# OPTIMAL CONTROL OF ADVECTIVE DIRECTION IN REACTION-DIFFUSION POPULATION MODELS 

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#### Abstract

We investigate optimal control of the advective coefficient in a class of parabolic partial differential equations, modeling a population with nonlinear growth. This work is motivated by the question: Does movement toward a better resource environment benefit a population? Our objective functional is formulated with interpreting "benefit" as the total population size integrated over our finite time interval. Results on existence, uniqueness, and characterization of the optimal control are established. Our numerical illustrations for several growth functions and resource functions indicate that movement along the resource spatial gradient benefits the population, meaning that the optimal control is close to the spatial gradient of the resource function.


1. Introduction. The reaction of a species to spatially heterogeneous resources is an important ecological issue (see $[1,3,12,13]$ ). The reaction may result in movement with two features: directed advection and random diffusion [23, 21]. If a species could choose the direction for advective movement, how would such a choice be made? One would expect directed movement to be beneficial to the species, perhaps to increase the population level. Recent papers have given some answers to such a question (see $[1,5]$ ) and we consider this issue using tools of optimal control theory $([17,18,10])$.

In [1], Belgacem and Cosner studied the effects of advection along an environmental resource gradient in a logistic model

$$
\begin{equation*}
u_{t}=\nabla \cdot[D \nabla u-\alpha u \nabla m(x)]+u[m(x)-u], \quad \text { on } \quad \Omega \times(0, \infty), \tag{1.1}
\end{equation*}
$$

[^0]with no-flux boundary condition
$$
D \frac{\partial u}{\partial \nu}-\alpha u \frac{\partial m}{\partial \nu}=0, \quad \text { on } \quad \partial \Omega \times(0, \infty)
$$

For $\alpha$ sufficiently large, they showed persistence of the population. This result was in contrast to the results in [22], which imply that the advection constant $\alpha$ is not beneficial for a population modeled with lethal zero Dirichlet boundary condition and constant resource function $m$. Note that Belgacem and Cosner obtain "mixed" results in the Dirichlet boundary case, meaning sometimes advection is beneficial or sometimes not. Continuing along this line of investigation, Cosner and Lou [9] asked the question: Does movement towards better environments always benefit a population? They showed that advection in the direction of gradient of the resource is beneficial to the population if $\Omega$ is convex and there is a specific derivative condition on $m$. "Beneficial" for the population means the existence of a unique positive globally attracting steady state. They showed that for thin nonconvex domains, advection may not be beneficial, perhaps such a domain restricts the movement.

In related work, Cantrell, Cosner and Lou [5, 6] investigated both the effect of diffusion coefficient and advection along the "fitness" gradient, i.e. $\nabla[m(x)-u]$. In some cases, this mechanism is advantageous due to allowing populations to track resource approximately. They also showed that if the set of critical points of $m(x)$ has Lebesgue measure 0 , then the population size tends to 0 as $\alpha \rightarrow \infty$. See also $[8,15,16]$ for related work regarding $\alpha$. Cantrell et. al [7] proved that $m$ is the positive steady state if the advection term is $\ln (m(x))+C$, a constant. Also in [7], a two species competition model was considered to investigate whether one species can be invaded by other, with advection term, $\ln (m(x))$.

In a finite temporal setting, we seek to use the tools of optimal control to investigate a "different goal". In a parabolic PDE, we investigate the choice of advection direction to maximize the population over a region $\Omega$ while minimizing the cost of choosing such a direction. In our model, we use a more realistic spatial-temporal resource function $m(x, t)$ and a growth function more general than logistic. Movement usually comes with some risk or "cost" to populations. Optimal control methods can treat two competing goals - increasing population level while minimizing the costs incurred. The techniques of optimal control theory are extensions of Pontryagin's Maximum Principle [4] to partial differential equations. To justify the existence and characterization of an optimal control, using adjoint functions, one needs appropriate apriori estimates on solutions of the partial differential equations and analysis results. Interested readers can see the books by Lions [19] or Li and Yong [18] for background on such techniques.

Ding et. al [10] choose the control as the resource function $m(x)$ in an elliptic PDE (like the equation (1.1) with no advection) to maximize the population with an
integral constraint on $m$ (fixed amount of resource). The optimal control depended on the diffusion coefficient and the amount of total resource.

In the next section, we formulate our control problem giving the model with PDE with general nonlinearity and setting the objective functional. Section 3 gives apriori estimates and existence of non-negative solutions to our PDE problems. The existence of an optimal control is shown in section 4. The necessary conditions satisfied by an optimal control are derived in section 5. Uniqueness of optimal control and the stability dependence of the optimal control on the resource function $m$ are in section 6. Our numerical results are illustrated in section 7 followed by the last section with conclusions.
2. Problem Formulation. Let $u$ represent the population density of a species occupying a spatial open bounded domain $\Omega$ in $d$-dimensional space $\mathbb{R}^{d}, d \in \mathbb{N}$, and with smooth boundary $\partial \Omega$ of $\Omega$. For some given fixed time $0<T<\infty$, let us denote $Q_{T}=\Omega \times[0, T)$, and $\mathcal{S}_{T}=\partial \Omega \times[0, T)$. We consider the following model in population dynamics with nonlinear growth, and zero flux boundary condition:

$$
\left\{\begin{array}{cccc}
u_{t}-\nabla \cdot[\mu \nabla u-u \vec{h}] & = & u[m-f(x, t, u)], & Q_{T}  \tag{2.1}\\
\mu \frac{\partial u}{\partial \nu}-u \vec{h} \cdot \nu & = & 0, & \mathcal{S}_{T} \\
u(\cdot, 0) & = & u_{0}, & \Omega
\end{array}\right.
$$

Here $\mu>0$ is the diffusion coefficient which is fixed, $\nu$ is the outward normal vector on $\partial \Omega$, and $m=m(x, t)$ is the "resource" function which is assumed to be bounded. Unlike in the previous work, our $m$ is allowed to depend on time. Our control function $\vec{h} \in L^{\infty}\left(Q_{T}\right)^{d}$ is an advective vector field which again is allowed to be time dependent

$$
\vec{h}: Q_{T} \rightarrow \mathbb{R}^{d}, \quad \vec{h}(x, t)=\left(h_{1}(x, t), h_{2}(x, t), \cdots, h_{d}(x, t)\right), \quad(x, t) \in Q_{T}
$$

The initial function $u_{0}$ is given and assumed to be non-negative bounded and is in the Sobolev space $H^{1}(\Omega)$. In the growth term, we assume that the function $f: Q_{T} \times \mathbb{R} \mapsto \mathbb{R}$ satisfies the following natural conditions:
(i) $f$ is continuous on $Q_{T} \times \mathbb{R}$ and there are positive constants $C$ and $\alpha$ such that $0<\alpha<\frac{4}{(d-2)^{+}}$, where $(d-2)^{+}=\max \{d-2,0\}$, and

$$
0 \leq f(x, t, u) \leq C\left[1+|u|^{\alpha}\right], \quad \forall(x, t) \in Q_{T}, \quad \forall u \in \mathbb{R} .
$$

(ii) $f$ is locally Lipschitz with respect to $u \in[0, \infty)$, i.e. for all $R>0$, there is $C_{R}>0$ such that

$$
|f(x, t, u)-f(x, t, v)| \leq C_{R}|u-v|, \forall(x, t) \in Q_{T}, \forall u, v \in[0, R]
$$

(iii) For each $(x, t) \in Q_{T}$, the function $u \mapsto u f(x, t, u)$ is differentiable on $[0, \infty)$, and

$$
g(x, t, u)=\frac{\partial[u f(x, t, u)]}{\partial u}, \quad|g(x, t, u)| \leq C\left[1+|u|^{\alpha}\right], \quad(x, t) \in Q_{T}, \quad u \geq 0
$$

(iv) $g$ is locally Lipschitz with respect to $u \in[0, \infty)$, i.e.

$$
|g(x, t, u)-g(x, t, v)| \leq C_{R}|u-v|, \forall(x, t) \in Q_{T}, \forall u, v \in[0, R] .
$$

Throughout this paper, we shall denote by $H^{1}(\Omega)$ the usual Sobolev space and its dual space is $H^{1}(\Omega)^{*}$. Also, we denote by $V_{2}\left(Q_{T}\right)$ the space of all functions $u \in L^{2}\left((0, T), H^{1}(\Omega)\right)$ such that its norm

$$
\|u\|_{V_{2}\left(Q_{T}\right)}=\left\{\sup _{0 \leq t \leq T} \int_{\Omega} u(x, t)^{2} d x+\int_{Q_{T}}|\nabla u(x, t)|^{2} d x d t\right\}^{1 / 2}<\infty
$$

The function $u \in L^{2}\left((0, T), H^{1}(\Omega)\right)$ with $u_{t} \in L^{2}\left((0, T), H^{1}(\Omega)^{*}\right)$ and $u(0, \cdot)=u_{0}$ is said to be a solution (in weak sense) of (2.1) if and only if for a.e. $t \in(0, T)$

$$
\begin{equation*}
\int_{\Omega} u_{t} \phi d x+\int_{\Omega}[\mu \nabla u-\vec{h} u] \cdot \nabla \phi d x=\int_{\Omega}[m-f(x, t, u)] u \phi d x, \quad \forall \phi \in H^{1}(\Omega) . \tag{2.2}
\end{equation*}
$$

The reaction-diffusion equation (2.1) is a general model which arises naturally in many mathematical and biological settings. In particular, when $f(x, t, u)=|u|$, the equation (2.1) has logistic growth and it has been studied extensively (e.g. see $[3,4,17,21]$ and references therein). Also, we would like to remark that for this type of nonlinearity, we require that our dimensional space satisfies $1 \leq d \leq 5$ (recall that $\left.\alpha<\frac{4}{(d-2)^{+}}\right)$.

