A Remark on Global Existence of Solutions of Shadow Systems

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Abstract

Let Ω be an open, bounded domain in $\mathbb{R}^n$ ($n \in \mathbb{N}$) with smooth boundary $\partial\Omega$. Let p, q, r, d_1, τ be positive real numbers and s be a non-negative number which satisfies $0 < \frac{p-1}{r} < \frac{q}{s+1}$. We consider the shadow system of the well-known Gierer-Meinhardt system:

$$\begin{cases} u_t = d_1 \Delta u - u + \dfrac{u^p}{\xi^q}, & \text{in} \quad \Omega \times (0,T), \\ \tau \xi_t = -\xi + \dfrac{1}{|\Omega|} \displaystyle\int_\Omega \dfrac{u^r}{\xi^s} dx, & \text{in} \quad (0,T), \\ \dfrac{\partial u}{\partial \nu} = 0, & \text{on} \quad \partial\Omega \times (0,T), \\ \xi(0) = \xi_0 > 0, \quad u(\cdot, 0) = u_0(\cdot) \geq 0 & \text{in} \quad \Omega. \end{cases}$$

We prove that solutions of this system exist globally in time under some conditions on the coefficients. Our results are based on a priori estimates of the solutions and improve the global existence results of F. Li and W.-M. Ni in [4].

1 Introduction

Let Ω be an open, bounded domain in $\mathbb{R}^n$ (for $n \in \mathbb{N}$) with smooth boundary $\partial\Omega$. We are interested in the global (in time) existence of solutions (u, ξ) of the following shadow system:

$$\begin{cases} u_t = d_1 \Delta u - u + \dfrac{u^p}{\xi^q}, & \text{in} \quad \Omega \times (0,T), \\ \tau \xi_t = -\xi + \dfrac{1}{|\Omega|} \displaystyle\int_\Omega \dfrac{u^r}{\xi^s} dx, & \text{in} \quad (0,T), \\ \dfrac{\partial u}{\partial \nu} = 0, & \text{on} \quad \partial\Omega \times (0,T), \\ \xi(0) = \xi_0 > 0, \quad u(\cdot, 0) = u_0(\cdot) \geq 0 & \text{in} \quad \Omega, \end{cases} \tag{1.1}$$

where ν is the outward normal vector on $\partial\Omega$, $|\Omega|$ is the volume of Ω, $0 < T \leq \infty$, Δ is the Laplace operator on $\mathbb{R}^n$, and ξ_0, u_0 are given initial data. Moreover, p, q, r, τ, d_1 are given fixed positive numbers and s is a given fixed non-negative number satisfying

$$0 < \frac{p-1}{r} < \frac{q}{s+1}. \tag{1.2}$$

The system (1.1) is the shadow system corresponding to a famous model proposed by Gierer-Meinhardt [2] in 1972:

$$\begin{cases} u_t = d_1\Delta u - u + \dfrac{u^p}{v^q}, & \text{in} \quad \Omega \times (0,T), \\ \tau v_t = d_2\Delta v - v + \dfrac{u^r}{v^s}, & \text{in} \quad \Omega \times (0,T), \\ \dfrac{\partial u}{\partial \nu} = \dfrac{\partial v}{\partial \nu} = 0, & \text{on} \quad \partial\Omega \times (0,T), \\ u(\cdot,0) = u_0(\cdot) \geq 0, \quad v(\cdot,0) = v_0(\cdot) \geq 0 & \text{in} \quad \Omega. \end{cases} \tag{1.3}$$

The problem on global (in time) existence of solutions (u,v) to (1.3) is almost completely understood. Precisely, solutions of (1.3) exist globally if $\frac{p-1}{r} < 1$ and blow up in finite time if $\frac{p-1}{r} > 1$. However, in the borderline case $\frac{p-1}{r} = 1$, the question of global existence of solutions of (1.3) remains open. Interested readers can find more details in [3, 4], and references therein.

