little bit of elementary algebra that T is extremal if and only if

$$(\lambda_{31} - \lambda_{11})(\lambda_{22} - \lambda_{12}) \neq (\lambda_{21} - \lambda_{11})(\lambda_{32} - \lambda_{12}).$$

Hence there are extremal row contractions on finite dimensional spaces that are not spherical unitaries. We also note that the above examples of extremals where the defect operator is not a projection show that part (iii) of Theorem 1.6 does not cover all extremals. This is in contrast to the family $\mathcal{F}_{sc}$ where for all extremals the defect operator $D = \left(I - \sum_{i=1}^d T_i^* T_i\right)^{1/2}$ must be a projection (see Theorem 1.4).

If $d = 1$ and if $\mathcal{F}$ is either the family of contractions or the family of isometries, then any non-extremal operator $T \in \mathcal{F}$ has a nontrivial rank one extension in $\mathcal{F}$. This is well-known and easy to see (compare Lemma 7.2). If $d > 1$, then for each of the families $\mathcal{F}_{sc}, \mathcal{F}_{si}$ and $\mathcal{F}_{rc}$ there is a difference between extremals and operator tuples that allow nontrivial finite rank or rank one extensions. We shall show in Corollary 5.4 that a spherical isometry V has no nontrivial rank one extensions in $\mathcal{F}_{si}$ if and only if the Koszul complex for $V - b$ is exact at Λ^1 for all $b \in \mathbb{B}_d$. We will review definitions and elementary properties of the Koszul complex in Section 3. Furthermore, in Theorem 6.1 we will show that if $T \in \mathcal{F}_{sc}$, then T has no nontrivial finite rank extensions if and only if T has no nontrivial rank one extensions and this happens if and only if $T = S^* \oplus V$, where S is a direct sum of d-shifts and V is a spherical isometry with no nontrivial rank one extension. For the row contractions we only have the following technical condition, and we note that all extension results of Theorem 1.6 follow from this result, i.e. if either D_* is onto, or if $T \in \mathcal{F}_{rc}$ is nonextremal and D_* is a projection or a rank one operator, then T has a nontrivial rank 1 extension in $\mathcal{F}_{rc}$.

Theorem 1.8. *Let $\mathcal{F}_{rc}$ be the family of commuting row contractions, and let $T \in \mathcal{F}_{rc}$.*

The following are equivalent:

(a) T has a nontrivial rank 1 extension in $\mathcal{F}_{rc}$,

(b) T has a nontrivial finite rank extension in $\mathcal{F}_{rc}$,

(c) there are $a_1, .., a_d \in \operatorname{ran} D_*$, $\sum_i \|a_i\| > 0$, and $b = (b_1, .., b_d) \in \mathbb{B}_d$ such that $(T_i - b_i)a_j = (T_j - b_j)a_i$ for all i, j.

In Section 9 we will present an example of a nonextremal commuting row contraction which has no nontrivial finite dimensional extensions.

The remainder of the paper is structured as follows. In Section 2 we will prove Theorem 1.3 and we will see that spherical unitaries are extremal spherical contractions. Section 3 contains a proof that the adjoint of the d-shift is an extremal among the spherical contractions. A basic proposition about spherical contractions of the form $S^* \oplus V$, where S is a sum of d-shifts and V is a spherical isometry will be presented and proved in Section 4. Section 5 contains a theorem characterizing the spherical isometries that have nontrivial rank one extensions (Corollary 5.4) and it also has some preliminary results about rank one and finite rank extensions of spherical contractions. Theorem 6.1 characterizes spherical contractions with nontrivial finite rank extensions and Corollaries 6.2 and 6.3 are Theorems 1.4. and Corollary 1.5. In Section 7 we give our results about finite rank extensions of row contractions and in Section 8 we present our main results about extremals of $\mathcal{F}_{rc}$.

At various places throughout the paper we will use multinomial notation. If $\alpha \in \mathbb{N}_0^d$, then $\alpha = (\alpha_1, ..., \alpha_d)$, $|\alpha| = \alpha_1 + ... + \alpha_d$, $\alpha! = \alpha_1! \cdot ... \cdot \alpha_d!$, and $\begin{pmatrix} |\alpha| \\ \alpha \end{pmatrix} = \frac{|\alpha|!}{\alpha!}$. If $z \in \mathbb{C}^d$ and if $T = (T_1, .., T_d) \in \mathcal{B}(\mathcal{H})^d$, then $z^\alpha = z_1^{\alpha_1} \cdot ... \cdot z_d^{\alpha_d}$ and $T^\alpha = T_1^{\alpha_1} ... T_d^{\alpha_d}$. Furthermore, we will use $e_j = (0, .., 0, 1, 0, .., 0)$ where the 1 is in the jth spot.

The reproducing kernel for the Drury-Arveson space H_d^2 satisfies

$$k_\lambda(z) = \frac{1}{1 - \langle z, \lambda \rangle} = \sum_{n=0}^{\infty} \langle z, \lambda \rangle^n = \sum_{n=0}^{\infty} \sum_{|\alpha|=n} \begin{pmatrix} n \\ \alpha \end{pmatrix} z^\alpha \overline{\lambda}^\alpha = \sum_{\alpha \in \mathbb{N}_0^d} \begin{pmatrix} |\alpha| \\ \alpha \end{pmatrix} z^\alpha \overline{\lambda}^\alpha.$$

Since we also have

$$k_\lambda(z) = \langle k_\lambda, k_z \rangle = \sum_{\alpha \in \mathbb{N}_0^d} \sum_{\beta \in \mathbb{N}_0^d} \begin{pmatrix} |\alpha| \\ \alpha \end{pmatrix} \begin{pmatrix} |\beta| \\ \beta \end{pmatrix} z^\beta \overline{\lambda}^\alpha \langle w^\alpha, w^\beta \rangle_{H_d^2}$$

it follows that

$$\langle w^\alpha, w^\beta \rangle_{H_d^2} = \delta_{\alpha\beta} \frac{1}{\begin{pmatrix} |\alpha| \\ \alpha \end{pmatrix}}, \quad \text{where } \delta_{\alpha\beta} = 0 \text{ if } \alpha \neq \beta, \delta_{\alpha\beta} = 1 \text{ if } \alpha = \beta.$$

From this one deduces that for $f \in \mathrm{Hol}(\mathbb{B}_d)$, $f(z) = \sum_{\alpha \in \mathbb{N}_0^d} \hat{f}(\alpha) z^\alpha$ one has

$$\|f\|_{H_d^2}^2 = \sum_{\alpha \in \mathbb{N}_0^d} \frac{|\hat{f}(\alpha)|^2}{\begin{pmatrix} |\alpha| \\ \alpha \end{pmatrix}} = \sum_{\alpha \in \mathbb{N}_0^d} \frac{\alpha!}{|\alpha|!} |\hat{f}(\alpha)|^2. \tag{1.2}$$

If $\alpha \in {\mathbb{N}_0}^d$, then $\sum_{i=1}^d \|z_i z^\alpha\|^2 = \frac{\alpha!}{|\alpha|!} \frac{|\alpha|+d}{|\alpha|+1}$, hence it follows that

$$\sum_{i=1}^d \|z_i f\|^2 \geq \|f\|^2 \tag{1.3}$$

for all $f \in H_d^2$. Furthermore, one calculates that for $\alpha \in \mathbb{N}_0^d$ and $1 \leq i \leq d$

$$M_{z_i}^* z^\alpha = \frac{\alpha_i}{|\alpha|} z^{\alpha - e_i} \text{ whenever } \alpha_i > 0, \tag{1.4}$$

and $M_{z_i}^* z^\alpha = 0$ if $\alpha_i = 0$.

2. Spherical Isometries

In this Section we will prove Theorem 1.3 and part of the proof of $(ii) \Rightarrow (i)$ of Theorem 1.4.

The fact that extremals of the spherical isometries must be jointly normal follows easily from the arguments of Attele and Lubin [8], who presented an alternate proof of Athavale's Theorem. In fact, let $T =$

$(T_1, ..., T_d)$ be a commuting spherical isometry acting on $\mathcal{H}$ and assume that T_1 is not normal. We must show that T is not extremal, i.e. we have to construct a commuting spherical isometry S that extends T nontrivially. By Corollary 6 of [8] T_1 is a subnormal contraction. Thus we can let $S_1 \in \mathcal{B}(\mathcal{K})$ be the minimal normal extension of T_1. Since we assumed that T_1 is not normal, it is clear that any extension of T of the form $S = (S_1, S_2, ..., S_d)$ will be nontrivial. In order to define $S_2, ..., S_d$ we use the standard extensions

$$S_i\left(\sum_{k=0}^{n} S_1^{*k}x_k\right) = \sum_{k=0}^{n} S_1^{*k}T_i x_k, \ i = 2, ..., d, \ x_0, ..., x_n \in \mathcal{H},$$

see the proof of Proposition 7 of [8]. Since S_1 is normal it is easy to verify that

$$\sum_{i=1}^{d} \|\sum_{k=0}^{n} S_1^{*k}T_i x_k\|^2 = \|\sum_{k=0}^{n} S_1^{*k}x_k\|^2.$$

This implies that $S_2, ..., S_d$ are well defined and extend to $\mathcal{K}$ and $S = (S_1, S_2, ..., S_d)$ forms a spherical isometry. Finally, we see that for all $1 \le i, j \le d$

$$\begin{aligned} S_j S_i \sum_{k=0}^{n} S_1^{*k}x_k &= S_j \sum_{k=0}^{n} S_1^{*k}T_i x_k \\ &= \sum_{k=0}^{n} S_1^{*k}T_j T_i x_k \\ &= S_i S_j \sum_{k=0}^{n} S_1^{*k}x_k. \end{aligned}$$

Thus S forms a tuple of commuting operators, and this proves that the extremals of the spherical isometries must be commuting normals.

For later reference we make some simple observations about extremals.

Lemma 2.1. *Let $\mathcal{F}$ and $\mathcal{G}$ be families.*

(a) Let $U \in \operatorname{ext}(\mathcal{F})$ and $V \in \mathcal{F}$. If $R \in \mathcal{F}$ with $R \ge U \oplus V$, then $R = U \oplus R'$ for some $R' \ge V$, $R' \in \mathcal{F}$.

(b) Finite or infinite direct sums of extremals are extremal.

(c) If $\mathcal{F} \subseteq \mathcal{G}$, then $ext(\mathcal{G}) \cap \mathcal{F} \subseteq \operatorname{ext}(\mathcal{F})$.

Proof. (a) is obvious and it easily implies (b) for finite direct sums. In order to prove (b) for infinite direct sums let $U_n \in \operatorname{ext}(\mathcal{F}) \cap \mathcal{B}(\mathcal{H}_n)^d$, $V = U_1 \oplus U_2 \oplus ...$ and $R \in \mathcal{F} \cap \mathcal{B}(\mathcal{K})^d$ with $R \ge V$. For $n \in \mathbb{N}$ we let P_n be the projection from $\mathcal{K}$ onto $\mathcal{H}_1 \oplus ... \oplus \mathcal{H}_n$. By the finite case we have

$R_iP_n = P_nR_i$ for all i and n. The sequence P_n converges in the strong operator topology to P, the projection from $\mathcal{K}$ onto $\mathcal{H}_1 \oplus \mathcal{H}_2 \oplus ....$ It follows that P commutes with R. Hence V must be extremal.