Turning to the optimal control formulation, for a given "cost constant" $B>0$, we search for $\vec{h}^{*} \in U$ such that

$$
J\left(\vec{h}^{*}\right)=\sup _{\vec{h} \in U} J(\vec{h})
$$

where the objective functional is defined by

$$
\begin{equation*}
J(\vec{h})=\int_{Q_{T}}\left[u-B|\vec{h}|^{2}\right] d x d t \tag{2.3}
\end{equation*}
$$

with $u=u(\vec{h})$ is the solution of the equations (2.1) for the corresponding given $\vec{h}$, and our control set $U$ is defined as

$$
\begin{equation*}
U=\left\{\vec{h} \in\left[L^{\infty}\left(Q_{T}\right)\right]^{d}:\left|h_{i}\right| \leq M \text { for }, i=1,2, \cdots, d\right\} \tag{2.4}
\end{equation*}
$$

for some given fixed constant $M>0$. Our objective functional represents the benefit less the cost where the benefit is the total population integrated over time and the cost of the control reflects the risk due to the movement of individuals. Note that a similar problem for the elliptic case where the resource $m$ is considered to be the control was considered in [10] when $f(x, t, u)=|u|$. Also, note that unlike in the mentioned works $[3,4,5,9,10]$, our resource function $m$ and the advective vector field $\vec{h}$ are allowed to be time-dependent.
3. Apriori Estimates and Existence of Solutions. In this section, we establish some preliminary results needed for proving the existence and characterizing our optimal solutions. In the first lemma, we prove that all solutions $u$ of (2.1) must be positive.

Lemma 3.1. Assume that $|\vec{h}|$ and $m$ are in $L^{\infty}\left(Q_{T}\right)$ and $u_{0} \geq 0$. Then, any solution $u$ of (2.1) must be non-negative on $Q_{T}$.

Proof. First of all, note that in $[3,4,5,9]$, the advection term $\vec{h}$ is assumed to be the gradient of some time-independent function. Because of this and by a simple change of variables, it follows from the classical maximum principle that solutions $u$ must be non-negative. In our case here, it seems impossible to have such a change of variables. Therefore, we need a different approach. In what follows, we use Stampacchia's truncation method (see [2, p. 301] for the elliptic version of this method). For arbitrary $0 \leq T^{\prime}<T$, and for $(x, t) \in Q_{T^{\prime}}$, let $w(x, t)=$ $\max \{-u(x, t), 0\}$. Multiplying (2.1) with $w$ and using integration by parts, we get $\int_{\Omega} u_{t} w d x+\mu \int_{\Omega} \nabla u \nabla w d x=\int_{\Omega}[u \nabla w \cdot \vec{h}] d x+\int_{\Omega} u w[m-f(x, t, u)] d x, \quad 0 \leq t \leq T^{\prime}$. Note that on the set $\left\{(x, t) \in Q_{T^{\prime}}: u(x, t)<0\right\}$,

$$
w=-u, \quad u_{t}=-w_{t}, \quad \nabla u=-\nabla w .
$$

Thus, it follows from Young's inequality and Hölder's inequality that

$$
\begin{aligned}
\frac{1}{2} \frac{d}{d t} \int_{\Omega} w^{2} d x+\mu \int_{\Omega}|\nabla w|^{2} d x & =\int_{\Omega}\left[w \nabla w \cdot \vec{h}+w^{2}(m-f(x, t, u)] d x\right. \\
& \leq \int_{\Omega}\left[w|\nabla w||\vec{h}| d x+w^{2} m^{+}\right] d x \\
& \leq C_{T^{\prime}} \int_{\Omega}\left[w|\nabla w|+w^{2}\right] d x \\
& \leq C_{T^{\prime}} \int_{\Omega}\left[\epsilon|\nabla w|^{2}+\left(1+\frac{1}{4 \epsilon}\right) w^{2}\right] d x, \quad 0 \leq t \leq T^{\prime}
\end{aligned}
$$

Here $C_{T^{\prime}}:=\max \left\{\|\vec{h}\|_{L^{\infty}\left(Q_{T^{\prime}}\right)},\left\|m^{+}\right\|_{L^{\infty}\left(Q_{T^{\prime}}\right)}\right\}<\infty$ and $\epsilon$ is any positive number. By choosing $\epsilon$ so small that $C_{T^{\prime}} \epsilon<\mu / 2$, we obtain

$$
\frac{d}{d t} \int_{\Omega} w^{2} d x \leq \frac{d}{d t} \int_{\Omega} w^{2} d x+\mu \int_{\Omega}|\nabla w|^{2} d x \leq C_{T^{\prime}}\left(2+\frac{1}{2 \epsilon}\right) \int_{\Omega} w^{2} d x, \quad 0 \leq t \leq T^{\prime} .
$$

Since $u_{0} \geq 0$, it follows that $w(\cdot, 0)=0$ on $\Omega$. Applying Grownwall's inequality, we get

$$
\int_{\Omega} w^{2}(x, t) d x=0, \quad \forall t \in\left[0, T^{\prime}\right] .
$$

Hence, $w=0$ a.e on $Q_{T^{\prime}}$, and we conclude that $u \geq 0$ a.e. on $Q_{T^{\prime}}$. Since $T^{\prime}$ is arbitrary, it follows that $u \geq 0$ a.e. on $Q_{T}$.

Next, we show that our objective functional $J$ is bounded above.

Lemma 3.2. Assume that $C_{m}=\sup _{Q_{T}}|m|<\infty, u_{0} \in L^{\infty}\left(Q_{T}\right)$ and $u_{0} \geq 0$. Then for each $\vec{h} \in U$, any solution $u$ of (2.1) satisfies

$$
\int_{\Omega} u(x, t) d x \leq e^{t C_{m}} \int_{\Omega} u_{0}(x) d x, \forall t \in[0, T)
$$

In particular,

$$
J(\vec{h}) \leq \frac{\left[e^{T C_{m}}-1\right]\left\|u_{0}\right\|_{L^{\infty}}}{C_{m}}, \quad \forall \vec{h} \in U .
$$

Proof. From Lemma 3.1, we see that $u \geq 0$. Then, the lemma follows directly by integrating (2.1) on $\Omega$ and using integration by parts.

The rest of the section is devoted to prove the existence and apriori estimates of the solutions $u$ of the equations (2.1). Our first result is a simple energy estimate.

Lemma 3.3. Assume that $\|m\|_{L^{\infty}\left(Q_{T}\right)} \leq \beta<\infty, u_{0} \in L^{\infty}\left(Q_{T}\right)$ and $u_{0} \geq 0$. Then, for every $\vec{h}$ in $U$, there exists a constant $C>0$ depending on $\beta, M, \mu,|\Omega|$ and $T$ such that the following estimate holds for any solution $u$ of (2.1)

$$
\|u\|_{V_{2}\left(Q_{T}\right)}^{2}=\sup _{t \in[0, T]} \int_{\Omega}|u(x, t)|^{2} d x+\int_{Q_{T}}|\nabla u(x, t)|^{2} d x d t \leq C\left\|u_{0}\right\|_{L^{2}(\Omega)}^{2} .
$$

Proof. It follows from Lemma 3.1 that $u \geq 0$. Thus, multiplying the first equation of (2.1) with $u$ and applying integration by parts, we get

$$
\begin{aligned}
\frac{1}{2} \frac{d}{d t} \int_{\Omega} u^{2} d x+\mu \int_{\Omega}|\nabla u|^{2} d x & =\int_{\Omega}\left[u \nabla u \cdot \vec{h}+m u^{2}-u^{2} f(x, t, u)\right] d x \\
& \leq \int_{\Omega}\left[u \nabla u \cdot \vec{h}+m^{+} u^{2}\right] d x
\end{aligned}
$$

Then, from Young's inequality, it follows

$$
\frac{1}{2} \frac{d}{d t} \int_{\Omega} u^{2} d x+\mu \int_{\Omega}|\nabla u|^{2} d x \leq \frac{\mu}{2} \int_{\Omega}|\nabla u|^{2} d x+\int_{\Omega}\left(\frac{2}{\mu}|\vec{h}|^{2}+m^{+}\right) u^{2} d x
$$

This implies that there is a constant $C_{1}$ depending on $M, \beta, \mu$ such that

$$
\begin{equation*}
\frac{d}{d t} \int_{\Omega} u(x, t)^{2} d x+\mu \int_{\Omega}|\nabla u(x, t)|^{2} d x \leq C_{1} \int_{\Omega} u^{2} d x \tag{3.1}
\end{equation*}
$$

By Grownwall's inequality, we obtain

$$
\int_{\Omega} u(x, t)^{2} d x \leq e^{C_{1} T} \int_{\Omega} u_{0}(x) d x, \quad \forall 0 \leq t<T .
$$

The lemma then follows from this and (3.1).
Next, we show that all solutions $u$ must be bounded. Moreover, the bounds only depend on $M$ and the upper bound of the resource $m$, not on the structure of the resource $m$ nor the advective vector field $\vec{h}$. This is essential in studying our optimal control problem and it is also the most important result of this section. To prove this apriori result, we adapt a well-known iteration technique in [20].