For the shadow system (1.1), little is known on the global (in time) existence of its solutions. In a very recent paper [4], Li and Ni prove that solutions of (1.1) exist for all $t > 0$ provided that $\frac{p-1}{r} < \frac{2}{n+2}$ (strict inequality). They also construct finite time blow-up solutions of (1.1) when $p = r, \tau = s + 1 - q$, $0 < \frac{2}{n} < \frac{p-1}{r} < \frac{q}{s+1} < 1$, $n \geq 3$ and Ω is an open unit ball in $\mathbb{R}^n$. Therefore, comparing (1.1) with (1.3), we see that the results in [4] demonstrate that there are serious discrepancies between the dynamics of the reaction-diffusion systems and that of their corresponding shadow systems. Moreover, as it is posed in [4], for the case $\frac{2}{n+2} \leq \frac{p-1}{r} \leq \frac{2}{n}$, the question of global existence of solutions of (1.1) remains open and needs to be investigated. In this short note, we show that in the borderline case of [4] (i.e. $\frac{2}{n+2} = \frac{p-1}{r}$), solutions of (1.1) also exist globally in time. Precisely, we prove the following theorem:

Theorem 1.1. *Assume that $\xi_0 > 0$, and u_0 is non-negative bounded function. Assume also that $\frac{p-1}{r} \leq \frac{2}{n+2}$ and (1.2) holds. Then every solution of (1.1) exists globally in time.*

The rest of this paper is devoted to the proof of Theorem 1.1. Our proof is carried using standard techniques of parabolic equations: suitable applications of Sobolev's embedding theorems, the Hölder, Young and Gronwall inequalities. Particularly, for the case $\frac{p-1}{r} = \frac{2}{n+2}$ which is critical for the approach in [4], besides those standard techniques, we use a key observation that any non-negative solution g of the ordinary differential equation $\dot{g}(t) = m(t)g(t)$ does not blow up in finite time if $m(t)$ is integrable on $(0,t')$ for all $t' > 0$.

2 Proof of Theorem 1.1

For given initial data ξ_0, u_0 and the parameters p, q, r, s, τ, d_1 satisfying conditions in Theorem 1.1, let T be the maximal existence time of the solution (u, ξ) of (1.1). It follows from the standard theory of parabolic equations that $T > 0$. We argue by contradiction to prove Theorem 1.1. We assume that $T < \infty$ and we shall derive a contradiction. Without loss of generality, we assume that $|\Omega| = 1$. We first estimate $\xi(t)$ from below by the following simple lemma which is due to [4]:

Lemma 2.1. $\xi(t) \geq \xi_0 e^{-t/\tau}$ *for all* $0 \leq t < T$.

Proof. From the second equation of (1.1) and since $u \geq 0$ on $\Omega \times [0, T)$, we get

$$\frac{d}{dt}(e^{\frac{t}{\tau}}\xi) = e^{\frac{t}{\tau}}[\xi_t + \frac{1}{\tau}\xi] = \frac{e^{\frac{t}{\tau}}}{\tau}\int_\Omega \frac{u^r}{\xi^s}dx \geq 0.$$

Therefore, $e^{\frac{t}{\tau}}\xi \geq \xi_0$. Thus, Lemma 2.1 follows. $\square$

From Lemma 2.1 and standard parabolic regularity theory, we shall obtain a contradiction if we can prove that for sufficiently large $l > 0$, under the conditions of Theorem 1.1 and $T < \infty$, there exists a finite constant $C_l(T) > 0$ such that

$$||u(\cdot, t)||_{L^l(\Omega)} \leq C_l(T), \ \forall \ t \in [0, T). \tag{2.1}$$

The estimate (2.1) follows directly from Lemma 2.3 below. To prove Lemma 2.3, we need the following a priori estimate:

Lemma 2.2. *For each $a > 0$, let $g_a(x,t) = \frac{u^r(x,t)}{\xi^{s+1+a}(t)}$, for all $(x,t) \in \Omega \times [0,T)$. Then, we have*

$$\int_{t_1}^{t_2}\int_\Omega g_a(x,t)dxdt \leq \frac{\tau}{a}\left\{\frac{e^{\frac{a}{\tau}t_2} - e^{\frac{a}{\tau}t_1}}{\xi_0^a} + \frac{1}{\xi^a(t_1)} - \frac{1}{\xi^a(t_2)}\right\}, \quad \forall \ 0 \leq t_1 \leq t_2 < T.$$

In particular,

$$\int_0^{t'}\int_\Omega g_a(x,t)dxdt \leq C_a(t') \overset{\text{def}}{=\!=} \frac{\tau e^{\frac{at'}{\tau}}}{a\xi_0^a}, \quad 0 \leq t' < T.$$

Proof. Let us denote $\zeta = \xi^{-a}$. Then, we have

$$\dot{\zeta} = -a\frac{\zeta}{\xi}\xi_t = -\frac{a\zeta}{\tau\xi}\left[-\xi + \int_\Omega \frac{u^r}{\xi^s}dx\right] = \frac{a}{\tau}\zeta - \frac{a}{\tau}\int_\Omega g_a(x,t)dx.$$

From this and Lemma 2.1, it follows that

$$\begin{aligned}\int_{t_1}^{t_2}\int_\Omega g_a(x,t)dxdt &= \int_{t_1}^{t_2}\zeta dt - \frac{\tau}{a}\int_{t_1}^{t_2}\dot{\zeta}dt \leq \frac{1}{\xi_0^a}\int_{t_1}^{t_2} e^{\frac{a}{\tau}t}dt + \frac{\tau}{a}\left[\zeta(t_1) - \zeta(t_2)\right] \\ &= \frac{\tau}{a}\left[\frac{e^{\frac{a}{\tau}t_2} - e^{\frac{a}{\tau}t_1}}{\xi_0^a} + \zeta(t_1) - \zeta(t_2)\right].\end{aligned}$$

Therefore, the first claim of Lemma 2.2 follows. The last claim of Lemma 2.2 follows from its first claim when $t_1 = 0$ and $t_2 = t'$. $\square$

Lemma 2.3. *For any $\alpha, \beta \geq 0$, let*

$$g_{\alpha,\beta}(t) = \int_\Omega \frac{u^\alpha(x,t)}{\xi^\beta(t)}dx, \quad 0 \leq t < T.$$

Assume that $T < \infty$ and all conditions in Theorem 1.1 hold true. Then there exits a constant $C(T) = C_{\alpha,\beta}(T) < \infty$ such that

$$g_{\alpha,\beta}(t) \leq C(T), \quad \textit{for all} \quad 0 \leq t < T.$$

Proof. We follow the ideas in [5, 4]. Since $\frac{p-1}{r} \leq \frac{2}{n+2}$ and $\frac{p-1}{r} < \frac{q}{s+1}$, we can choose a constant $a > 0$ such that

$$\rho \overset{\text{def}}{=\!=} \frac{p-1}{r} = \frac{q}{s+1+a} \leq \frac{2}{n+2}.$$

Note that it is sufficient to prove Lemma 2.3 for $\alpha > 1$, which we shall assume from now on. Let $w = u^{\alpha/2}$, from (1.1) we have

$$\begin{aligned}
\dot{g}_{\alpha,\beta}(t) &= -\beta \frac{g_{\alpha,\beta}}{\xi}\xi_t + \frac{\alpha}{\xi^\beta}\int_\Omega u^{\alpha-1}u_t dx \\
&= -\beta \frac{g_{\alpha,\beta}}{\tau\xi}\left[-\xi + \int_\Omega \frac{u^r}{\xi^s}dx\right] + \frac{\alpha}{\xi^\beta}\int_\Omega u^{\alpha-1}\left[d_1\Delta u - u + \frac{u^p}{\xi^q}\right]dx \\
&= \left(\frac{\beta}{\tau} - \alpha\right) g_{\alpha,\beta}(t) - \frac{\beta g_{\alpha,\beta}(t)}{\tau}\int_\Omega \frac{u^r}{\xi^{s+1}}dx - \frac{4d_1(\alpha-1)}{\alpha\xi^\beta}\int_\Omega |\nabla w|^2 dx + \alpha\int_\Omega \frac{u^{p-1+\alpha}}{\xi^{\beta+q}}dx \\
&\leq \left(\frac{\beta}{\tau} - \alpha\right) g_{\alpha,\beta}(t) - \frac{4d_1(\alpha-1)}{\alpha\xi^\beta}\int_\Omega |\nabla w|^2 dx + \frac{\alpha}{\xi^\beta}\int_\Omega \frac{u^{p-1+\alpha}}{\xi^q}dx. \qquad (2.2)
\end{aligned}$$