(c) is immediate. ■

Theorem 2.2. *Commuting spherical unitary tuples are extremal for the families of commuting spherical contractions and commuting spherical isometries.*

Proof. Let $U = (U_1, ..., U_d)$ be a commuting spherical unitary tuple. By Lemma 2.1(c) it suffices to show that U is extremal for the commuting spherical contractions.

Thus let $S \geq U$ be a commuting spherical contraction. Then for each $1 \leq i \leq d$ we have

$$S_i = \begin{pmatrix} U_i & A_i \\ 0 & B_i \end{pmatrix} \in \mathcal{B}(\mathcal{H} \oplus \mathcal{K})$$

with

$$U_iA_j + A_iB_j = U_jA_i + A_jB_i \tag{2.1}$$

for all $1 \leq i, j \leq d$ and

$$\sum_{i=1}^{d} S_i^*S_i = \begin{pmatrix} I & \sum_{i=1}^d U_i^*A_i \\ \sum_{i=1}^d A_i^*U_i & \sum_{i=1}^d A_i^*A_i + B_i^*B_i \end{pmatrix} \leq \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix}.$$

It follows that $\sum_{i=1}^d U_i^*A_i = 0$ and

$$\sum_{i=1}^{d} A_i^*A_i + B_i^*B_i \leq I. \tag{2.2}$$

Let $j \in \{1, ..., d\}$. We shall establish Lemma 2.2 by showing that $A_j = 0$.

By hypothesis each U_i is normal and $U_iU_j = U_jU_i$. Hence it follows from Fuglede's theorem [19] that $U_i^*U_j = U_jU_i^*$. We now apply U_i^* on the left in equation (2.1), sum in i, and obtain

$$\sum_{i=1}^{d} U_i^*U_iA_j + \sum_{i=1}^{d} U_i^*A_iB_j = \sum_{i=1}^{d} U_i^*U_jA_i + \sum_{i=1}^{d} U_i^*A_jB_i.$$

Since $\sum_{i=1}^d U_i^*U_i = I$ and $\sum_{i=1}^d U_i^*A_i = 0$ this implies $A_j = \sum_{i=1}^d U_i^*A_jB_i$. Since U is a spherical contraction it follows that $U^* = (U_1^*, ..., U_d^*)$ is a row contraction. Hence for $x \in \mathcal{K}$, $\|x\| \leq 1$ we have

$$\begin{aligned}\|A_j x\|^2 &= \|\sum_{i=1}^d U_i^* A_j B_i x\|^2 \leq \sum_{i=1}^d \|A_j B_i x\|^2 \\ &\leq \|A_j\|^2 \sum_{i=1}^d \|B_i x\|^2\end{aligned}$$

By equation (2.2) this implies

$$\|A_j x\|^2 \leq \|A_j\|^2 \left(\|x\|^2 - \sum_{i=1}^d \|A_i x\|^2 \right) \leq \|A_j\|^2 (1 - \|A_j x\|^2).$$

We now rearrange the terms to get $\|A_j x\|^2(1 + \|A_j\|^2) \leq \|A_j\|^2$ and after taking the sup over $\|x\| \leq 1$ we obtain $\|A_j\|^2(1+\|A_j\|^2) \leq \|A_j\|^2$ which implies that $A_j = 0$. ∎

3. Extremality of the adjoint of the d-shift

Let $T = (T_1, \ldots, T_d)$ be a commuting tuple of operators on a Hilbert space $\mathcal{H}$. We will now define the Koszul complex of T. We will follow [5]. For more information of a general type on the Koszul complex and its relationship to invertible and Fredholm tuples, the reader is also referred to [10] and [21].

Let $\Lambda = \Lambda[e] = \Lambda_d[e]$ be the exterior algebra generated by the d symbols $e_1, \ldots, e_d$, along with the identity e_0 defined by $e_0 \wedge \xi = \xi$ for all ξ. Then Λ is the algebra of forms in $e_1, \ldots, e_d$ with complex coefficients, subject to the anti-commutative property $e_i \wedge e_j + e_j \wedge e_i = 0$ $(1 \leq i, j \leq d)$. In fact, we can make Λ into a 2^d-dimensional Hilbert space with orthonormal basis

$$\{e_0\} \bigcup \{e_{i_1} \wedge \cdots \wedge e_{i_k} |\, i_j \in \{1, \ldots, d\},\ i_1 < i_2 < \cdots < i_k\}.$$

For each $i = 0, 1, \ldots, d$ let $E_i : \Lambda \to \Lambda$ be given by $E_i \xi = e_i \wedge \xi$. E_0 is thus the identity on Λ. For $i = 1, \ldots, d$ the E_i are called the creation operators and they satisfy the following anticommutation relations

$$E_i E_j + E_j E_i = 0 \quad \text{and} \quad E_i^* E_j + E_j E_i^* = \delta_{ij} E_0.$$

Let $\Lambda(\mathcal{H}) := \mathcal{H} \otimes_{\mathbb{C}} \Lambda$ and define $\partial_T : \Lambda(\mathcal{H}) \to \Lambda(\mathcal{H})$ by

$$\partial_T := \sum_{i=1}^d T_i \otimes E_i.$$

It follows easily from the anticommutation relationships that $\partial_T^2 = 0$. Thus, the Koszul complex of the tuple T can be defined by

$$K(T) : 0 \to \Lambda^0(\mathcal{H}) \xrightarrow{\partial_{T,0}} \Lambda^1(\mathcal{H}) \xrightarrow{\partial_{T,1}} \cdots \xrightarrow{\partial_{T,d-1}} \Lambda^d(\mathcal{H}) \to 0$$

where $\Lambda^p(\mathcal{H})$ is the collection of p forms in $\Lambda(\mathcal{H})$ and $\partial_{T,p} := \partial_T|\Lambda^p(\mathcal{H})$. For purposes of notation we also define $\Lambda^{-1}(\mathcal{H}) = 0$ and $\partial_{T,-1}$ and $\partial_{T,d}$ to be the zero maps at the two ends of the complex.

The identity $\partial_T^2 = 0$ implies that for each $p = 0, 1, ..., d$ ran $\partial_{T,p-1} \subseteq \ker \partial_{T,p}$ and ran $\partial_{T,p}^* \subseteq \ker \partial_{T,p-1}^*$, and one says that the Koszul complex $K(T)$ is exact at $\Lambda^p(\mathcal{H})$, if ran $\partial_{T,p-1} = \ker \partial_{T,p}$. In particular, if $K(T)$ is exact at $\Lambda^p(\mathcal{H})$, then ran $\partial_{T,p-1}$ must be closed, hence $\partial_{T,p-1}\partial_{T,p-1}^*$ is 1-1 and onto when restricted to ran $\partial_{T,p-1} = (\ker \partial_{T,p-1}^*)^\perp$. Furthermore, in this case one also has that ran $\partial_{T,p}^*$ is dense in $\ker \partial_{T,p-1}^*$. It follows that the operator

$$D_{T,p} : \Lambda^p(\mathcal{H}) \to \Lambda^p(\mathcal{H}), \quad D_{T,p} = \partial_{T,p-1}\partial_{T,p-1}^* + \partial_{T,p}^*\partial_{T,p} \tag{3.1}$$

is 1-1 and has dense range whenever $K(T)$ is exact at $\Lambda^p(\mathcal{H})$, and $D_{T,p}$ is invertible if it is also known that ran $\partial_{T,p}$ or what is the same ran $\partial_{T,p}^*$ is closed.

In order to relate properties of the Koszul complex for T with the Koszul complex for T^* we define the Hodge $*$-operator (see [16] for more information on this topic). For $p = 0, 1, ..., d$ we have $\dim \Lambda^p = \dim \Lambda^{d-p}$ and $*$ establishes a conjugate linear isomorphism between Λ^p and Λ^{d-p} that is compatible with ∂. Indeed,

$$*(e_{i_1} \wedge e_{i_2} \wedge ... \wedge e_{i_p}) = (-1)^\varepsilon e_{j_1} \wedge e_{j_2} \wedge ... \wedge e_{j_{d-p}}$$

where $\{e_{i_1}, e_{i_1}, ..., e_{i_p}, e_{j_1}, ..., e_{j_{d-p}}\} = \{e_1, ..., e_d\}$ and $\varepsilon \in \{0, 1\}$ is chosen so that $\eta \wedge *\omega = \langle \eta, \omega \rangle e_1 \wedge ... \wedge e_d$ for all p-forms $\eta, \omega \in \Lambda^p$.

If $\eta \in \Lambda^{p-1}$ and $\omega \in \Lambda^p$, then

$$\begin{aligned}
\eta \wedge (*E_i^*\omega) &= \langle \eta, E_i^*\omega \rangle e_1 \wedge ... \wedge e_d \\
&= \langle E_i\eta, \omega \rangle e_1 \wedge ... \wedge e_d \\
&= E_i\eta \wedge *\omega \\
&= e_i \wedge \eta \wedge *\omega = (-1)^{p-1}\eta \wedge e_i \wedge *\omega \\
&= (-1)^{p-1}\eta \wedge E_i(*\omega)
\end{aligned}$$

Thus for $x \in \mathcal{H}$ and $\omega \in \Lambda^p$ we have

$$\begin{aligned}(I \otimes *)\partial^*_{T,p-1}(x \otimes \omega) &= \sum_{i=1}^d T_i^* x \otimes (*E_i^* \omega) \\ &= (-1)^{p-1} \sum_{i=1}^d T_i^* x \otimes E_i(*\omega) \\ &= (-1)^{p-1} \partial_{T^*,d-p}(I \otimes *)(x \otimes \omega)\end{aligned}$$

Now note that elementary functional analysis results imply that for any p ran $\partial_{T,p}$ is closed if and only if ran $\partial^*_{T,p}$ is closed, and by the above this happens if and only if ran $\partial_{T^*,d-(p+1)}$ is closed. Thus, if it is known for some p that ran $\partial_{T,p}$ is closed and $K(T)$ is exact at Λ^p, then $K(T^*)$ is exact at Λ^{d-p}.

It is known that if $\alpha > 0$ and $\mathcal{K}_\alpha$ is the Hilbert space of analytic functions on $\mathbb{B}_d$ with reproducing kernel $k_\lambda(z) = (1-\langle z, \lambda\rangle)^{-\alpha}$, then the Koszul complex for $S = (M_z, \mathcal{K}_\alpha)$ is exact at all stages $p = 0, .., d-1$ and at the last stage we have $\dim \ker \partial_{S,d} / \text{ran } \partial_{S,d-1} = 1$ (see Proposition 2.6 of [15]). Thus, ran $\partial_{S,p}$ is closed for all p. From this and the above remarks about the Hodge $*$-operator it follows that for $T = S^*$ we have ran $\partial_{T,p}$ is closed for all p and $K(T)$ is exact at $\Lambda^p(\mathcal{H})$ for each $p \geq 1$.