Lemma 3.4. Assume that $\|m\|_{L^{\infty}\left(Q_{T}\right)} \leq \beta<\infty$. Then, for every non-negative bounded $u_{0}$ and for every $\vec{h} \in U$, there exists $C>0$ depending on $\beta, M, \mu,|\Omega|, T, d$ and $\left\|u_{0}\right\|_{L^{\infty}(\Omega)}$ such that every weak solution $u$ of (2.1) satisfies

$$
\|u\|_{L^{\infty}\left(Q_{T}\right)} \leq C .
$$

Proof. For any fixed $K \in \mathbb{N}$, let $0=t_{0}<t_{1}<\cdots<t_{K}=T$ be a partition of [0,T] which will be determined. For each $i=1,2, \cdots, K$ let $Q_{i}=\Omega \times\left[t_{i-1}, t_{i}\right]$ and

$$
\|u\|_{V_{2}\left(Q_{i}\right)}^{2}=\sup _{t \in\left[t_{i-1}, t_{i}\right]} \int_{\Omega} u^{2}(x, t) d x+\int_{Q_{i}}|\nabla u(x, t)|^{2} d x d t .
$$

By Lemma 3.1, we have $u \geq 0$. Therefore, we only need to show that $u$ is bounded above. It suffices to show that $u$ is bounded above on $Q_{i}$ for all $i=1,2, \cdots, K$. From Lemma 3.3, we see that there exists a finite constant $C_{0}=C\left(\mu, M, T,|\Omega|, u_{0}\right)$ such that

$$
\begin{equation*}
\|u\|_{V_{2}\left(Q_{i}\right)} \leq C_{0}, \quad \forall i=1,2, \cdots, K \tag{3.2}
\end{equation*}
$$

Next, for each $k>\hat{k} \xlongequal{\text { def }}\left\|u_{0}\right\|_{L^{\infty}(\Omega)}+1$, let us denote

$$
u^{(k)}(x, t)=\max \{u(x, t)-k, 0\} .
$$

Also, denote the sets

$$
A_{k}(t)=\{x \in \Omega: u(x, t)>k\}, \quad \mathcal{Q}_{i}(k)=\left\{(x, t) \in Q_{i}: u(x, t)>k\right\}, \quad i=1, \cdots, K .
$$

Multiplying the first equation of (2.1) by $u^{(k)}$ and using integration by parts, we get

$$
\begin{aligned}
\frac{1}{2} \frac{d}{d t} \int_{\Omega} u^{(k)}(x, t)^{2} d x+\mu \int_{\Omega}\left|\nabla u^{(k)}\right|^{2} d x & =\int_{\Omega}\left[u \vec{h} \cdot \nabla u^{(k)}+u u^{(k)}[m-f(x, t, u)] d x\right. \\
& =\int_{A_{k}(t)}\left[u \vec{h} \cdot \nabla u^{(k)}+u u^{(k)}[m-f(x, t, u)] d x\right. \\
& \leq \int_{A_{k}(t)}\left[u \vec{h} \cdot \nabla u^{(k)}+u u^{(k)} m^{+}\right] d x .
\end{aligned}
$$

As in the proof of Lemma 3.3, we can use the Young's inequality and the fact that $\vec{h}$ and $m^{+}$are in $L^{\infty}\left(Q_{T}\right)$ to get

$$
\begin{align*}
\frac{1}{2} \frac{d}{d t} \int_{\Omega} u^{(k)}(x, t)^{2} d x+\frac{\mu}{2} \int_{\Omega}\left|\nabla u^{(k)}\right|^{2} d x & \leq \int_{A_{k}(t)}\left[\left.\frac{2}{\mu} \right\rvert\, \vec{h}^{2} u^{2}+u u^{(k)} m^{+}\right] d x \\
& \leq 4 \int_{A_{k}(t)}\left(\frac{2}{\mu}|\vec{h}|^{2}+m^{+}\right)\left[(u-k)^{2}+k^{2}\right] d x \\
& \leq C_{2} \int_{A_{k}(t)}\left[(u-k)^{2}+k^{2}\right] d x \tag{3.3}
\end{align*}
$$

for some constants $C_{2}>0$ depending only on $\mu, M, \beta$. Note that $u^{(k)}(\cdot, 0)=0$. Thus, by integrating this equation in time on $[0, t]$ with $0<t<t_{1}$, we obtain

$$
\begin{equation*}
\left\|u^{(k)}\right\|_{V_{2}\left(Q_{1}\right)}^{2} \leq C_{3} \int_{\mathcal{Q}_{1}(k)}\left[(u-k)^{2}+k^{2}\right] d x d t, \quad \text { with } \quad C_{3}=2 \min \{1, \mu\}^{-1} C_{2} \tag{3.4}
\end{equation*}
$$

Note that

$$
\begin{aligned}
\int_{\mathcal{Q}_{1}(k)}(u-k)^{2} d x d t & =\int_{\mathcal{Q}_{1}(k)}\left[u^{(k)}\right]^{2} d x d t \\
& \leq t_{1} \sup _{0<t<t_{1}} \int_{\Omega}\left[u^{(k)}(x, t)\right]^{2} d x \leq t_{1}\left\|u^{(k)}\right\|_{V_{2}\left(Q_{1}\right)}^{2}
\end{aligned}
$$

Therefore, choosing $t_{1}$ sufficiently small such that $t_{1} C_{3}<\frac{1}{2}$ yields

$$
\left\|u^{(k)}\right\|_{V_{2}\left(Q_{1}\right)}^{2} \leq 2 C_{3} k^{2} \sigma(k), \quad \text { where } \quad \sigma(k)=\left|\mathcal{Q}_{1}(k)\right|=\int_{0}^{t_{1}}\left|A_{k}(t)\right| d t
$$

Equivalently,

$$
\begin{equation*}
\left\|u^{(k)}\right\|_{V_{2}\left(Q_{1}\right)} \leq \mu_{1} k \sigma(k)^{\frac{1}{2}}, \quad \forall k>\hat{k} \tag{3.5}
\end{equation*}
$$

for some positive constant $\mu_{1}$ which depends on $\beta, \mu$ and $M$ only. From (3.5), we claim that there is a constant $c_{1}>0$ depending only on $\beta, \mu, T,|\Omega|, d$ and $\left\|u_{0}\right\|_{L^{\infty}(\Omega)}$ such that

$$
\begin{equation*}
\sup _{Q_{1}} u(x, t) \leq c_{1} . \tag{3.6}
\end{equation*}
$$

The proof of (3.6) is standard (see [20, Theorem 6.1, p. 102]). However, since we need to keep track all of the constants in the estimates to make sure that they only depend on the upper bound of the control function $\vec{h}$, but not on its structure, and also for completeness, we provide the details of the proof here. First of all, for all $2 \leq r \leq \frac{2(d+2)}{d}$, by the Sobolev embedding theorem (see [20, eqn (3.8), p. 77], we can find a constant $\beta_{0}>0$ depending only on $|\Omega|, d, r$ and $T$ such that

$$
\begin{equation*}
\|w\|_{L^{r}\left(Q_{k}\right)} \leq \beta_{0}\|w\|_{V_{2}\left(Q_{k}\right)}, \quad \forall w \in V_{2}\left(Q_{k}\right), \quad \forall k \in\{1,2, \cdots, K\} . \tag{3.7}
\end{equation*}
$$

Let $M_{0}=m_{0} \hat{k}$ for some $m_{0}>1$ which will be determined. Also, for $i=0,1,2, \cdots$, let us denote $k_{i}=M_{0}\left(2-2^{-i}\right)$. It follows directly from the definition of $\sigma$ that

$$
\begin{equation*}
\left(k_{i+1}-k_{i}\right) \sigma^{\frac{1}{r}}\left(k_{i+1}\right) \leq\left\|u^{\left(k_{i}\right)}\right\|_{L^{r}\left(Q_{1}\right)}, \quad \forall i \in \mathbb{N} \cup\{0\} . \tag{3.8}
\end{equation*}
$$