Now, we write

$$\frac{u^{p-1+\alpha}}{\xi^q} = \left(\frac{u^r}{\xi^{s+1+a}}\right)^\rho u^\alpha = (g_a)^\rho w^2.$$

Here, g_a is defined in Lemma 2.2. Then, applying Hölder's inequality, we get

$$\int_\Omega \frac{u^{p-1+\alpha}}{\xi^q}dx \leq \left(\int_\Omega g_a dx\right)^\rho \left(\int_\Omega w^{\frac{2}{1-\rho}}dx\right)^{1-\rho} = \|g_a\|^\rho_{L^1(\Omega)}\|w\|^2_{L^{\frac{2}{1-\rho}}(\Omega)}.$$

Since $0 < \rho \leq \frac{2}{n+2} < \frac{2}{n}$, using Gagliardo-Nirenberg's inequality (see [1]), we can find a constant C_1 depending only on Ω, n and ρ such that

$$\|w\|_{L^{\frac{2}{1-\rho}}(\Omega)} \leq C_1[\|\nabla w\|^\gamma_{L^2(\Omega)}\|w\|^{1-\gamma}_{L^2(\Omega)} + \|w\|_{L^2(\Omega)}], \quad \text{where} \quad \gamma = \frac{n\rho}{2} \in (0,1).$$

Thus, it follows that

$$\int_\Omega \frac{u^{p-1+\alpha}}{\xi^q}dx \leq 2C_1^2\|g_a\|^\rho_{L^1(\Omega)}\left[\|\nabla w\|^{2\gamma}_{L^2(\Omega)}\|w\|^{2(1-\gamma)}_{L^2(\Omega)} + \|w\|^2_{L^2(\Omega)}\right].$$

From this last inequality and (2.2), we obtain

$$\begin{aligned}
\dot{g}_{\alpha,\beta}(t) \leq \left(\frac{\beta}{\tau} - \alpha\right) g_{\alpha,\beta}(t) - \frac{4d_1(\alpha-1)}{\alpha\xi^\beta}\int_\Omega |\nabla w|^2 dx + \\
+ \frac{2\alpha C_1^2}{\xi^\beta}\|g_a\|^\rho_{L^1(\Omega)}\left[\|\nabla w\|^{2\gamma}_{L^2(\Omega)}\|w\|^{2(1-\gamma)}_{L^2(\Omega)} + \|w\|^2_{L^2(\Omega)}\right].
\end{aligned} \qquad (2.3)$$

Now, let ϵ be a positive number. Using Young's inequality, we get

$$\begin{aligned}
\|g_a\|^\rho_{L^1(\Omega)}\|\nabla w\|^{2\gamma}_{L^2(\Omega)}\|w\|^{2(1-\gamma)}_{L^2(\Omega)} &= \left[\epsilon\|\nabla w\|^{2\gamma}_{L^2(\Omega)}\right]\left[\frac{1}{\epsilon}\|g_a\|^\rho_{L^1(\Omega)}\|w\|^{2(1-\gamma)}_{L^2(\Omega)}\right] \\
&\leq \gamma\left[\epsilon\|\nabla w\|^{2\gamma}_{L^2(\Omega)}\right]^{1/\gamma} + (1-\gamma)\left[\frac{1}{\epsilon}\|g_a\|^\rho_{L^1(\Omega)}\|w\|^{2(1-\gamma)}_{L^2(\Omega)}\right]^{\frac{1}{1-\gamma}} \\
&= \gamma\epsilon^{\frac{1}{\gamma}}\|\nabla w\|^2_{L^2(\Omega)} + \frac{(1-\gamma)}{\epsilon^{\frac{1}{1-\gamma}}}\|g_a\|^{\frac{\rho}{1-\gamma}}_{L^1(\Omega)}\|w\|^2_{L^2(\Omega)}. \qquad (2.4)
\end{aligned}$$