In the following let T be a commuting d-tuple of operators on $\mathcal{H}$. We will later take T so that $(T^*, \mathcal{H}) = (M_z, H^2_d)$ is the d-shift. Suppose that we have an extension $R \in \mathcal{B}(\mathcal{H} \oplus \mathcal{K})^d$ of T. Then

$$R_i = \begin{pmatrix} T_i & A_i \\ 0 & B_i \end{pmatrix}.$$

Define

$$\begin{aligned}\partial_T &: \Lambda(\mathcal{H}) \to \Lambda(\mathcal{H}), \quad \partial_T = \sum_{i=1}^d T_i \otimes E_i \\ \partial_A &: \Lambda(\mathcal{K}) \to \Lambda(\mathcal{H}), \quad \partial_A = \sum_{i=1}^d A_i \otimes E_i \\ \partial_B &: \Lambda(\mathcal{K}) \to \Lambda(\mathcal{K}), \quad \partial_B = \sum_{i=1}^d B_i \otimes E_i\end{aligned}$$

Lemma 3.1. *If R is a tuple of commuting operators, then*

$$\partial_T \partial_A + \partial_A \partial_B = 0.$$

Proof. Since $\Lambda(\mathcal{H} \oplus \mathcal{K}) = \Lambda(\mathcal{H}) \oplus \Lambda(\mathcal{K})$ we can write

$$\partial_R = \begin{pmatrix} \partial_T & \partial_A \\ 0 & \partial_B \end{pmatrix}.$$

The Lemma follows easily, because the expression equals the (1,2)-entry of the matrix for $\partial_R^2 = 0$. ∎

Proposition 3.2. *Let $d \geq 2$ and let $T = (T_1, ..., T_d) \in \mathcal{B}(\mathcal{H})^d$ be a commuting tuple of operators which is graded in the sense that there is a decomposition of $\mathcal{H}$ as a direct sum of mutually orthogonal subspaces,*

$$\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus ...$$

such that $T_j(\mathcal{H}_0) = (0)$ and $T_j(\mathcal{H}_n) \subseteq \mathcal{H}_{n-1}$ for all $n \geq 1$ and all $1 \leq j \leq d$. Assume that the Koszul complex $K(T)$ is exact at $\Lambda^p(\mathcal{H})$ for $p = 1$ and $p = 2$.

If $R \in \mathcal{B}(\mathcal{H} \oplus \mathcal{K})^d$ is a commuting extension of T of the form

$$R_i = \begin{pmatrix} T_i & A_i \\ 0 & B_i \end{pmatrix}, \quad i = 1, ..., d,$$

and if $\sum_{j=1}^d A_j^* T_j = 0$, then $A_i = 0$ for all $i = 1, ..., d$.

Proof. We start by noting that in terms of the Koszul complex the hypothesis $\sum_{j=1}^d A_j^* T_j = 0$ can be restated as $\partial_{A,0}^* \partial_{T,0} = 0$ and that we have to show that $\partial_{A,0}^* = 0$.

Since $T_j(\mathcal{H}_n) \subseteq \mathcal{H}_{n-1}$ we see that $\partial_{T,p}(\Lambda^p(\mathcal{H}_n)) \subseteq \Lambda^{p+1}(\mathcal{H}_{n-1})$ for all $n \geq 1$. Furthermore, one easily checks that for all $n \geq 0$ and all $1 \leq j \leq d$ we have $T_j^*(\mathcal{H}_n) \subseteq \mathcal{H}_{n+1}$. Thus for each p the selfadjoint operators $\partial_{T,p-1}\partial_{T,p-1}^*$ and $D_{T,p}$ (see equation (3.1)) leave $\Lambda^p(\mathcal{H}_n)$ invariant, and the hypothesis implies that

$$D_{T,1}(\Lambda^1(\mathcal{H}_n)) = \Lambda^1(\mathcal{H}_n) \tag{3.2}$$

and that

$$D_{T,2}(\Lambda^2(\mathcal{H}_n)) \text{ is dense in } \Lambda^2(\mathcal{H}_n) \tag{3.3}$$

for each $n \geq 0$.

Define $C = \partial_{T,1}\partial_{A,0}$, then

$$C^* \partial_{T,1} = \partial_{A,0}^* D_{T,1}, \tag{3.4}$$

because $\partial_{A,0}^* \partial_{T,0} \partial_{T,0}^* = 0$. We also have

$$\begin{aligned} C^* D_{T,2} &= \partial_{A,0}^* \partial_{T,1}^* (\partial_{T,1}\partial_{T,1}^* + \partial_{T,2}^*\partial_{T,2}) = \partial_{A,0}^* \partial_{T,1}^* \partial_{T,1} \partial_{T,1}^* \\ &= -\partial_{B,0}^* \partial_{A,1}^* \partial_{T,1} \partial_{T,1}^*, \end{aligned} \tag{3.5}$$

by Lemma 3.1 and because $\partial_{T,1}^* \partial_{T,2}^* = 0$.

We shall now show inductively that

$$\partial^*_{A,0}(\Lambda^1(\mathcal{H}_n)) = (0) \text{ and } C^*(\Lambda^2(\mathcal{H}_n)) = (0) \text{ for each } n \geq 0.$$

We start by applying (3.4) to $\Lambda^1(\mathcal{H}_0)$. Since $\partial_{T,1}(\Lambda^1(\mathcal{H}_0)) = 0$ and $D_{T,1}(\Lambda^1(\mathcal{H}_0)) = \Lambda^1(\mathcal{H}_0)$ we have $\partial^*_{A,0}(\Lambda^1(\mathcal{H}_0)) = 0$. This implies that for each i we have $A_i^*(\mathcal{H}_0) = 0$, thus $\partial^*_{A,1}\partial_{T,1}\partial^*_{T,1}(\Lambda^2(\mathcal{H}_0)) = 0$. In light of (3.5) and (3.3) this means that $C^* = 0$ on a dense subset of $\Lambda^2(\mathcal{H}_0)$, hence $C^*(\Lambda^2(\mathcal{H}_0)) = (0)$.

Next suppose that for some $n \geq 0$ we have $\partial^*_{A,0}(\Lambda^1(\mathcal{H}_n)) = (0)$ and $C^*(\Lambda^2(\mathcal{H}_n)) = 0$. Then since $\partial_{T,1}(\Lambda^1(\mathcal{H}_{n+1})) \subseteq (\Lambda^2(\mathcal{H}_n))$ we can use (3.4) and the induction hypothesis to see that $0 = \partial^*_{A,0}D_{T,1}(\Lambda^1(\mathcal{H}_{n+1})) = \partial^*_{A,0}(\Lambda^1(\mathcal{H}_{n+1}))$. Thus, for each i we have $A_i^*(\mathcal{H}_{n+1}) = 0$ and so

$$\partial^*_{A,1}\partial_{T,1}\partial^*_{T,1}(\Lambda^2(\mathcal{H}_{n+1})) = 0.$$

In light of (3.5) and (3.3) this means that $C^* = 0$ on a dense subset of $\Lambda^2(\mathcal{H}_{n+1})$, hence $C^*(\Lambda^2(\mathcal{H}_{n+1})) = (0)$. ∎

Theorem 3.3. *If $S = (M_z, H^2_d)$, then S^* is extremal for the family of spherical contractions.*

Proof. Set $T = S^*$ and let $R \in \mathcal{B}(H^2_d \oplus \mathcal{K})^d$ be a commuting spherical contraction which extends T. Then R is of the form $R = (R_1, ..., R_d)$,

$$R_i = \begin{pmatrix} T_i & A_i \\ 0 & B_i \end{pmatrix}.$$

We shall use Proposition 3.2 to show that each A_i equals 0. To this end let $\mathcal{H}_n$ be the homogeneous polynomials of degree n, so that $H^2_d = \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus ...$, $T_j(\mathcal{H}_0) = (0)$ and $T_j(\mathcal{H}_n) \subseteq \mathcal{H}_{n-1}$ for all $n \geq 1$ and all $1 \leq j \leq d$ (see (1.4)). We noted earlier in this Section that the Koszul complex $K(T) = K(S^*)$ is exact at stages 1 and 2. Thus by Proposition 3.2 it suffices to show that $\sum_{j=1}^d A_j^* T_j = 0$.

The condition $I - \sum_{i=1}^d R_i^* R_i \geq 0$ implies that for all $f \in H^2_d$ and $y \in \mathcal{K}$ we have
(3.6)

$$\|f\|^2 - \sum_{i=1}^d \|T_i f\|^2 - 2 \text{ Re } \langle \sum_{i=1}^d A_i^* T_i f, y\rangle + \|y\|^2 - \sum_{i=1}^d \left(\|A_i y\|^2 + \|B_i y\|^2\right) \geq 0.$$

Now we recall that $I - \sum_{i=1}^d T_i^* T_i = I - \sum_{i=1}^d S_i S_i^*$ equals the projection onto $\mathcal{H}_0$ (the constants, see equation (1.1)). Thus, if $f \perp \mathcal{H}_0$, then (3.6) implies that $\sum_{i=1}^d A_i^* T_i f = \sum_{i=1}^d A_i^* S_i^* f = 0$. If $f \in \mathcal{H}_0$, then $S_i^* f = 0$ for all $1 \leq i \leq d$, hence $\sum_{i=1}^d A_i^* T_i = 0$. ∎

4. A proposition about tuples of the type $S^* \oplus V$

In this Section we shall prove the following Proposition which will easily imply one of the implications of Theorem 1.4.

Proposition 4.1. *Let $T \in \mathcal{B}(\mathcal{H})^d$ be a commuting operator tuple which satisfies the following two conditions:*
(a) $\sum_{i=1}^d T_i^* T_i$ is a projection, and
(b) if $x_1, .., x_d \in \mathcal{H}$ with $T_i x_j = T_j x_i$ for all i, j, then there is an $x \in \mathcal{H}$ with $x_i = T_i x$ for all i.
Then T is unitarily equivalent to $S^* \oplus V$, where S is a direct sum of d-shifts, and V is a spherical isometry.

Note that for $d = 1$ condition (b) says that T is surjective, while condition (a) implies that T is a partial isometry. Thus T must be a coisometry, and if $T^* = S \oplus V^*$, then V^* must be isometric. This means that for $d = 1$ the operator V in the Proposition is automatically unitary. If $d > 1$, then the operator tuple $T = (M_z, H^2(\partial B_d))$ provides an example of a d-tuple that satisfies conditions (a) and (b) of the Proposition, but is not a spherical unitary tuple.

Let $\mathcal{E}_0 = \bigcap_{i=1}^d \ker T_i$. Inductively define a sequence of positive operators by

$$P_0 = I \text{ and } P_{N+1} = \sum_{i=1}^d T_i^* P_N T_i \text{ for } N = 0, 1, ... \tag{4.1}$$

One verifies that for $N \geq 1$

$$P_N = \sum_{|\alpha|=N} \binom{N}{\alpha} T^{*\alpha} T^\alpha.$$

Hence it follows that $\ker P_N = \bigcap_{|\alpha|=N} \ker T^\alpha$, and $\mathcal{E}_0 = \ker P_1$.

Note that for $N \geq 1$ we have $P_N - P_{N+1} = \sum_{i=1}^d T_i^*(P_{N-1} - P_N)T_i$. Hence part (a) of the hypothesis and an induction argument imply that $\{P_N\}_{N \in \mathbb{N}}$ is a non-increasing sequence of positive operators which thus converges strongly to a positive operator P. Our first step will be to show that P and each P_N are projections, and that $T_i P = P T_i$ for all $1 \leq i \leq d$. This means that $\mathcal{M} = \operatorname{ran} P$ reduces each T_i and we will see that $T|\mathcal{M}$ is a spherical isometry and that $T^*|\mathcal{M}^\perp$ is unitarily equivalent to the d-shift acting on $H^2_d(\mathcal{E}_0)$.