From now, we fix $2<r<\frac{2(d+2)}{d}$ and write $r=2(1+\kappa)$ with some $\kappa>0$. Since $k_{i}>\hat{k}$ for all $i$, it follows from the inequalities (3.5) and (3.7) that

$$
\begin{equation*}
\left\|u^{\left(k_{i}\right)}\right\|_{L^{r}\left(Q_{1}\right)} \leq \beta_{0}\left\|u^{\left(k_{i}\right)}\right\|_{V_{2}\left(Q_{1}\right)} \leq \beta_{0} \mu_{1} k_{i} \sigma\left(k_{i}\right)^{\frac{1+\kappa}{r}}, \quad \forall i \in \mathbb{N} \cup\{0\} \tag{3.9}
\end{equation*}
$$

Then, combining the inequalities (3.8) and (3.9), we obtain

$$
\begin{equation*}
\sigma\left(k_{i+1}\right)^{\frac{1}{r}} \leq \frac{\beta_{0} \mu_{1} k_{i}}{k_{i+1}-k_{i}} \sigma\left(k_{i}\right)^{\frac{1+\kappa}{r}} \leq 4 \beta_{0} \mu_{1} 2^{i} \sigma\left(k_{i}\right)^{\frac{1+\kappa}{r}}, \quad \forall i \in \mathbb{N} \cup\{0\} \tag{3.10}
\end{equation*}
$$

For all $i=0,1, \cdots$, let $y_{i}=\sigma\left(k_{i}\right)^{\frac{1}{r}}$. Then, it follows directly from the recursion formula (3.10) and a simple calculation that

$$
\begin{equation*}
y_{i} \leq\left[4 \beta_{0} \mu_{1}\right]^{\frac{(1+\kappa)^{i}-1}{\kappa}} 2^{\frac{(1+\kappa)^{i}-1}{\kappa^{2}}-\frac{i}{\kappa}} y_{0}^{(1+\kappa)^{i}}, \quad \forall i=0,1,2, \cdots \tag{3.11}
\end{equation*}
$$

On the other hand, by substituting $\hat{k}$ for $k_{i}$ and $M_{0}$ for $k_{i+1}$ in (3.8) and using (3.9), we obtain

$$
\sigma\left(M_{0}\right)^{\frac{1}{r}} \leq \frac{\beta_{0} \mu_{1}}{m_{0}-1} \sigma(\hat{k})^{\frac{1+\kappa}{r}} \leq \frac{\beta_{0} \mu_{1}}{m_{0}-1}[T|\Omega|]^{\frac{1}{2}}
$$

Thus, by choosing

$$
m_{0}=1+\beta_{0} \mu_{1}[T|\Omega|]^{\frac{1}{2}}\left(4 \beta_{0} \mu_{1}\right)^{\frac{1}{\kappa}} 2^{\frac{1}{\kappa^{2}}}
$$

we have

$$
\begin{equation*}
y_{0}=\sigma\left(k_{0}\right)^{\frac{1}{\kappa}}=\sigma\left(M_{0}\right)^{\frac{1}{\kappa}} \leq\left(4 \beta_{0} \mu_{1}\right)^{-\frac{1}{\kappa}} 2^{-\frac{1}{\kappa^{2}}} \tag{3.12}
\end{equation*}
$$

Then, it follows from the inequalities (3.11) and (3.12) that

$$
y_{i} \leq\left[4 \beta_{0} \mu_{1}\right]^{-\frac{1}{\kappa}} 2^{-\frac{1}{\kappa}} 2^{-\frac{i}{\kappa}}, \quad \text { for all } \quad i=0,1,2, \cdots
$$

In particular, $y_{i}=\sigma\left(k_{i}\right)^{\frac{1}{\kappa}} \rightarrow 0$ as $i \rightarrow \infty$. Hence, $\sigma\left(2 M_{0}\right)=0$ and therefore, on $Q_{1}$,

$$
\begin{equation*}
u \leq c_{1} \xlongequal{\text { def }} 2 m_{0} \hat{k}=2\left\{1+\beta_{0} \mu_{1}[T|\Omega|]^{\frac{1}{2}}\left(4 \beta_{0} \mu_{1}\right)^{\frac{1}{\kappa}} 2^{\frac{1}{\kappa^{2}}}\right\}\left\{\left\|u_{0}\right\|_{L^{\infty}\left(Q_{T}\right)}+1\right\} \tag{3.13}
\end{equation*}
$$

This proves (3.6). Next, note that similar to the choice of $t_{1}$, we choose $K \in \mathbb{N}$ sufficiently large so that

$$
\begin{equation*}
C_{3}\left|t_{k}-t_{k-1}\right|<\frac{1}{2} \quad \text { and } \quad k=2, \cdots, K \tag{3.14}
\end{equation*}
$$

where $C_{3}$ is defined in (3.4) which only depends on $\beta, \mu, M$. Therefore, by the same proof as that of $(3.6)$ using $u_{1}(\cdot) \xlongequal{\text { def }} u\left(\cdot, t_{1}\right)$ as $u_{0}$, we can prove that $u$ is bounded above on $Q_{2}$ by a constant $c_{2}$. Keep doing this, we arrive at

$$
\sup _{Q_{i}} u(x, t) \leq c_{i}, \quad \text { for all } \quad i=2,3, \cdots, K
$$

where all of the constants $c_{i}$ can be explicitly defined as

$$
\begin{equation*}
c_{i}=2\left\{1+\beta_{0} \mu_{1}[T|\Omega|]^{\frac{1}{2}}\left(4 \beta_{0} \mu_{1}\right)^{\frac{1}{\kappa}} 2^{\frac{1}{\kappa^{2}}}\right\}\left(c_{i-1}+1\right) \tag{3.15}
\end{equation*}
$$

Note that at the $i^{\text {th }}$ step, for $i=2, \cdots, K, u_{i}=u\left(\cdot, t_{i-1}\right)$ is used as the initial data. That is how we derived (3.15). Also, it follows from (3.15) that $c_{1} \leq c_{2} \leq \cdots \leq c_{K}$. Moreover, from (3.14), we see that we can chose $K$ that depends only on $\beta, \mu, M$ and $T$, explicitly $K>2 T C_{3}$. Therefore, it follows from (3.13) and (3.15) that $c_{K}$ depends only on $\beta, \mu, M,|\Omega|, T,\left\|u_{0}\right\|_{L^{\infty}}$ and the dimension $d$. Moreover,

$$
\sup _{Q_{T}} u \leq C, \quad \text { with } \quad C=c_{K}=\max \left\{c_{i}, i=1,2, \cdots, K\right\} .
$$

The proof of the theorem is therefore complete.
We conclude this section with the following important theorem:
Theorem 3.1. Let $0<T<\infty, m \in L^{\infty}\left(Q_{T}\right)$ and $u_{0}$ be non-negative, bounded and in $H^{1}(\Omega)$. Then, for each $\vec{h} \in U$, there is a unique weak solution $u=$ $u(\vec{h})$ of (2.1). Moreover, there is a finite constant $C>0$ depending only on $|\Omega|, \mu, d, T,\|m\|_{L^{\infty}},\left\|u_{0}\right\|_{L^{\infty}}$ and $M$ such that

$$
\begin{equation*}
\|u(\vec{h})\|_{V_{2}\left(Q_{T}\right)} \leq C, \quad \text { and } \quad 0 \leq u(\vec{h}) \leq C, \quad \forall(x, t) \in Q_{T} \tag{3.16}
\end{equation*}
$$

Proof. We look for $u \in L^{2}\left((0, T), H^{1}(\Omega)\right), u_{t} \in L^{2}\left((0, T), H^{1}(\Omega)^{*}\right)$ such that $u(\cdot, 0)=u_{0}$ and for a.e. $0 \leq t \leq T$,

$$
\begin{equation*}
\int_{\Omega} u_{t} \phi d x+\int_{\Omega}[\mu \nabla u-\vec{h} u] \cdot \nabla \phi d x=\int_{\Omega}[m-f(x, t, u)] u \phi d x, \quad \forall \phi \in H^{1}(\Omega) \tag{3.17}
\end{equation*}
$$

Note that since $\alpha<\frac{4}{(d-2)^{+}}$, we can choose constants $p>1, q>1$ such that

$$
\frac{1}{p}+\frac{1}{q}=1, \quad \text { and } \quad p(1+\alpha), q<\frac{2 d}{(d-2)^{+}}
$$

Thus, by using the Hölder's inequality, (i) and the Sobolev's embedding theorem, we get

$$
\left|\int_{\Omega} f(x, t, u) u \phi d x\right| \leq C\left[\|\phi\|_{L^{2}}\|u\|_{L^{2}}+\|\phi\|_{L^{q}}\|u\|_{L^{p(1+\alpha)}}^{1+\alpha}\right] \leq C\|\phi\|_{H^{1}}\|u(\cdot, t)\|_{H^{1}}^{\alpha+1}
$$

This implies that all of the integrals in (3.17) are all well-defined. From this and the apriori estimates (Lemmas 3.1-3.4), it follows easily from the Galerkin's method (e.g. see [11]) that there exists at least one weak solution of (2.1) which satisfies (3.16).