Then, by choosing ϵ sufficiently small so that $\alpha\gamma C_1^2\epsilon^{1/\gamma} < \frac{d_1(\alpha-1)}{\alpha}$, it follows from (2.3) and (2.4) that we can find a universal $C_2 > 0$ such that

$$\begin{aligned}\dot{g}_{\alpha,\beta}(t) &\leq \left(\frac{\beta}{\tau}-\alpha\right) g_{\alpha,\beta}(t) - \frac{2d_1(\alpha-1)}{\alpha\xi^\beta}\int_\Omega |\nabla w|^2 dx + \frac{C_2}{\xi^\beta}\left[\|g_a\|_{L^1(\Omega)}^{\frac{\rho}{1-\gamma}} + \|g_a\|_{L^1(\Omega)}^{\rho}\right]\|w\|_{L^2(\Omega)}^2 \\ &\leq \left\{\frac{\beta}{\tau}-\alpha + C_2\left[\|g_a\|_{L^1(\Omega)}^{\frac{\rho}{1-\gamma}} + \|g_a\|_{L^1(\Omega)}^{\rho}\right]\right\} g_{\alpha,\beta}(t).\end{aligned}$$

Now, let us denote

$$m(t) = \frac{\beta}{\tau} - \alpha + C_2\left[\|g_a(\cdot,t)\|_{L^1(\Omega)}^{\frac{\rho}{1-\gamma}} + \|g_a(\cdot,t)\|_{L^1(\Omega)}^{\rho}\right], \quad 0 \leq t < T.$$

We obtain

$$\dot{g}_{\alpha,\beta}(t) \leq m(t) g_{\alpha,\beta}(t). \tag{2.5}$$

Since $\gamma = \frac{n\rho}{2} \in (0,1)$ and $\rho = \frac{p-1}{r} \leq \frac{2}{n+2}$, it follows that $\frac{\rho}{1-\gamma} \leq 1$. By using Lemma 2.2 and Hölder's inequality, we can find a constant $\widetilde{C}_a = \widetilde{C}_a(T) < \infty$ such that

$$\int_0^{t'}\left[\|g_a(\cdot,t)\|_{L^1(\Omega)}^{\frac{\rho}{1-\gamma}} + \|g_a(\cdot,t)\|_{L^1(\Omega)}^{\rho}\right] dt \leq \widetilde{C}_a, \quad \text{for all} \quad 0 \leq t' < T.$$

Thus, for all $0 \leq t' < T$, we get

$$\int_0^{t'} m(t)dt = \left(\frac{\beta}{\tau}-\alpha\right)t' + C_2\int_0^{t'}\left[\|g_a(\cdot,t)\|_{L^1(\Omega)}^{\frac{\rho}{1-\gamma}} + \|g_a(\cdot,t)\|_{L^1(\Omega)}^{\rho}\right] dt \leq \left(\frac{\beta}{\tau}-\alpha\right)T + C_2\widetilde{C}_a.$$

Hence, it follows from (2.5) that

$$g_{\alpha,\beta}(t') \leq g_{\alpha,\beta}(0)e^{\int_0^{t'} m(t)dt} \leq C(T) < \infty, \quad \text{for all} \quad 0 \leq t' < T.$$

This completes the proof of Lemma 2.3. $\square$

Finally, by taking $\beta = 0$, it follows from Lemma 2.3 that (2.1) holds for all $l > 0$. The proof of Theorem 1.1 is now complete.

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