Lemma 4.2. *Let $T \in \mathcal{B}(\mathcal{H})^d$ be as in Proposition 4.1. Then for each $N \in \mathbb{N}$ the operator P_N is a projection such that $T_i P_N = P_{N-1} T_i$ for all $i \in \{1, ..., d\}$.*

Proof. We will start by using induction on $N \in \mathbb{N}$ to show that $T_i P_N = P_{N-1} T_i$ for all $i \in \{1, ..., d\}$. The hypothesis (a) of Proposition 4.1 implies that P_1 is a projection, hence ran $(I - P_1) = \ker P_1 = \bigcap_{i=1}^d \ker T_i$. This implies $T_i(I - P_1) = 0$ for all $i \in \{1, ..., d\}$. Thus $P_0 T_i = T_i = T_i P_1$ and the statement is true for $N = 1$.

Next suppose that $N > 1$ and that $T_i P_{N-1} = P_{N-2} T_i$ for all $i \in \{1, ..., d\}$. For $x \in \mathcal{H}$ and $j \in \{1, .., d\}$ set $z_j = P_{N-1} T_j x$. Then for all i and j we have

$$T_i z_j = T_i P_{N-1} T_j x = P_{N-2} T_i T_j x = P_{N-2} T_j T_i x = T_j z_i.$$

Thus the hypothesis (b) of Proposition 4.1 implies that there exists $y \in \mathcal{H}$ such that $P_{N-1} T_j x = z_j = T_j y$ for all j. Then for all i we have

$$T_i P_N x = T_i \sum_{j=1}^d T_j^* P_{N-1} T_j x = T_i \sum_{j=1}^d T_j^* T_j y = T_i P_1 y = T_i y = P_{N-1} T_i x.$$

Thus $T_i P_N = P_{N-1} T_i$ for all $N \in \mathbb{N}$ and $i \in \{1, ..., d\}$. For $N > 1$ we can iterate this to obtain

$$T_i T_j P_N = T_i P_{N-1} T_j = P_{N-2} T_i T_j$$

for all i, j. Continuing the same way and using that $P_0 = I$ we see that for all $N \in \mathbb{N}$ and all multiindices $\alpha \in \mathbb{N}_0^d$ with $|\alpha| = N$ we have $T^\alpha P_N = T^\alpha$. Thus

$$P_N^2 = \left(\sum_{|\alpha|=N} \binom{N}{\alpha} T^{*\alpha} T^\alpha \right) P_N = \sum_{|\alpha|=N} \binom{N}{\alpha} T^{*\alpha} T^\alpha = P_N,$$

which shows that each P_N is a projection. ■

Lemma 4.3. *Let T be a commuting operator tuple on $\mathcal{H}$ which satisfies condition (b) of Proposition 4.1, and let $N \in \mathbb{N}$. Suppose that for each $\alpha \in \mathbb{N}^d$ with $|\alpha| = N$ we are given an element $x_\alpha \in \mathcal{H}$.*

Then there is an $x \in \mathcal{H}$ such that $x_\alpha = T^\alpha x$ for all $|\alpha| = N$ if and only if

(4.2)
$$T_i x_{\beta+e_j} = T_j x_{\beta+e_i} \text{ for all } 1 \le i, j \le d \text{ and all } \beta \in \mathbb{N}_0^d \text{ with } |\beta| = N - 1.$$

Proof. It is clear that if $x_\alpha = T^\alpha x$ for all $|\alpha| = N$, then $T_i x_{\beta+e_j} = T_j x_{\beta+e_i}$ for all $1 \le i, j \le d$ and all $\beta \in \mathbb{N}_0^d$ with $|\beta| = N - 1$. We will use induction on N to verify the sufficiency of condition (4.2). For $N = 1$ this is just the hypothesis of the lemma.

Suppose that the lemma holds for $N \ge 1$, and suppose that $\{x_\alpha\}_{|\alpha|=N+1}$ satisfies (4.2) with $N + 1$ instead of N. Let $|\beta| = N$. For $i = 1, .., d$ set $z_i = x_{\beta+e_i}$. Then we have $T_j z_i = T_i z_j$ for all i and j. Thus by our hypothesis on T there exists $x_\beta \in \mathcal{H}$ with $x_{\beta+e_i} = z_i = T_i x_\beta$ for all i. The collection $\{x_\beta\}_{|\beta|=N}$ satisfies (4.2). Indeed, let $|\gamma| = N-1$ and $1 \le i, j \le d$, then $T_j x_{\gamma+e_i} = x_{(\gamma+e_i)+e_j} = x_{(\gamma+e_j)+e_i} = T_i x_{\gamma+e_j}$. Hence the induction hypothesis and the construction imply that there is a $x \in \mathcal{H}$ such that $x_\beta = T^\beta x$ and $x_{\beta+e_i} = T_i x_\beta = T^{\beta+e_i}$ for all $|\beta| = N$ and $1 \le i \le d$. Thus, $x_\alpha = T^\alpha x$ for all $|\alpha| = N+1$. ■

Lemma 4.4. *Let $T \in \mathcal{B}(\mathcal{H})^d$ be as in Proposition 4.1 and let $N \in \mathbb{N}$.*

Then for all $x \in \mathcal{E}_0$ and all $\alpha, \beta \in \mathbb{N}_0^d$ with $|\alpha| = |\beta| = N$ we have

$$\sqrt{\binom{N}{\alpha}}\sqrt{\binom{N}{\beta}}T^\alpha T^{*\beta}x = \delta_{\alpha\beta}x,$$

where $\delta_{\alpha\beta} = 1$ if $\alpha = \beta$ and $\delta_{\alpha\beta} = 0$ otherwise.

Of course, the definition of $\mathcal{E}_0$ then immediately implies that $T^\gamma T^{*\beta}x = 0$ for all $x \in \mathcal{E}_0$ and $\beta, \gamma \in \mathbb{N}_0^d$ with $|\gamma| > N = |\beta|$.

Proof. Define the column operator $T^{(N)} : \mathcal{H} \to \oplus_{|\alpha|=N}\mathcal{H}$ by

$$T^{(N)}x = \left\{\sqrt{\binom{N}{\alpha}}T^\alpha\right\}_{|\alpha|=N}.$$

Then

$$T^{(N)^*}T^{(N)} = \sum_{|\alpha|=N}\binom{N}{\alpha}T^{*\alpha}T^\alpha = P_N.$$

Lemma 4.2 implies that $T^{(N)^*}T^{(N)}$ is a projection and hence it follows that $T^{(N)}T^{(N)^*}$ is the orthogonal projection onto ran $T^{(N)}$.

Now let $x \in \mathcal{E}_0$, fix $\beta \in \mathbb{N}_0^d$ with $|\beta| = N$, and define a column vector $z = \{x_\alpha\}_{|\alpha|=N} \in \oplus_{|\alpha|=N}\mathcal{H}$ by $x_\alpha = 0$ if $\alpha \ne \beta$ and $x_\beta = x$. Then $T_i x_{\gamma+e_j} = 0 = T_j x_{\gamma+e_i}$ for all $1 \le i, j \le d$ and all $|\gamma| = N-1$. Thus it follows from Lemma 4.3 that there is a $w \in \mathcal{H}$ such that $x_\alpha = T^\alpha w$ for all $|\alpha| = N$ and hence $z \in$ ran $T^{(N)}$ and

$$z = T^{(N)}T^{(N)^*}z = \left\{\sqrt{\binom{N}{\alpha}}\sqrt{\binom{N}{\beta}}T^\alpha T^{*\beta}x\right\}_{|\alpha|=N}.$$

The lemma now follows from the definition of z. ■

Proof of Proposition 4.1. From Lemma 4.2 and the remarks preceding it we know that the sequence $\{P_N\}_{\mathbb{N}\in\mathbb{N}}$ forms a decreasing sequence of projections. Let P denote the strong limit of this sequence. Then P is a projection and the assertion $P_{N-1}T_i = T_i P_N$ of Lemma 4.2 implies that $PT_i = T_i P$ for all $1 \le i \le d$. Thus $\mathcal{M} =$ ran P reduces T and the identity $P = \sum_{i=1}^d T_i^* P T_i$ shows that $T|\mathcal{M}$ is a spherical isometry.

Let S denote the d-shift acting on $H^2_d(\mathcal{E}_0)$, then S is unitarily equivalent to $M_z\otimes I$ acting on $H^2_d\otimes\mathcal{E}_0$. We will show that $M_z\otimes I$ is unitarily equivalent to $T^*|\mathcal{M}^\perp$. Since $\mathcal{E}_0\subseteq\mathcal{M}^\perp$ we can define a linear transformation $U: H^2_d\otimes\mathcal{E}_0\to\mathcal{M}^\perp$ by setting $U(p\otimes x)=p(T^*)x$ for every polynomial p and $x\in\mathcal{E}_0$. Note that for $x,y\in\mathcal{E}_0$ and $\alpha,\beta\in\mathbb{N}_0^d$ Lemma 4.4 implies

$$\sqrt{\binom{|\alpha|}{\alpha}}\sqrt{\binom{|\beta|}{\beta}}\langle T^{*\alpha}x,T^{*\beta}y\rangle=\delta_{\alpha\beta}\langle x,y\rangle.$$

Thus if $p(z)=\sum_\alpha\hat{p}(\alpha)z^\alpha$ and $q(z)=\sum_\beta\hat{q}(\beta)z^\beta$ are polynomials, then for all $x,y\in\mathcal{E}_0$ we have

$$\begin{aligned}\langle p(T^*)x,q(T^*)y\rangle&=\sum_{\alpha,\beta}\hat{p}(\alpha)\overline{\hat{q}(\beta)}\langle T^{*\alpha}x,T^{*\beta}y\rangle\\&=\sum_\alpha\frac{\hat{p}(\alpha)\overline{\hat{q}(\alpha)}}{\binom{|\alpha|}{\alpha}}\ \langle x,y\rangle=\langle p,q\rangle_{H^2_d}\ \langle x,y\rangle,\end{aligned}$$

where the identity for the H^2_d-inner product follows from (1.2). This implies that

$$\begin{aligned}\|U(\sum_j p_j\otimes x_j)\|^2&=\sum_{i,j}\langle p_i(T^*)x_i,p_j(T^*)x_j\rangle=\sum_{i,j}\langle p_i,p_j\rangle_{H^2_d}\ \langle x_i,x_j\rangle\\&=\|\sum_j p_j\otimes x_j\|^2_{H^2_d\otimes\mathcal{E}_0},\end{aligned}$$

thus U extends to be an isometric operator on $H^2_d\otimes\mathcal{E}_0$.

We shall now finish the proof by showing that U has dense range. Let $[\mathcal{E}_0]_{T^*}$ denote the smallest common invariant subspace for $T_1^*,...,T_d^*$ that contains $\mathcal{E}_0$. We have to show that $[\mathcal{E}_0]_{T^*}=\mathcal{M}^\perp=\text{ran }(I-P)$. For $k\ge 0$ we set $Q_k=P_k-P_{k+1}$, then each Q_k is a projection and $I-P=\lim_{N\to\infty}I-P_N=\sum_{k=0}^\infty Q_k$. Note that ran $Q_0=\mathcal{E}_0$. Thus if we define $\mathcal{E}_k=\text{ran }Q_k$ then we must show that for each $k\ge 0$ we have $\mathcal{E}_k\subseteq[\mathcal{E}_0]_{T^*}$. This is trivially true for $k=0$. Thus assume that $\mathcal{E}_k\subseteq[\mathcal{E}_0]_{T^*}$ for some $k\ge 0$. Then for $x\in\mathcal{H}$ we have

$$Q_{k+1}x=\sum_{i=1}^d T_i^*Q_kT_ix\in\sum_{i=1}^d T_i^*\mathcal{E}_k\subseteq[\mathcal{E}_0]_{T^*}.$$

Hence the density of ran U in $\mathcal{M}^\perp$ follows by induction. ■

5. Finite rank extensions of spherical contractions and isometries

We start out with a trivial lemma that will be used repeatedly and without further mention.