Next, we show that for each $\vec{h} \in U$, there is only one weak solution of (2.1). Again, we shall use Stampacchia's truncation method. Indeed, for each $\vec{h} \in U$ let us denote $u$ and $v$ be two solutions of (2.1). It follows from Lemmas 3.1-3.4 that

$$
\begin{equation*}
0 \leq u, v \leq C, \quad\|u\|_{V_{2}\left(Q_{T}\right)}, \quad\|v\|_{V_{2}\left(Q_{T}\right)} \leq C \tag{3.18}
\end{equation*}
$$

Let $w=u-v$, we obtain

$$
\left\{\begin{array}{cccc}
w_{t}-\nabla \cdot[\mu \nabla w-w \vec{h}] & = & w[m-b], & Q_{T}  \tag{3.19}\\
\nu \cdot[\mu w-w \vec{h}] & = & 0, & S_{T} \\
w(\cdot, 0) & = & 0, & \Omega
\end{array}\right.
$$

Here,

$$
b(x, t)=\frac{u(x, t) f(x, t, u(x, t))-v(x, t) f(x, t, v(x, t))}{u(x, t)-v(x, t)}, \quad \forall(x, t) \in Q_{T}
$$

From the assumptions (i)-(ii) and the $L^{\infty}$-estimates of $u$ and $v$ as in (3.18), it follows that $b$ is bounded on $Q_{T}$. Therefore, we can chose $K>0$ and sufficiently large such that $m-b-K<0$ on $Q_{T}$. Let $z(x, t)=e^{-K t} w(x, t)$. It follows from (3.19) that $z$ solves the equations

$$
\left\{\begin{array}{cccc}
z_{t}-\nabla \cdot[\mu \nabla z-z \vec{h}] & = & z[m-b-K], & Q_{T}, \\
\nu \cdot[\mu \nabla z-z \vec{h}] & = & 0, & S_{T} \\
z(\cdot, 0) & = & 0, & \Omega
\end{array}\right.
$$

Multiplying this equation by $z$ and using integration by parts gives

$$
\frac{1}{2} \frac{d}{d t} \int_{\Omega} z^{2} d x+\mu \int_{\Omega}|\nabla z(x, t)|^{2} d x=\int_{\Omega}[m-b-K] z^{2}-\int_{\Omega} \nabla z \cdot \vec{h} z(x, t) d x
$$

As in the proof of Lemma 3.1, we can find a constant $C>0$ such that

$$
\frac{d}{d t} \int_{\Omega} z^{2}(x, t) d x \leq C \int_{\Omega} z^{2}(x, t) d x, \quad \forall t \in[0, T)
$$

From this and since $z(\cdot, 0)=0$, it follows that $z=0$ and then $w=0$. Therefore, $u=v$ and this completes the proof of Theorem 3.1.
4. Existence of an Optimal Control. To investigate the maximum of our objective functional, we first show the existence of an optimal control.

Theorem 4.1. Assume that $0<T<\infty, m \in L^{\infty}\left(Q_{T}\right), u_{0} \in L^{\infty}(\Omega) \cap H^{1}(\Omega)$ and $u_{0}$ is non-negative. There exists an optimal control $\vec{h}^{*} \in U$ maximizing the objective functional $J(\vec{h})$.

Proof. By Lemma 3.2, $J(\vec{h}) \leq C$ where $C$ is a constant depending on $\|m\|_{L^{\infty}\left(Q_{T}\right)}$, $\left\|u_{0}\right\|_{L^{\infty}\left(Q_{T}\right)}$ and $T$ only. Therefore, $\sup _{\vec{h} \in U} J(\vec{h})$ exists. Let us denote $\left\{\vec{h}^{n}\right\} \subset U$, be a maximizing sequence, i.e.

$$
\begin{equation*}
\lim _{n \rightarrow \infty} J\left(\vec{h}^{n}\right)=\sup _{\vec{h} \in U} J(\vec{h}) . \tag{4.1}
\end{equation*}
$$

Let $u^{n}=u\left(\vec{h}^{n}\right)$, the corresponding solutions of (2.1) when the control $\vec{h}$ is $\vec{h}^{n}$. It follows from Lemma 3.3 and Lemma 3.4 that

$$
\begin{equation*}
\left\|u^{n}\right\|_{V_{2}\left(Q_{T}\right)}, \quad\left\|u^{n}\right\|_{L^{\infty}\left(Q_{T}\right)} \leq C<\infty, \quad \forall n \in \mathbb{N} \tag{4.2}
\end{equation*}
$$

for some constant $C>0$ depending only on $\mu, d,|\Omega|, T,\left\|u_{0}\right\|_{L^{\infty}},\|m\|_{L^{\infty}\left(Q_{T}\right)}$ and $M$, where $M$ is a fixed constant defined in (2.4).

From (4.2) and by passing to a subsequence, we can assume that

$$
\begin{equation*}
u^{n} \rightharpoonup u^{*} \text { in } L^{2}\left(0, T, H^{1}(\Omega)\right) . \tag{4.3}
\end{equation*}
$$

On the other hand, for each $n$ and each $\phi \in L^{2}\left(0, T, H^{1}(\Omega)\right)$, the weak form of the solution $u^{n}$ is

$$
\begin{aligned}
\int_{Q_{T}} u_{t}^{n} \phi d x d t=-\mu & \int_{Q_{T}} \nabla u^{n} \cdot \nabla \phi d x d t+\int_{Q_{T}}\left(\overrightarrow{h^{n}} \cdot \nabla \phi\right) u^{n} d x d t \\
& +\int_{Q_{T}} m u^{n} \phi d x d t-\int_{Q_{T}} u^{n} f\left(x, t, u^{n}\right) \phi d x d t .
\end{aligned}
$$

Thus, it follows from this and (4.2) that

$$
\left|\int_{Q_{T}} u_{t}^{n} \phi d x d t\right| \leq C\|\phi\|_{L^{2}\left(0, T, H^{1}(\Omega)\right)}, \quad \forall n \in \mathbb{N}
$$

Therefore,

$$
\begin{equation*}
\left\|u_{t}^{n}\right\|_{L^{2}\left(0, T, H^{1}(\Omega)^{*}\right)} \leq C, \quad \forall n \in \mathbb{N} \tag{4.4}
\end{equation*}
$$

Again, the constant $C$ here depends only on $\mu, d,|\Omega|, T, M,\left\|u_{0}\right\|_{L^{\infty}}$ and $\|m\|_{L^{\infty}}$. From the estimates (4.2) - (4.4), the results in [24], and by passing to a subsequence, it follows that

$$
\begin{gather*}
u^{n} \rightarrow u^{*} \quad \text { in } L^{2}\left(Q_{T}\right), \quad \nabla u^{n} \rightharpoonup \nabla u^{*} \quad \text { in } \quad L^{2}\left(Q_{T}\right), \quad \text { and } \\
u_{t}^{n} \rightharpoonup u_{t}^{*} \text { in } L^{2}\left((0, T), H^{1}(\Omega)^{*}\right) . \tag{4.5}
\end{gather*}
$$

By the definition of the control set $U$ in (2.4) and the fact that $L^{2}\left(Q_{T}\right)$ is weakly compact, there exists $\vec{h}^{*} \in U$ such that

$$
\begin{equation*}
\vec{h}^{n} \rightharpoonup \vec{h}^{*} \text { in } L^{2}\left(Q_{T}\right) \tag{4.6}
\end{equation*}
$$

Again, for each fixed $v \in L^{2}\left(0, T, H^{1}(\Omega)\right)$, the solution $u^{n}$ satisfies

$$
\begin{align*}
& \int_{Q_{T}} u_{t}^{n} v d x d t=-\int_{Q_{T}} \mu \nabla u^{n} \cdot \nabla v d x d t+\int_{Q_{T}}\left(\vec{h}^{n} \cdot \nabla v\right) u^{n} d x d t \\
&+\int_{Q_{T}} m v u^{n} d x d t-\int_{Q_{T}} u^{n} f\left(x, t, u^{n}\right) v d x d t, \quad \forall n . \tag{4.7}
\end{align*}
$$

From the weak convergences in (4.3), (4.5) and (4.6), it follows

$$
\begin{align*}
& \lim _{n \rightarrow \infty} \int_{Q_{T}} u_{t}^{n} v d x d t=\int_{Q_{T}} u_{t}^{*} v d x d t \\
& \lim _{n \rightarrow \infty} \int_{Q_{T}} \nabla u^{n} \cdot \nabla v d x d t=\int_{Q_{T}} \nabla u^{*} \cdot \nabla v d x d t  \tag{4.8}\\
& \lim _{n \rightarrow \infty} \int_{Q_{T}} m v u^{n} d x d t=\int_{Q_{T}} m v u^{*} d x d t
\end{align*}
$$