Lemma 5.1. *Let T be a commuting d-tuple of operators acting on a Hilbert space $\mathcal{H}$ and let $R = (R_1, ..., R_d)$ be a nontrivial rank one extension of T acting on $\mathcal{H} \oplus \mathbb{C}$ i.e.*

$$R_i = \begin{pmatrix} T_i & A_i \\ 0 & B_i \end{pmatrix},$$

where $A_i 1 = \varepsilon x_i$ and $B_i 1 = b_i$ for some $\varepsilon > 0$, $x_1, ..., x_d \in \mathcal{H}$, $\sum_{i=1}^d \|x_i\|^2 = 1$, and $b = (b_1, ..., b_d) \in \mathbb{C}^d$.

Then R is a commuting d-tuple if and only if for all i, j we have $(T_i - b_i)x_j = (T_j - b_j)x_i$. ■

The following two lemmas are only preliminary results. A more definitive result for spherical contractions will be presented in Theorem 6.1, the result about spherical isometries will follow in Corollary 5.4.

Lemma 5.2. *Let $\mathcal{F}_{sc}$ be the family of commuting spherical contractions, let $T \in \mathcal{F}_{sc} \cap \mathcal{B}(\mathcal{H})^d$, and let $D = (I - \sum_{i=1}^d T_i^* T_i)^{1/2}$.*

Then T has a nontrivial rank one extension in $\mathcal{F}_{sc}$ if and only if there exist $b = (b_1, ..., b_d) \in \mathbb{C}_d$, $x_1, ..., x_d \in \mathcal{H}$ such that

(i) $\sum_{i=1}^d \|x_i\|^2 = 1$,
(ii) $(T_i - b_i)x_j = (T_j - b_j)x_i$ *for all* i, j
(iii) $|b| < 1$, *and*
(iv) $\sum_{i=1}^d T_i^* x_i \in \operatorname{ran} D$.

Proof. Let R be a commuting rank 1 extension of T as in Lemma 5.1. Let $x \in \mathcal{H}$ and $y \in \mathbb{C}$ and calculate

$$\begin{aligned}\sum_{i=1}^d \|R_i \begin{pmatrix} x \\ y \end{pmatrix}\|^2 &= \sum_{i=1}^d \|T_i x + \varepsilon x_i y\|^2 + |b|^2|y|^2 \\ &= \sum_{i=1}^d \|T_i x\|^2 + 2\varepsilon \operatorname{Re} \overline{y}\langle x, \sum_{i=1}^d T_i^* x_i\rangle + (\varepsilon^2 \sum_{i=1}^d \|x_i\|^2 + |b|^2)|y|^2\end{aligned}$$

We now set $x_0 = \sum_{i=1}^d T_i^* x_i$ and recall $\sum_{i=1}^d \|x_i\|^2 = 1$. Then we see that R is a spherical contraction if and only if for all $x \in \mathcal{H}$ and $y \in \mathbb{C}$ we have

$$2\varepsilon \operatorname{Re} \overline{y}\langle x, x_0\rangle + (\varepsilon^2 + |b|^2)|y|^2 \leq \|Dx\|^2 + |y|^2. \tag{5.1}$$

Assume now that R is a spherical contraction for some $\varepsilon > 0$, then clearly $|b| < 1$. By changing the argument of y if necessary, it follows from (5.1) that $2\varepsilon|y||\langle x, x_0\rangle| \le \|Dx\|^2 + |y|^2$ for all $x \in \mathcal{H}$ and $y \in \mathbb{C}$. Thinking of this as a quadratic inequality in $|y|$ we conclude that $\varepsilon^2|\langle x, x_0\rangle|^2 \le \|Dx\|^2$ for all $x \in \mathcal{H}$. By the Douglas lemma ([12]) $\sum_{i=1}^d T_i^* x_i = x_0 \in \operatorname{ran} D$. This proves the necessity of the four conditions.

Conversely assume (i)-(iv). In particular, $x_0 = Dz_0$ for some $z_0 \in \mathcal{H}$. Then

$$\begin{aligned} 2\varepsilon \operatorname{Re} \overline{y}\langle x, x_0\rangle + (\varepsilon^2 + |b|^2)|y|^2 &\le 2\varepsilon|y\langle Dx, z_0\rangle| + (\varepsilon^2 + |b|^2)|y|^2 \\ &\le \varepsilon\|Dx\|^2\|z_0\|^2 + (\varepsilon + \varepsilon^2 + |b|^2)|y|^2, \end{aligned}$$

which will be $\le \|Dx\|^2 + |y|^2$ for sufficiently small $\varepsilon > 0$. ■

Lemma 5.3. *Let $\mathcal{F}_{si}$ be the family of commuting spherical isometries and let $T \in \mathcal{F}_{si} \cap \mathcal{B}(\mathcal{H})^d$. Then the following are equivalent:*

(a) *T has a nontrivial rank one extension in $\mathcal{F}_{si}$,*
(b) *T has a nontrivial rank one extension in $\mathcal{F}_{sc}$,*
(c) *there exist $b = (b_1, ..., b_d) \in \mathbb{C}_d$ and $x_1, ..., x_d \in \mathcal{H}$ such that*
  (i) *$\sum_{i=1}^d \|x_i\|^2 = 1$,*
  (ii) *$(T_i - b_i)x_j = (T_j - b_j)x_i$ for all i, j*
  (iii) *$|b| < 1$, and*
  (iv) *$\sum_{i=1}^d T_i^* x_i = 0$.*

Proof. (i) $\Rightarrow$ (ii) is trivial and (ii) $\Rightarrow$ (iii)follows immediately from Lemma 5.2, because $D = 0$. The implication (iii) $\Rightarrow$ (i) follows from the proof of Lemma 5.2. Indeed since $D = 0$ and $x_0 = 0$ we can take $\varepsilon = \sqrt{1 - |b|^2}$ to obtain equality in (5.1). ■

Using the definition as given e.g. in Section 3 one checks that the Koszul complex for a commuting operator tuple $R = (R_1, ..., R_d)$ acting on $\mathcal{H}$ is exact at $\Lambda^1(\mathcal{H})$ if and only if whenever $x_1, ..., x_d \in \mathcal{H}$ are such that $R_i x_j = R_j x_i$ for all i, j, then there is an $x \in \mathcal{H}$ such that $x_i = R_i x$ for all i.

Corollary 5.4. *Let $T \in \mathcal{B}(\mathcal{H})^d$ be a commuting spherical isometry. Then the following are equivalent:*

(i) *T has a nontrivial rank one extension in $\mathcal{F}_{si}$,*
(ii) *T has a nontrivial rank one extension in $\mathcal{F}_{sc}$,*
(iii) *there exists $b \in \mathbb{B}_d$ such that the Koszul complex for $T - b$ is not exact at $\Lambda^1(\mathcal{H})$.*

For example, if $M_z = (M_z, H^2(\partial\mathbb{D}))$ is the unilateral shift, then $T = (M_z, 0)$ is a commuting spherical isometry. One easily checks that for $(b_1, b_2) \in \mathbb{B}_2$ the Koszul complex for $T - b$ is exact at $\Lambda^1(H^2(\partial\mathbb{D}))$ if and only if $b_2 \neq 0$, and since M_z has a nontrivial rank 1 extension it is of course clear that T has a nontrivial rank one extension. On the other hand, if $d > 1$ and if $T = M_z = (M_{z_1}, ..., M_{z_d})$ acting on $H^2(\partial\mathbb{B}_d)$, then it is known that the Koszul complex for $M_z - b$ is exact at $\Lambda^1(H^2(\partial\mathbb{B}_d))$ for all $b \in \mathbb{B}_d$ (see e.g. Proposition 2.6 of [15]). Thus M_z does not have a nontrivial rank one extension in $\mathcal{F}_{si}$.

Proof. We have already seen the equivalence of (i) and (ii). To prove (i) $\Rightarrow$ (iii) suppose that T has a nontrivial rank one extension in $\mathcal{F}_{si}$, then there exist a $b \in \mathbb{B}_d$ and $x_1, ..., x_d \in \mathcal{H}$ such that (i)-(iv) of Lemma 5.3 (c) are satisfied. We will show that the Koszul complex for $T - b$ is not exact at $\Lambda^1(\mathcal{H})$. If it were exact, then by (ii) and the definition of exactness at $\Lambda^1(\mathcal{H})$ there is an $x \in \mathcal{H}$ such that $x_i = (T_i - b_i)x$ for all i. By (i) we have $x \neq 0$, and by (iv) we have

$$0 = \sum_{i=1}^{d} T_i^* x_i = \sum_{i=1}^{d} T_i^*(T_i - b_i)x = x - \sum_{i=1}^{d} T_i^* b_i x.$$

Thus $\|x\|^2 = \|\sum_{i=1}^{d} T_i^* b_i x\|^2 \leq \sum_{i=1}^{d} \|b_i x\|^2 = |b|^2\|x\|^2$, because the adjoints of spherical isometries must be row contractions. Since $x \neq 0$ we conclude $|b| \geq 1$ which is a contradiction. Hence the Koszul complex for $T - b$ cannot be exact at $\Lambda^1(\mathcal{H})$.

We now prove (iii) $\Rightarrow$ (i). Suppose that $b \in \mathbb{B}_d$ such that the Koszul complex for $T - b$ is not exact at $\Lambda^1(\mathcal{H})$. First we will assume that $b = 0$. Since T is a spherical isometry, the range of $\partial_0 : \mathcal{H} \to \mathcal{H} \oplus ... \oplus \mathcal{H}$, $x \to (T_1 x, ..., T_d x)$ must be closed. Thus if the Koszul complex is not exact at $\Lambda^1(\mathcal{H}) \simeq \mathcal{H} \oplus ... \oplus \mathcal{H}$, then there is $(x_1, ..., x_d) \perp \operatorname{ran} \partial_0$ such that $\sum_{i=1}^{d} \|x_i\|^2 = 1$ and $T_i x_j = T_j x_i$ for all i, j. Then $\sum_{i=1}^{d} T_i^* x_i = \partial_0^*(x_1, ..., x_d) = 0$, so (i)-(iv) of Lemma 5.3 (c) are satisfied with $b = 0$ and hence T has a nontrivial rank one extension in $\mathcal{F}_{si}$.

If $b \neq 0$, then we consider a ball automorphism φ_b that takes b to 0. As in the paragraph preceding Lemma 2.4 of [15] we can define $S = \varphi_b(T)$. Then one checks that S is a commuting spherical isometry. Thus it is clear that T has a nontrivial rank one extension in $\mathcal{F}_{si}$ if and only if S has a nontrivial rank one extension in $\mathcal{F}_{si}$. By Lemma 2.4 of [15] the Koszul complex for S is isomorphic to the Koszul complex for $T - b$. Hence the result follows from the case $b = 0$. ∎

If $\mathcal{F}$ is a family and if an operator tuple $T \in \mathcal{F}$ has a nontrivial finite rank extension $R \in \mathcal{F}$ acting on $\mathcal{H} \oplus \mathcal{K}$, then the compressions of the R_i to $\mathcal{K}$ will have a common eigenvector x_0 and $R' = R|(\mathcal{H} \oplus \mathbb{C}x_0)$ will be a rank one extension of T in $\mathcal{F}$. However, it may happen that R'

is a trivial extension of T. For such situations the following lemma is useful.

Lemma 5.5. *Let $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$ be Hilbert spaces, let $\mathcal{K} = \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus \mathcal{H}_3$, and let $R = (R_1, ..., R_d)$ be a operator tuple acting on $\mathcal{K}$ with matrix representation of the form*

$$R_i = \begin{pmatrix} T_i & 0 & A_i \\ 0 & B_i & C_i \\ 0 & 0 & D_i \end{pmatrix}.$$

If R is a commuting spherical contraction (resp. commuting row contraction), then $S = (S_1, ..., S_d)$,

$$S_i = \begin{pmatrix} T_i & A_i \\ 0 & D_i \end{pmatrix}$$

is a commuting spherical contraction (resp. commuting row contraction).

Proof. Write $\mathcal{H}_{13} = \mathcal{H}_1 \oplus \mathcal{H}_3$ and note that $S = P_{\mathcal{H}_{13}} R | \mathcal{H}_{13}$. The contractiveness assertions thus follow immediately. The commutativity follows from the special form of R. For all i, j we have

$$\begin{aligned} S_i S_j &= P_{\mathcal{H}_{13}} R_i P_{\mathcal{H}_{13}} R_j | \mathcal{H}_{13} \\ &= P_{\mathcal{H}_{13}} R_i R_j | \mathcal{H}_{13} - P_{\mathcal{H}_{13}} R_i P_{\mathcal{H}_2} R_j | \mathcal{H}_{13} \\ &= P_{\mathcal{H}_{13}} R_i R_j | \mathcal{H}_{13} \text{ since } P_{\mathcal{H}_{13}} R_i P_{\mathcal{H}_2} = 0 \\ &= P_{\mathcal{H}_{13}} R_j R_i | \mathcal{H}_{13} = S_j S_i. \end{aligned}$$

■

Corollary 5.6. *Let $\mathcal{F} = \mathcal{F}_{sc}$ or $\mathcal{F} = \mathcal{F}_{rc}$ and let $T \in \mathcal{F}$. Then T has a nontrivial finite rank extension in $\mathcal{F}$ if and only if T has a nontrivial rank one extension in $\mathcal{F}$.*

Proof. Suppose T acts on a Hilbert space $\mathcal{H}$ and let R be a nontrivial finite rank extension of T in $\mathcal{F}$ acting on $\mathcal{H} \oplus \mathcal{K}$ with $1 < \dim \mathcal{K} < \infty$. Let $B = P_{\mathcal{K}} R | \mathcal{K}$ be the compression of R to $\mathcal{K}$. Then B is a commuting tuple of linear transformations on a finite dimensional space, thus the transformations B_i will have a common eigenvector $x_0 \neq 0$. Then either $R' = R | (\mathcal{H} \oplus \mathbb{C} x_0)$ is a nontrivial rank one extension of T in $\mathcal{F}$ or R is of the form as in Lemma 5.5 with $\mathcal{H}_1 = \mathcal{H}$ and $\mathcal{H}_2 = \mathbb{C} x_0$. In the latter case we can use the lemma to get a nontrivial extension R' of T acting on $\mathcal{H} \oplus \mathcal{K}'$ with $R' \in \mathcal{F}$ and $\dim \mathcal{K}' = \dim \mathcal{K} - 1$. Thus the result follows by an induction argument. ■

6. Extensions of spherical contractions

Theorem 6.1. *Let T be a commuting spherical contraction. Then the following are equivalent:*

(i) *T has only trivial rank one extensions in $\mathcal{F}_{sc}$,*
(ii) *T has only trivial finite rank extensions in $\mathcal{F}_{sc}$,*
(iii) *$T = S^* \oplus V$, where S is a direct sum of d-shifts and V is a spherical isometry such that for all $b \in \mathbb{B}_d$ the Koszul complex for $V - b$ is exact at Λ^1,*
(iv) (a) *$\sum_{i=1}^d T_i^* T_i$ is a projection, and*
(b) *for all $b \in \mathbb{B}_d$ the Koszul complex for $T - b$ is exact at Λ^1.*

Proof. (i) and (ii) are equivalent by Corollary 5.6.

(iii) $\Rightarrow$ (i): If R is a rank one extension of $T = S^* \oplus V$ in $\mathcal{F}_{sc}$, then since S^* is extremal in $\mathcal{F}_{sc}$ Lemma 2.1 (a) implies that $R = S^* \oplus R'$ for some $R' \in \mathcal{F}_{sc}$ with $R' \geq V$. Then clearly R' is a rank one extension of V and (i) follows from the hypothesis and the equivalence of (ii) and (iii) of Corollary 5.4.

(iv) $\Rightarrow$ (iii): By Proposition 4.1 the conditions (iv) (a) and (b) with $b = 0$ imply that T is of the form $T = S^* \oplus V$, where S is a direct sum of d-shifts and V is a spherical isometry. The Koszul complex of $S^* \oplus V$ splits into a direct sum of Koszul complexes. Thus it is clear that (iv) (b) implies that for each $b \in \mathbb{B}_d$ the Koszul complex for $V - b$ is exact at Λ^1.

(i) $\Rightarrow$ (iv): We will show the contrapositive. First suppose that (iv)(a) is not satisfied, i.e. $\sum_{i=1}^d T_i^* T_i$ is not a projection. We will use Lemma 5.2 with $b = 0$.

If E is the spectral measure for $\sum_{i=1}^d T_i^* T_i$, then there are real numbers r, s such that $0 < r < s < 1$ and such that $Q = E([r, s]) \neq 0$. Let $x_0 \in \operatorname{ran} Q$, $\|x_0\| \neq 0$ and set $x_i = T_i x_0$. Then

$$\sum_{i=1}^d \|x_i\|^2 = \sum_{i=1}^d \|T_i x_0\|^2 = \int_0^1 t \; d\langle E_t x_0, x_0\rangle \geq r\|x_0\|^2 \neq 0.$$

Thus by scaling x_0 we may assume that $\sum_{i=1}^d \|x_i\|^2 = 1$, and we have (i), (ii), and (iii) of Lemma 5.2. Furthermore, since $s < 1$ we have $Q = D \int_r^s \frac{1}{\sqrt{1-t}} dE$, so $\operatorname{ran} Q \subseteq \operatorname{ran} D$. Hence

$$\sum_{i=1}^d T_i^* x_i = \sum_{i=1}^d T_i^* T_i x_0 = \sum_{i=1}^d T_i^* T_i Q x_0 = Q \sum_{i=1}^d T_i^* T_i x_0 \in \operatorname{ran} D$$

and (iv) of Lemma 5.2 is also satisfied. Thus T must have a nontrivial rank one extension in $\mathcal{F}_{sc}$ i.e. condition (i) of Theorem 6.1 does not hold.

Next suppose that $\sum_{i=1}^d T_i^* T_i = P_1$ is a projection, but that the Koszul complex for T is not exact at Λ^1. This implies that the column operator $T^{(1)} : \mathcal{H} \to \mathcal{H}^d$ defined by

$$T^{(1)}x = \begin{pmatrix} T_1 x \\ . \\ . \\ . \\ T_d x \end{pmatrix}$$

satisfies $P_1 = T^{(1)^*} T^{(1)}$ and hence $T^{(1)}$ is a partial isometry and in particular has closed range. Furthermore, there exist $x_1, ..., x_d \in \mathcal{H}$ such that $T_i x_j = T_j x_i$ for all i, j, but $\begin{pmatrix} x_1 \\ . \\ . \\ . \\ x_d \end{pmatrix} \notin \text{ran } T^{(1)}$. Since ran $T^{(1)}$ is closed we may assume that $\begin{pmatrix} x_1 \\ . \\ . \\ . \\ x_d \end{pmatrix} \perp \text{ran } T^{(1)}$ and $\sum_{i=1}^d \|x_i\|^2 = 1$. But this means that $\begin{pmatrix} x_1 \\ . \\ . \\ . \\ x_d \end{pmatrix} \in \ker T^{(1)^*}$ or $\sum_{i=1}^d T_i^* x_i = 0 \in \text{ran } D$. Thus again we can use Lemma 5.2 to see that condition (i) of Theorem 6.1 does not hold.

Finally we suppose that $\sum_{i=1}^d T_i^* T_i$ is a projection, that the Koszul complex for T is exact at Λ^1, but that there is a $b \in \mathbb{B}_d$, $b \neq 0$ such that the Koszul complex for $T - b$ is not exact at Λ^1. Then Proposition 4.1 implies that $T = S^* \oplus V$, where S is a direct sum of d-shifts and V is a spherical isometry. Since the Koszul complex of $S^* - b$ is exact at Λ^1, it follows that the Koszul complex for $V - b$ cannot be exact at Λ^1. Thus in this case it follows from Corollary 5.4 that T would have a nontrivial rank one extension in $\mathcal{F}_{sc}$. The concludes the proof of (iv) $\Rightarrow$ (i). ∎

Corollary 6.2. *Theorem 1.4 holds.*

Proof. The implication $(ii) \Rightarrow (i)$ follows from Theorems 3.3 and 2.2 and Lemma 2.1.

We now show (iii) $\Rightarrow$ (ii). If T satisfies the conditions (a), (b), and (c) of part (iii) of Theorem 1.4, then by Proposition 4.1 T is unitarily equivalent to $S^* \oplus V$, where S is a direct sum of d-shifts and $V = (V_1, ..., V_d)$ is a spherical isometry. We will see that V is a spherical unitary. Condition (b) implies that

$$0 \leq \sum_{i=1}^{d} V_i V_i^* - I = \sum_{i=1}^{d} V_i V_i^* - V_i^* V_i.$$

We already mentioned that each operator in a spherical isometric tuple must be subnormal (also see Theorem 1.3), thus for each i we have $V_i V_i^* - V_i^* V_i \leq 0$. Hence $V_i V_i^* = V_i^* V_i$ for all $1 \leq i \leq d$.

Finally we prove $(i) \Rightarrow (iii)$. If T is extremal then it has no nontrivial rank one extension, hence conditions (iii) (a) and (c) follow from the equivalence of (i) and (iv) in Theorem 6.1. Then it follows from Proposition 4.1 that $T = S^* \oplus V$ for a spherical isometry V. Since T is extremal Theorem 2.2 implies that V must in fact be a spherical unitary tuple. Then $T^* = S \oplus U$ for some spherical unitary tuple U and a direct sum of d-shifts S. Condition (iii)(b) follows easily (see (1.3)). ∎

Corollary 6.3. *Corollary 1.5 holds.*

Proof. Let $T = S \oplus U$, where S is a direct sum of d-shifts and U is a spherical unitary tuple. It follows from (1.1) and (1.3) that T satisfies conditions (a) and (b) of Corollary 1.5. Furthermore, in Section 3 we mentioned that the Koszul complex for the d-shift is exact at $\Lambda^p(\mathcal{H})$ for all p with $1 \leq p \leq d-1$. The Taylor spectrum of any spherical unitary tuple U must be contained in $\partial\mathbb{B}_d$. Since such U is normal this can easily be deduced from Proposition 7.2 of [11]. Hence the Koszul complex of $T = S \oplus U$ is exact at $\Lambda^{d-1}(\mathcal{H})$, i.e. (c) of Corollary 1.5 holds as well.