Moreover, from the strong convergence of the sequence $\left\{u^{n}\right\}_{n \in \mathbb{N}}$ (see (4.5)) and the weak convergence of the sequence $\left\{h^{n}\right\}_{n \in \mathbb{N}}$ (see (4.6)), we obtain

$$
\begin{aligned}
& \left|\int_{Q_{T}}\left(\vec{h}^{n} \cdot \nabla v\right) u^{n} d x d t-\int_{Q_{T}}\left(\vec{h}^{*} \cdot \nabla v\right) u^{*} d x d t\right| \\
& =\left|\int_{Q_{T}}\left(\vec{h}^{n} \cdot \nabla v\right)\left(u^{n}-u^{*}\right) d x d t-\int_{Q_{T}}\left(\left(\vec{h}^{n}-\vec{h}^{*}\right) \cdot \nabla v\right) u^{*} d x d t\right| \\
& \leq M\|\nabla v\|_{L^{2}\left(Q_{T}\right)}\left\|u^{n}-u^{*}\right\|_{L^{2}\left(Q_{T}\right)}+\left|\int_{Q_{T}}\left(\left[\vec{h}^{n}-\vec{h}^{*}\right] \cdot \nabla v\right) u^{*} d x d t\right| \\
& \rightarrow 0, \quad \text { as } \quad n \rightarrow \infty .
\end{aligned}
$$

In other words,

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \int_{Q_{T}}\left(\vec{h}^{n} \cdot \nabla v\right) u^{n} d x d t=\int_{Q_{T}}\left(\vec{h}^{*} \cdot \nabla v\right) u^{*} d x d t \tag{4.9}
\end{equation*}
$$

Finally, from the assumption (ii) on $f$, the uniform boundedness of the sequence $\left\{u^{n}\right\}_{n \in \mathbb{N}}$ (see (4.2)), and its strong convergence in $L^{2}\left(Q_{T}\right)$ (see (4.5)), the following convergence holds

$$
\begin{align*}
& \left|\int_{Q_{T}} u^{n} f\left(x, t, u^{n}\right) v d x d t-\int_{Q_{T}} u^{*} f\left(x, t, u^{*}\right) v d x d t\right| \\
& \leq\left|\int_{Q_{T}} u^{n} v\left[f\left(x, t, u^{n}\right)-f\left(x, t, u^{*}\right)\right] d x d t\right|+\left|\int_{Q_{T}} v f\left(x, t, u^{*}\right)\left[u^{n}-u^{*}\right] d x d t\right| \\
& \leq C\left\|u^{n}-u^{*}\right\|_{L^{2}\left(Q_{T}\right)}\|v\|_{L^{2}\left(Q_{T}\right)}+\left|\int_{Q_{T}} v f\left(x, t, u^{*}\right)\left[u^{n}-u^{*}\right] d x d t\right| \\
& \rightarrow 0, \quad \text { as, } \quad n \rightarrow \infty . \tag{4.10}
\end{align*}
$$

Collecting those convergence terms in (4.8), (4.9) and (4.10), and using the equation (4.7), we arrive at

$$
\begin{align*}
& \int_{Q_{T}} u_{t}^{*} v d x d t=-\int_{Q_{T}} \mu \nabla u^{*} \cdot \nabla v d x d t+\int_{Q_{T}}\left(\vec{h}^{*} \cdot \nabla v\right) u^{*} d x d t \\
&+\int_{Q_{T}} m v u^{*} d x d t-\int_{Q_{T}} u^{*} f\left(x, t, u^{*}\right) v d x d t \tag{4.11}
\end{align*}
$$

This implies that $u^{*}$ is the solution of (2.1) with respect to the control $h^{*}$. In other words, $u^{*}=u\left(h^{*}\right)$. On the other hand, using the strong convergence in $L^{2}\left(Q_{T}\right)$ of the sequence $\left\{u^{n}\right\}_{n \in \mathbb{N}}\left(\right.$ see (4.5)), and the fact that the function $\vec{h} \mapsto \int_{Q_{T}}|\vec{h}|^{2} d x d t$ is weakly lower semi-continuous in $L^{2}\left(Q_{T}\right)$ (see e.g. [2]), we also get

$$
\begin{aligned}
\sup _{\vec{h} \in U} J(\vec{h}) & =\lim _{n \rightarrow \infty} J\left(\vec{h}^{n}\right)=\lim _{n \rightarrow \infty}\left[\int_{Q_{T}} u^{n}-B\left|\vec{h}^{n}\right|^{2} d x d t\right] \\
& \leq \int_{Q_{T}} u^{*}-B\left|\vec{h}^{*}\right|^{2} d x d t=J\left(\vec{h}^{*}\right) .
\end{aligned}
$$

This implies that $J\left(h^{*}\right)=\sup _{\vec{h} \in U} J(\vec{h})$. Therefore, $h^{*} \in U$ is an optimal control and the proof of Theorem 4.1 is complete.
5. Necessary Conditions. In order to characterize the optimal control, we need to differentiate the map $h \rightarrow J(h)$ with respect to the control $\vec{h}$. We denote by $u=u(\vec{h})$ the unique, positive solution of (2.1). Since $u=u(\vec{h})$ is involved in $J(\vec{h})$, we first must prove appropriate differentiability of the mapping $\vec{h} \longrightarrow u(\vec{h})$ whose derivative is called the sensitivity.

Lemma 5.1 (Sensitivity). The mapping $\vec{h} \in U \longrightarrow u(\vec{h})$ is differentiable in the following sense: for each $\vec{h}, \vec{l}$ in $U$ such that $\vec{h}+\epsilon \vec{l} \in U$ for all $\epsilon$ sufficiently small, then there is a uniform constant $C>0$ such that $\psi^{\epsilon}=\frac{u(\vec{h}+\epsilon \vec{l})-u(\vec{h})}{\epsilon}$ satisfies

$$
\left\|\psi^{\epsilon}\right\|_{V_{2}\left(Q_{T}\right)}, \quad\left\|\psi^{\epsilon}\right\|_{L^{\infty}\left(Q_{T}\right)} \leq C .
$$

Moreover, there exists $\psi=\psi(\vec{h}, \vec{l}) \in L^{2}\left((0, T), H^{1}(\Omega)\right)$, such that

$$
\psi^{\epsilon} \rightharpoonup \psi \text { weakly in } L^{2}\left((0, T), H^{1}(\Omega)\right) \text { as } \epsilon \rightarrow 0
$$

and the sensitivity $\psi$ satisfies

$$
\left\{\begin{array}{cccc}
\psi_{t}-\nabla \cdot(\mu \nabla \psi-\vec{h} \psi)-[m-g(x, t, u)] \psi & = & -\nabla \cdot(u \vec{l}), & Q_{T},  \tag{5.1}\\
\mu \frac{\partial \psi}{\partial \nu}-\psi \vec{h} \cdot \nu & & u \vec{l} \cdot \nu, & S_{T} \\
\psi(x, 0) & & 0, & \Omega .
\end{array}\right.
$$

Proof. For each $\vec{h} \in U$ and $\vec{h}+\epsilon \vec{l} \in U$, let us denote $u^{\epsilon}=u(\vec{h}+\epsilon \vec{l}), u=u(\vec{h})$. Recall that $u$ solves (2.1) and $u^{\epsilon}$ solves

$$
\left\{\begin{array}{cccc}
u_{t}^{\epsilon}-\nabla \cdot\left[\mu \nabla u^{\epsilon}-u^{\epsilon}(\vec{h}+\epsilon \vec{l})\right] & = & m u^{\epsilon}-u^{\epsilon} f\left(x, t, u^{\epsilon}\right), & Q_{T}  \tag{5.2}\\
\mu \frac{\partial u^{\epsilon}}{\partial \nu}-u^{\epsilon}(\vec{h}+\epsilon \vec{l}) \cdot \nu & = & 0, & S_{T} \\
u^{\epsilon}(\cdot, 0) & = & u_{0}, & \Omega
\end{array}\right.
$$

Also, note that it follows from Lemma 3.3 and Lemma 3.4 that there is a constant depending on $\mu, d, T,|\Omega|,\|m\|_{L^{\infty}},\left\|u_{0}\right\|_{L^{\infty}}$ and $M$ such that

$$
\begin{equation*}
\left\|u^{\epsilon}\right\|_{V_{2}\left(Q_{T}\right)}, \quad\|u\|_{V_{2}\left(Q_{T}\right)}, \quad\left\|u^{\epsilon}\right\|_{L^{\infty}\left(Q_{T}\right)}, \quad\|u\|_{L^{\infty}\left(Q_{T}\right)} \leq C<\infty, \quad \forall \epsilon>0 . \tag{5.3}
\end{equation*}
$$

Let us define

$$
c^{\epsilon}(x, t)=\frac{u^{\epsilon}(x, t) f\left(x, t, u^{\epsilon}(x, t)\right)-u(x, t) f(x, t, u(x, t))}{u^{\epsilon}(x, t)-u(x, t)}, \quad \forall(x, t) \in Q_{T}
$$