Conversely suppose that T satisfies (a), (b), and (c) of Corollary 1.5. Then the adjoint tuple T^* satisfies (iii)(a) and (iii)(b) of Theorem 1.4. Since $\sum_{i=1}^{d} T_i T_i^*$ is a projection the operator $\mathcal{H} \to \mathcal{H}^d$ defined by $x \to (T_1^* x, ..., T_d^* x)$ is a partial isometry and thus has closed range. This operator is unitarily equivalent to $\partial_{T^*,0}$, hence $\partial_{T^*,0}$ has closed range in $\Lambda^1(\mathcal{H})$. Thus, as the hypothesis (c) is that $K(T)$ is exact at $\Lambda^{d-1}(\mathcal{H})$ the discussion about the Hodge $*$-operator at the beginning of Section 3 implies that $K(T^*)$ is exact at $\Lambda^1(\mathcal{H})$, i.e. T^* satsfies (iii)(c) of Theorem 1.4. Hence Theorem 1.4 implies that $T^* = S^* \oplus U$, where S

is a direct sum of d-shifts and U (and hence U^*) is a spherical unitary tuple. This concludes the proof of the Corollary. ∎

7. Rank one extensions of row contractions

Next we consider row contractions. Let $\mathcal{F}_{rc}$ denote the family of commuting row contractions, let $T = (T_1, .., T_d) \in \mathcal{F}_{rc} \cap \mathcal{B}(\mathcal{H})^d$ and write $D_* = (I - \sum_{i=1}^d T_i T_i^*)^{1/2}$ for the defect operator. If $R \in \mathcal{B}(\mathcal{H} \oplus \mathcal{K})$ is an operator tuple that extends T, then we will use the notation

$$R_i = \begin{pmatrix} T_i & A_i \\ 0 & B_i \end{pmatrix}, i = 1, .., d. \tag{7.1}$$

Note that R will be a commuting tuple if and only if

$$T_i A_j - T_j A_i = A_j B_i - A_i B_j \text{ and } B_i B_j = B_j B_i \text{ for all } i, j. \tag{7.2}$$

Lemma 7.1. *Let T be a commuting row contraction.*

(a) If $b = (b_1, ..., b_d) \in \mathbb{C}^d$, $|b| = 1$, then $\ker(I - \sum_{i=1}^d \overline{b_i} T_i) \subseteq \ker D_*$.

(b) If R is a row contraction that extends T, then $\sum_{i=1}^d A_i A_i^* \le D_*^2$ and for each i we have ran $A_i \subseteq$ ran D_*.

Proof. (a) Let $|b| = 1$ and let $x \in \ker(I - \sum_{i=1}^d \overline{b_i} T_i)$. Then

$$\begin{aligned} \|x\|^4 =& |\langle x, \sum_{i=1}^d \overline{b_i} T_i x\rangle|^2 = |\sum_{i=1}^d b_i \langle T_i^* x, x\rangle|^2 \\ &\le |b|^2 \sum_{i=1}^d \|T_i^* x\|^2 \|x\|^2 = (\|x\|^2 - \|D_* x\|^2)\|x\|^2. \end{aligned}$$

This implies $\|D_* x\| = 0$.

(b) Recall that R is a row contraction if and only if R^* is a spherical contraction, i.e. for all $x \in \mathcal{H}, y \in \mathcal{K}$ we have

$$\sum_{i=1}^d \|R_i^* \begin{pmatrix} x \\ y \end{pmatrix}\|^2 \le \|x\|^2 + \|y\|^2.$$

A short calculation shows that this happens if and only if

$$\sum_{i=1}^d \|A_i^* x + B_i^* y\|^2 \le \|D_* x\|^2 + \|y\|^2. \tag{7.3}$$

In particular, we see that if R is a row contraction, then $\sum_{i=1}^d A_i A_i^* \le D_*^2$ and hence for each i we must have ran $A_i \subseteq$ ran D_* (by the Douglas Lemma ([12]). This proves (b). ∎

Lemma 7.2. *Let $\mathcal{F}_{rc}$ be the family of commuting row contractions, let $T \in \mathcal{F}_{rc} \cap \mathcal{B}(\mathcal{H})^d$, and let $D_* = (I - \sum_{i=1}^d T_i T_i^*)^{1/2}$.*

Then T has a nontrivial rank one extension in $\mathcal{F}_{rc}$, if and only if there exist $b = (b_1, ..., b_d) \in \mathbb{C}_d$, $x_1, ..., x_d \in \mathcal{H}$ such that

(i) $\sum_{i=1}^d \|x_i\|^2 = 1$,
(ii) $(T_i - b_i)x_j = (T_j - b_j)x_i$ *for all* i, j
(iii) $|b| < 1$, *and*
(iv) $x_i \in \text{ran } D_*$ *for each* i,

Proof. First assume that we are given $b \in \mathbb{C}^d$ and $x_1, ..., x_d \in \mathcal{H}$ such that (i)-(iv) are satisfied. For any $\varepsilon > 0$ we define a rank one extension R of T as in Lemma 5.1. By (i) and (ii) it will be non-trivial and commutative. Thus by (7.3) R will be a row contraction if and only if $\sum_{i=1}^d |\varepsilon\langle x, x_i\rangle + \bar{b}_i y|^2 \le \|D_* x\|^2 + |y|^2$ for all $x \in \mathcal{H}, y \in \mathbb{C}$.

Since each $x_i \in \text{ran } D_*$, there are $z_i \in \mathcal{H}$ such that $x_i = D_* z_i$. Then for $x \in \mathcal{H}$ and $y \in \mathbb{C}$ we have

$$\begin{aligned}\sum_{i=1}^d |\varepsilon\langle x, x_i\rangle + \bar{b}_i y|^2 &\le \varepsilon^2 \sum_{i=1}^d |\langle D_* x, z_i\rangle|^2 + 2\varepsilon \sum_{i=1}^d |\langle D_* x, z_i\rangle||b_i||y| + |b|^2|y|^2 \\ &\le \varepsilon^2 \sum_{i=1}^d \|z_i\|^2 \|D_* x\|^2 + 2\varepsilon \left(\|D_* x\| \sqrt{\sum_{i=1}^d \|z_i\|^2} \right) |b||y| + |b|^2|y|^2 \\ &\le (\varepsilon^2 + \varepsilon) \sum_{i=1}^d \|z_i\|^2 \|D_* x\|^2 + (1+\varepsilon)|b|^2|y|^2 \le \|D_* x\|^2 + |y|^2,\end{aligned}$$

whenever ε is sufficiently small. Thus T has a nontrivial rank one extension in $\mathcal{F}_{rc}$.

Conversely, assume that T has a nontrivial rank one extension R in $\mathcal{F}_{rc}$. Then R can be written as in Lemma 5.1. Thus we have $\varepsilon > 0$, $b \in \mathbb{C}^d$ and $x_1, ..., x_d \in \mathcal{H}$ satisfying (i) and (ii). Furthermore, a calculation similar to what was done in the first part of the proof shows that since R is a row contraction we must have

$$\begin{aligned}\sum_{i=1}^d |\varepsilon\langle x, x_i\rangle + \bar{b}_i y|^2 &= \varepsilon^2 \sum_{i=1}^d |\langle x, x_i\rangle|^2 + 2\varepsilon \text{ Re } \sum_{i=1}^d \langle x, x_i\rangle b_i \bar{y} + |b|^2|y|^2 \\ &\le \|D_* x\|^2 + |y|^2\end{aligned}$$

for all $x \in \mathcal{H}$ and all $y \in \mathbb{C}$. By taking $y = 0$ we see that the Douglas Lemma ([12]) implies that (iv) must be satisfied, and by taking $x = 0$ it follows that $|b| \le 1$. We will be done if we can rule out the possibility that $|b| = 1$.

Note that R^* is a nontrivial extension in $\mathcal{F}_{sc}$ of the tuple of scalars $\overline{b} : \mathbb{C} \to \mathbb{C}$. Hence Theorem 2.2 implies $|b| < 1$. A somewhat more direct argument goes as follows: If $|b| = 1$, then the above inequality implies that $\operatorname{Re} \sum_{i=1}^d \langle x, x_i\rangle b_i \overline{y} = 0$ for all x and y. Hence $\sum_{i=1}^d \overline{b}_i x_i = 0$. Now we multiply (ii) by $\overline{b}_i$ and sum in i to obtain $(\sum_{i=1}^d \overline{b}_i T_i - I)x_j = (T_j - b_j)\sum_{i=1}^d \overline{b}_i x_i = 0$ for each j. Thus each $x_j \in \ker(I - \sum_{i=1}^d \overline{b}_i T_i)$ and hence by Lemma 7.1 (a) $x_j \in \ker D_*$. This contradicts (i) and (iv), which are already known to hold. ■

8. Extensions of row contractions

In this Section we shall prove Theorems 1.6 and 1.8 and Corollary 1.7.

Proposition 8.1. *(a) If $D_* = 0$, then $T \in ext(\mathcal{F}_{rc})$.*

(b) If D_* is onto, then T has a rank one extension in $ext(\mathcal{F}_{rc})$.

(c) If D_* is a projection, then the following are equivalent:

(i) *T has a nontrivial rank one extension in $\mathcal{F}_{rc}$,*

(ii) *$T \notin ext(\mathcal{F}_{rc})$,*

(iii) *there are $x_1, .., x_d \in \bigcap_{j=1}^d \ker T_j^*$ with $\sum_{i=1}^d \|x_i\|^2 > 0$ and $T_i x_j = T_j x_i$ for all i, j.*

Proof. (a) follows directly from Lemma 7.1 (b).

(b) Clearly the zero tuple, $T = (0, .., 0)$, is not extremal. Thus assume that D_* is onto and one of the T_i's is not zero. Then we can set $b = 0$ and choose $x \in \mathcal{H}$ such that the hypothesis of Lemma 7.2 is satisfied with $x_i = T_i x$.

(c) (iii) $\Rightarrow$ (i) follows directly from Lemma 7.2 with $b = 0$. (i) $\Rightarrow$ (ii) is trivial.