It follows from (ii), (iii) and (5.3) that $c^{\epsilon}(x, t)$ and $g(x, t, u)$ are uniformly bounded. On the other hand, as in the proof of Theorem 4.1 and by the uniqueness of solution of $u=u(\vec{h})$ of (2.1), we have
$u^{\epsilon} \rightarrow u$ in $L^{2}\left(Q_{T}\right) ; \quad u_{t}^{\epsilon} \rightharpoonup u_{t}$ in $L^{2}\left(Q_{T}\right) ; \quad$ and $\quad u^{\epsilon} \rightharpoonup u \quad$ in $\quad L^{2}\left(0, T, H^{1}(\Omega)\right)$.
In particular, this gives

$$
u^{\epsilon} \rightarrow u \text { a.e. in } \quad Q_{T}, \quad \text { and } \quad c^{\epsilon}(x, t) \rightarrow g(x, t, u(x, t)), \quad \text { a.e. in } \quad Q_{T} .
$$

By (5.3) and (iii), it follows that $\left|c^{\epsilon}(x, t)-g(x, t, u(x, t))\right|$ is uniformly bounded on $Q_{T}$ with respect to $\epsilon$. Thus, the Lebesgue Dominated Convergence Theorem implies

$$
\begin{equation*}
\lim _{\epsilon \rightarrow \infty} \int_{Q_{T}}\left|c^{\epsilon}(x, t)-g(x, t, u(x, t))\right|^{p} d x d t=0, \quad \forall 1 \leq p<\infty \tag{5.4}
\end{equation*}
$$

Recall that $\psi^{\epsilon}=\frac{u^{\epsilon}-u}{\epsilon}$. Then subtracting (5.2) from (2.1) and dividing by $\epsilon$, we obtain

$$
\left\{\begin{array}{cccc}
\psi_{t}^{\epsilon}-\nabla \cdot\left(\mu \nabla \psi^{\epsilon}-\psi^{\epsilon} \vec{h}-u^{\epsilon} \vec{l}\right) & = & {\left[m-c^{\epsilon}(x, t)\right] \psi^{\epsilon},} & Q_{T},  \tag{5.5}\\
\mu \frac{\partial \psi^{\epsilon}}{\partial \nu}-\psi^{\epsilon} \vec{h} \cdot \nu & = & u^{\epsilon} \vec{l} \cdot \nu, & S_{T} \\
\psi^{\epsilon}(\cdot, 0) & = & 0, & \Omega
\end{array}\right.
$$

As in the proofs of Lemma 3.3 and Lemma 3.4, we have

$$
\left\|\psi^{\epsilon}\right\|_{V_{2}\left(Q_{T}\right)} \leq C, \quad\left\|\psi^{\epsilon}\right\|_{L^{\infty}} \leq C, \quad \forall \epsilon
$$

Therefore, by passing to a subsequence, we can assume that that $\psi^{\epsilon} \rightharpoonup \psi$ in $L^{2}\left(0, T, H^{1}(\Omega)\right)$. Using the estimates (5.3)-(5.4), the equation (5.5), and as in the proof of Theorem 4.1, we obtain (5.1). This concludes the proof of Lemma 5.1.

Next, we characterize our optimal control solution $\vec{h}^{*}$ by differentiating the map $\vec{h} \rightarrow J(\vec{h})$. We use the sensitivity equation to find our adjoint equation and our characterization.

Theorem 5.1. Given an optimal control $\overrightarrow{h^{*}}$ and corresponding state $u^{*}$, there exists a solution $p$ in $L^{2}\left(0, T, H^{1}(\Omega)\right)$ which satisfies $p_{t} \in L^{2}\left((0, T), H^{1}(\Omega)^{*}\right)$ and

$$
\begin{cases}-p_{t}-\mu \Delta p-\overrightarrow{h^{*}} \cdot \nabla p-\left[m-g\left(x, t, u^{*}\right)\right] p=1, & \text { in } Q_{T}  \tag{5.6}\\ \frac{\partial p}{\partial \nu}=0, & \text { in } S_{T} \\ p(\cdot, T)=0 & \text { in } \Omega\end{cases}
$$

Furthermore, $\vec{h}^{*}$ is characterized by

$$
\begin{equation*}
h_{i}^{*}=\max \left\{\min \left\{M, \frac{u^{*} p_{x_{i}}}{2 B}\right\},-M\right\}, \quad \text { for each } \quad i \in\{1, \ldots, d\} . \tag{5.7}
\end{equation*}
$$

Proof. Suppose $\vec{h}^{*}$ is an optimal control. Let $\vec{l} \in U$ such that $\vec{h}^{*}+\epsilon \vec{l} \in U$ for sufficiently small $\epsilon>0$. Let us denote $u^{\epsilon}=u\left(\vec{h}^{*}+\epsilon \vec{l}\right)$ be the unique solution of (2.1) when the control term is $\vec{h}^{*}+\epsilon \vec{l}$.

The operator in the adjoint equation is the formal analysis "adjoint" of the operator in the sensitivity equation (5.1) at $\vec{h}^{*}$. The nonhomogeneous term, 1 , comes from differentiating the integrand of the objective functional with respect to the state. The final time condition on the adjoint function is the transversality condition.

The equation (5.6) is linear in $p$ and its coefficients are measurable and bounded. By the change of variable $t \rightarrow T-t$, the existence and uniqueness of the weak solution $p$ of (5.6) follows by Galerkin's method (e.g. see [11]).

Now, observe that the directional derivative of $J$ with respect to the control at $\vec{h}^{*}$ in the direction of $\vec{l}$ satisfies

$$
\begin{aligned}
0 & \geq \lim _{\epsilon \rightarrow 0^{+}} \frac{J\left(\vec{h}^{*}+\epsilon \vec{l}\right)-J\left(\vec{h}^{*}\right)}{\epsilon} \\
& =\lim _{\epsilon \rightarrow 0^{+}} \frac{1}{\epsilon}\left[\int_{Q_{T}} u^{\epsilon}-B\left|\vec{h}^{*}+\epsilon \vec{l}\right|^{2} d x d t-\left(\int_{Q_{T}} u^{*}-B\left|\vec{h}^{*}\right|^{2} d x d t\right)\right] \\
& =\lim _{\epsilon \rightarrow 0^{+}}\left[\int_{Q_{T}} \frac{u^{\epsilon}-u^{*}}{\epsilon} d x d t-\int_{Q_{T}} B\left(2 \vec{h}^{*} \cdot \vec{l}+\epsilon|\vec{l}|^{2}\right) d x d t\right] \\
& =\int_{Q_{T}} \psi d x d t-\int_{Q_{T}} 2 B \vec{h}^{*} \cdot \vec{l} d x d t
\end{aligned}
$$

Using the weak solution formulation for the adjoint problem with test function $\psi$, we obtain

$$
\begin{align*}
& 0 \geq \int_{Q_{T}} \psi d x d t-\int_{Q_{T}} 2 B \overrightarrow{h^{*}} \cdot \vec{l} d x d t \\
&= \int_{Q_{T}} p \psi_{t} d x d t+\int_{Q_{T}}\left\{\mu \nabla p \cdot \nabla \psi-\psi \vec{h}^{*} \cdot \nabla p-\left[m-g\left(x, t, u^{*}\right)\right] p \psi\right\} d x d t  \tag{5.8}\\
& \quad \quad-\int_{Q_{T}} 2 B \vec{h}^{*} \cdot \vec{l} d x d t \\
&= \int_{Q_{T}} u^{*} \vec{l} \cdot \nabla p-2 B \vec{h}^{*} \cdot \vec{l} d x d t=\int_{Q_{T}} \vec{l} \cdot\left(u^{*} \nabla p-2 B \vec{h}^{*}\right) d x d t
\end{align*}
$$

From Theorem 3.1 and Lemma 5.2 below, we see that there is a constant $C_{0}>0$ such that

$$
\left|u^{*} \nabla p-2 B \vec{h}^{*}\right| \leq C_{0}, \quad \text { on } \quad Q_{T} .
$$

Now, for each fixed $i=1,2, \cdots, d$, and each $0<\delta<1$, let $\Gamma_{\delta}$ be the set $\{(x, t) \in$ $\left.Q_{T}:\left|h_{i}(x, t)\right| \leq \delta M\right\}$. Then, let

$$
l_{i}=\left[u^{*} \frac{\partial p}{\partial x_{i}}-2 B h_{i}^{*} \chi_{\Gamma_{\delta}}, \quad l_{k}=0, \quad k \neq i\right.
$$

where $\chi_{\Gamma_{\delta}}(x, t)$ is a function which is one if $(x, t) \in \Gamma_{\delta}$ and zero otherwise. It follows that for all $\epsilon>0$ and sufficiently small such that $\epsilon C_{0}<(1-\delta) M$, we have $h^{*}+\epsilon l \in U$. Therefore, from (5.8), we see that

$$
\int_{\Gamma_{\delta}}\left|u^{*} \frac{\partial p}{\partial x_{i}}-2 B \vec{h}_{i}^{*}\right|^{2} d x d t \leq 0
$$