(ii) $\Rightarrow$ (iii): We assume that D_* is a projection and that we have a nontrivial extension in $\mathcal{F}_{rc}$. Then with the notation as in (7.1) we set $x_i = A_i x$, where x is chosen so that $\sum_{i=1}^d \|x_i\|^2 > 0$. Lemma 7.1 (b) implies that for all k we have $\operatorname{ran} A_k \subseteq \operatorname{ran} D_*$. Thus $x_1, .., x_d \in \operatorname{ran} D_*$. Furthermore, since D_* is a projection we have

$$\bigvee_{k=1}^d \operatorname{ran} A_k \subseteq \operatorname{ran} D_* = \bigcap_{j=1}^d \ker T_j^* = \left(\bigvee_{j=1}^d \operatorname{ran} T_j\right)^\perp.$$

Thus, commutativity implies that for all i and j

$$T_i x_j - T_j x_i = T_i A_j x - T_j A_i x = A_j B_i x - A_i B_j x \in \bigvee_{k=1}^d \operatorname{ran} A_k \cap \bigvee_{j=1}^d \operatorname{ran} T_j = (0).$$

This establishes (iii). ■

Proposition 8.2. *If $T \in \mathcal{F}_{rc}$ and if there is a $u \in \operatorname{ran} D_*$, $\|u\| = 1$, such that dim span $\{u, T_1 u, .., T_d u\} \leq 2$, then T has a nontrivial rank one extension in $\mathcal{F}_{rc}$.*

Proof. The hypothesis implies that there is $v \in \mathcal{H}$, $v \perp u$ and $\alpha = (\alpha_1, ..., \alpha_d)$, $\beta = (\beta_1, ..., \beta_d) \in \mathbb{C}^d$, $\beta \neq 0$ such that $T_i u = \alpha_i u + \beta_i v$ for $i = 1, ..., d$. Indeed, if dim span $\{u, T_1 u, .., T_d u\} = 2$, then we can find such a unit vector v satisfying this, while if dim span $\{u, T_1 u, .., T_d u\} = 1$ we take $v = 0$ and any $\beta \neq 0$.

We set $\gamma = P_{\beta^\perp} \alpha = \alpha - c\beta$, where $c = \frac{\langle \alpha, \beta \rangle}{|\beta|^2}$. Then $\langle \beta, \gamma \rangle = 0$ and $\langle \alpha, \gamma \rangle = \langle \alpha, P_{\beta^\perp} \alpha \rangle = |\gamma|^2$.

The conclusion will follow from Lemma 7.2 with $x_i = \frac{\beta_i}{|\beta|} u$ and $b = \gamma$. Conditions (i) and (iv) are obvious from the definition of the x_i. In order to verify (iii) we calculate

$$(T_i - \gamma_i) x_j = \frac{\beta_i \beta_j}{|\beta|} (v - cu) = (T_j - \gamma_j) x_i$$

for all i, j.

Finally, $\sum_{i=1}^d \overline{\gamma_i} T_i u = \langle \alpha, \gamma \rangle u + \langle \beta, \gamma \rangle v = |\gamma|^2 u$. Since T is a row contraction this implies

$$|\gamma|^4 = \| \sum_{i=1}^d T_i (\overline{\gamma_i} u) \|^2 \leq \sum_{i=1}^d \|\overline{\gamma_i} u\|^2 = |\gamma|^2.$$

Hence $|\gamma| \leq 1$.

If $|\gamma| = 1$, then $(I - \sum_{i=1}^d \overline{\gamma_i} T_i) u = 0$, thus Lemma 7.1 (a) implies that $u \in \ker D_*$. But since $u \in \operatorname{ran} D_*$ this would mean $u = 0$, which is impossible. Hence $|\gamma| < 1$. ■

We shall now prove part (iv) of Theorem 1.6.

Theorem 8.3. *Let T be a commuting row contraction with $D_* = u \otimes u$ for some $u \in \mathcal{H}$, $u \neq 0$. Then the following are equivalent*

(i) *$T \in ext(\mathcal{F}_{rc})$,*
(ii) *T has only trivial rank one extensions in $\mathcal{F}_{rc}$,*
(iii) *dim span $\{u, T_1 u, .., T_d u\} \geq 3$.*

Proof. (i) ⇒ (ii) is trivial and (ii) ⇒ (iii) follows directly from Proposition 8.2.

(iii) ⇒ (i): Suppose that dim span $\{u, T_1 u, .., T_d u\} \geq 3$. Since $u \neq 0$ we may without loss of generality assume that the set $\{u, T_1 u, T_2 u\}$ is linearly independent. Let R be an extension of T in $\mathcal{F}_{rc}$ and assume each R_i is of the form as in (7.1). We must show that each $A_i = 0$.

Since $D_* = u \otimes u$ Lemma 7.1 (b) implies the existence of $x_1, ..., x_d \in \mathcal{K}$ such that $A_i x = \langle x, x_i \rangle u$ for each i. Then commutativity (see (7.2)) implies for all $x \in \mathcal{K}$ and all i, j

(8.1)
$$\langle x, x_j \rangle T_i u - \langle x, x_i \rangle T_j u = \langle B_i x, x_j \rangle u - \langle B_j x, x_i \rangle u = \langle x, B_i^* x_j - B_j^* x_i \rangle u$$

Take $i = 1$ and $j = 2$. Then the linear independence of $\{u, T_1 u, T_2 u\}$ shows that $\langle x, x_1 \rangle = \langle x, x_2 \rangle = 0$ for all $x \in \mathcal{K}$. Thus $x_1 = x_2 = 0$. Next consider (8.1) with $i = 1$ and $j > 2$. Since $x_1 = 0$ we get

$$\langle x, x_j \rangle T_1 u = \langle x, B_1^* x_j \rangle u.$$

Again linear independence implies $x_j = 0$. Thus $A_j = 0$ for all j, and T must be extremal. ∎

Corollary 8.4. *Corollary 1.7 holds.*

Proof. Write $P = P_{\mathcal{M}^\perp}$ for the projection of H^2_d onto $\mathcal{M}^\perp$. Recall that if S denotes the d-shift on H^2_d, then $I - \sum_{i=1}^d S_i S_i^* = 1 \otimes 1$ is the projection onto the constants. For $i = 1, ..., d$ we have $T_i = P S_i | \mathcal{M}^\perp$, hence

$$D_*^2 = I_{\mathcal{M}^\perp} - \sum_{i=1}^d T_i T_i^* = P \left(I - \sum_{i=1}^d S_i S_i^* \right) P = \varphi \otimes \varphi,$$

where $\varphi = P1$. Since we are assuming $\mathcal{M} \neq H^2_d$ we have $1 \notin \mathcal{M}$, thus $\varphi \neq 0$ and rank $D_* = 1$.

Let $\alpha_0, ..., \alpha_d \in \mathbb{C}$, then

$$\alpha_0 \varphi + \sum_{i=1}^d \alpha_i T_i \varphi = 0 \quad \Leftrightarrow \quad \alpha_0 + \sum_{i=1}^d \alpha_i z_i \in \mathcal{M}.$$

This implies that span$\{\varphi, T_1 \varphi, ..., T_d \varphi\}$ is isomorphic to $\mathcal{L}/\mathcal{M} \cap \mathcal{L}$. But $\dim \mathcal{L}/\mathcal{M} \cap \mathcal{L} = d + 1 - \dim(\mathcal{M} \cap \mathcal{L})$. Thus Corollary 8.4 follows from Theorem 8.3. ∎

9. An example

Let $S = (M_z, M_w)$ be the 2-shift on H^2_2 and let $\mathcal{M} = \{f \in H^2_2 : f(z, 0) = 0\}$. $\mathcal{M}$ is invariant for S, thus $T = S|\mathcal{M}$ is a non-extremal row contraction. We claim that T has no nontrivial finite rank extensions in $\mathcal{F}_{rc}$.

Note that the linear span of monomials of the form $z^n w^m$, $n \geq 0, m > 0$ are dense in $\mathcal{M}$ and one computes

$$D_*^2 z^n w^m = \begin{cases} \frac{1}{n+1} z^n w & \text{if } m = 1 \\ \\ 0 & \text{if } m > 1 \end{cases}$$

From this one easily sees that there are no nonzero $f, g \in \operatorname{ran} D_*$ and $b = (b_1, b_2) \in \mathbb{B}_2$ such that $(z - b_1)f = (w - b_2)g$. Hence Theorem 1.8 applies to show the claim.

References

[1] Jim Agler. An abstract approach to model theory. In *Surveys of some recent results in operator theory, Vol. II*, volume 192 of *Pitman Res. Notes Math. Ser.*, pages 1–23. Longman Sci. Tech., Harlow, 1988.

[2] T. Andô. On a pair of commutative contractions. *Acta Sci. Math. (Szeged)*, 24:88–90, 1963.

[3] William Arveson. Subalgebras of C^*-algebras. II. *Acta Math.*, 128(3-4):271–308, 1972.

[4] William Arveson. Subalgebras of C^*-algebras. III. Multivariable operator theory. *Acta Math.*, 181(2):159–228, 1998.

[5] William Arveson. The Dirac operator of a commuting d-tuple. *J. Funct. Anal.*, 189(1):53–79, 2002.

[6] William Arveson. Notes on the unique extension property. *Unpublished Manuscript, see http://math.berkeley.edu/ arveson/texfiles.html*, 2003.

[7] Ameer Athavale. On the intertwining of joint isometries. *J. Operator Theory*, 23(2):339–350, 1990.

[8] K. R. M. Attele and A. R. Lubin. Dilations and commutant lifting for jointly isometric operators—a geometric approach. *J. Funct. Anal.*, 140(2):300–311, 1996.

[9] S. Brehmer. Über vetauschbare Kontraktionen des Hilbertschen Raumes. *Acta Sci. Math. Szeged*, 22:106–111, 1961.

[10] Raul E. Curto. Fredholm and invertible n-tuples of operators. The deformation problem. *Trans. Amer. Math. Soc.*, 266(1):129–159, 1981.

[11] Raúl E. Curto. Applications of several complex variables to multiparameter spectral theory. In *Surveys of some recent results in operator theory, Vol. II*, volume 192 of *Pitman Res. Notes Math. Ser.*, pages 25–90. Longman Sci. Tech., Harlow, 1988.

[12] R. G. Douglas. On majorization, factorization, and range inclusion of operators on Hilbert space. *Proc. Amer. Math. Soc.*, 17:413–415, 1966.

[13] Michael A. Dritschel and Scott McCullough. Model theory for hyponormal contractions. *Integral Equations Operator Theory*, 36(2):182–192, 2000.

[14] Michael A. Dritschel and Scott A. McCullough. Boundary representations for families of representations of operator algebras and spaces. *J. Operator Theory*, 53(1):159–167, 2005.

[15] Jim Gleason, Stefan Richter, and Carl Sundberg. On the index of invariant subspaces in spaces of analytic functions of several complex variables. *J. Reine Angew. Math.*, 587:49–76, 2005.

[16] Phillip Griffiths and Joseph Harris. *Principles of algebraic geometry*. Wiley Classics Library. John Wiley & Sons Inc., New York, 1994. Reprint of the 1978 original.

[17] Takasi Itô. On the commutative family of subnormal operators. *J. Fac. Sci. Hokkaido Univ. Ser. I*, 14:1–15, 1958.

[18] V. Müller and F.-H. Vasilescu. Standard models for some commuting multioperators. *Proc. Amer. Math. Soc.*, 117(4):979–989, 1993.

[19] Walter Rudin. *Functional analysis*. International Series in Pure and Applied Mathematics. McGraw-Hill Inc., New York, second edition, 1991.

[20] Béla Sz.-Nagy and Ciprian Foiaş. *Harmonic analysis of operators on Hilbert space*. Translated from the French and revised. North-Holland Publishing Co., Amsterdam, 1970.

[21] Joseph L. Taylor. A joint spectrum for several commuting operators. *J. Functional Analysis*, 6:172–191, 1970.

Department of Mathematics, University of Tennessee, Knoxville, TN 37996

E-mail address: `richter@math.utk.edu`

Department of Mathematics, University of Tennessee, Knoxville, TN 37996

E-mail address: `sundberg@math.utk.edu`