Thus, $h_{i}^{*}=\left[u^{*} \frac{\partial p}{\partial x_{i}}\right] /(2 B)$ on $\Gamma_{\delta}$. Since $\delta$ is arbitrary, we conclude that on the set where $\left|h_{i}^{*}\right|<M$,

$$
h_{i}^{*}=\frac{u^{*} \frac{\partial p}{\partial x_{i}}}{2 B} .
$$

Now assume that $h_{i}^{*}=M$ on some non-empty set $\Gamma \subseteq Q_{T}$. We can take $\vec{l} \in U$ to be such that its support lies in $\Gamma$ and $l_{j}=0$ if $i \neq j$, and $l_{i}<0$ on $\Gamma$. Then $\vec{h}^{*}+\epsilon \vec{l} \in U$ and we have

$$
\int_{\Gamma}\left|l_{i}\right|\left(u^{*} \frac{\partial p}{\partial x_{i}}-2 B M\right) d x d t \geq 0
$$

Since $l_{i}$ is arbitrary as long as $\vec{h}^{*}+\epsilon \vec{l} \in U$, we have $u^{*} \frac{\partial p}{\partial x_{i}}-2 B M \geq 0$ almost everywhere on $\Gamma$. Thus, $h_{i}^{*}=M \leq \frac{u^{*} \frac{\partial p}{\partial x_{i}}}{2 B}$ on $\Gamma$. Therefore, we conclude that $h_{i}^{*}=\min \left\{M, \frac{u^{*} \frac{\partial p}{\partial x_{i}}}{2 B}\right\}$ on $\Gamma \cup\left\{(x, t) \in Q_{T}:\left|h_{i}^{*}\right|<M\right\}$. We can continue similarly on the set $\left\{(x, t) \in Q_{T}: h_{i}^{*}(x, t)=-M\right\}$ to show that

$$
h_{i}^{*}=\max \left\{\min \left\{M, \frac{u^{*} p_{x_{i}}}{2 B}\right\},-M\right\} .
$$

This completes the proof of Theorem 5.1.
Finally, we conclude the section by establishing some estimates of $p$ which will be useful for the coming section.

Lemma 5.2. Let $p$ be the solution of (5.6). Then, there is constant $C$ depending on $d, \mu,|\Omega|, T, \beta, M$ such that

$$
0 \leq p(x, t) \leq C, \quad \text { and } \quad|\nabla p(x, t)| \leq C, \quad \forall(x, t) \in Q_{T} .
$$

Proof. Let us denote $q(x, t)=p(x, T-t)$, then it follows that $q$ solves the equation

$$
\begin{cases}q_{t}-\mu \Delta q+b \cdot \nabla q+c q=1, & Q_{T},  \tag{5.9}\\ \nabla q \cdot \nu=0, & S_{T} \\ q(\cdot, 0)=0, & \Omega\end{cases}
$$

Here, $b(x, t)=-\vec{h}^{*}(x, T-t)$ and $c(x, t)=g\left(x, T-t, u^{*}(x, T-t)\right)-m(x, T-t)$ for all $(x, t) \in Q_{T}$. From the assumption (iii) and Theorem 3.1, it follows the equation (5.9) is a linear parabolic equation with bounded coefficients. By the maximum principle, it follows that $p \geq 0$. On the other hand, by the parabolic regularity theory (see [20, Theorem 9.1, p. 341-342 and its Remark, p. 351]), we have

$$
\|p\|_{W_{l}^{2,1}\left(Q_{T}\right)}=\|q\|_{W_{l}^{2,1}\left(Q_{T}\right)} \leq C=C(l), \quad \forall l \geq 2 .
$$

Here,

$$
\|p\|_{W_{l}^{2,1}\left(Q_{T}\right)}=\|p\|_{L^{l}\left(Q_{T}\right)}+\left\|p_{t}\right\|_{L^{l}\left(Q_{T}\right)}+\|\nabla p\|_{L^{l}\left(Q_{T}\right)}+\sum_{i, j=1}^{d}\left\|p_{x_{i} x_{j}}\right\|_{L^{l}\left(Q_{T}\right)}
$$

Now, for $l>d+2$, it follows from the embedding theorem that

$$
|p(x, t)|, \quad|\nabla p(x, t)| \leq C(l), \quad \forall(x, t) \in Q_{T} .
$$

This yields the desired estimates.
6. Uniqueness and Stability Results. The equations (2.1), (5.6) with boundary conditions and the control characterization (5.7) form the optimality system. In this section, we show the uniqueness of solutions to this optimality system under some conditions on $B$ and $T$, which gives the uniqueness of the optimal control.

Theorem 6.1. There exist two positive numbers $T_{0}$ and $B_{0}$ such that if $0<T \leq T_{0}$ and $B \geq B_{0}$, then there is a unique solution of the optimality system.

Proof. By Theorem 4.1 and Theorem 5.1, there exist an optimal control, and corresponding adjoints and states satisfy the optimality system. Thus, we only need to prove the uniqueness. Let $\vec{h}^{*}$ and $\vec{h}_{*}$ be two controls corresponding to solutions of the optimal system. Also, let us denote $u=u\left(\vec{h}^{*}\right), p=p\left(\vec{h}^{*}\right)$ to be the state solution and the solution of the adjoint problem (5.6). Similarly, we also have $v$ and $q$ for $\vec{h}_{*}$. Now, for some $\lambda>0$ which will be determined, let us denote $w(x, t)=[u(x, t)-v(x, t)] e^{-\lambda t}$ and $z(x, t)=[p(x, t)-q(x, t)] e^{\lambda t}$, and
$b(x, t)=m(x, t)-g(x, t, u)-v(x, t) \frac{g(x, t, u(x, t))-g(x, t, v(x, t))}{u(x, t)-v(x, t)}$,
$d(x, t)=\left\{\left[\vec{h}_{*}(x, t)-\vec{h}^{*}(x, t)\right] \cdot \nabla q(x, t)+[g(x, t, u(x, t))-g(x, t, v(x, t))] q(x, t)\right\} e^{\lambda t}$.
In the sequel, $C, C_{k}, k=1,2, \cdots$ denote constants which may change from lines to lines and they depends on $T, M, \beta,|\Omega|, d$ but do not depend on $B$ and $\lambda$. From the assumptions (i) - (iii) and Theorem 3.1, it follows that

$$
\begin{equation*}
|b(x, t)| \leq C, \quad \forall(x, t) \in Q_{T} . \tag{6.1}
\end{equation*}
$$

On the other hand, from the characterization formula in Theorem 5.1 and Theorem 3.1, we see that

$$
\begin{align*}
\left|\vec{h}^{*}-\vec{h}_{*}\right| & \leq \frac{1}{2 B}|u \nabla p-v \nabla q|  \tag{6.2}\\
& \leq \frac{1}{2 B}\left[|u||\nabla z| e^{-\lambda t}+e^{\lambda t}|\nabla q||w|\right] \leq \frac{C}{B}\left[|\nabla z| e^{-\lambda t}+|w| e^{\lambda t}\right] .
\end{align*}
$$

Then, it follows from (6.2) and (iv) that

$$
\begin{equation*}
|d(x, t)| \leq\left|\vec{h}_{*}-\vec{h}^{*}\right| e^{\lambda t}+C|u-v| e^{\lambda t} \leq C\left[\frac{1}{B}|\nabla z|+|w| e^{2 \lambda t}\right] . \tag{6.3}
\end{equation*}
$$

Now, by subtracting the equations of $u$ and $v$, (see (2.1)), we see that $w$ solves

$$
\begin{cases}w_{t}-\nabla \cdot\left[\mu \nabla w-w \vec{h}^{*}+v\left(\vec{h}_{*}-\vec{h}^{*}\right) e^{-\lambda t}\right]=[b-\lambda] w, & Q_{T},  \tag{6.4}\\ {\left[\mu \nabla w-w \vec{h}^{*}+v\left(\vec{h}_{*}-\vec{h}^{*}\right) e^{-\lambda t}\right] \cdot \nu=0,} & S_{T}, \\ w(\cdot, 0)=0, & \Omega\end{cases}
$$

Similarly, $z$ also solves

$$
\begin{cases}z_{t}+\mu \Delta z+\vec{h}^{*} \cdot \nabla z=[m-g(x, t, u)+\lambda] z+d(x, t), & Q_{T},  \tag{6.5}\\ \nabla z \cdot \nu=0, & S_{T} \\ z(\cdot, T)=0, & \Omega\end{cases}
$$

Multiplying (6.4) by $w$ and using the integration by parts, Hölder's inquality, Young's inequality and the estimate (6.2), we get

$$
\begin{aligned}
& \frac{1}{2} \frac{d}{d t} \int_{\Omega} w^{2} d x+\mu \int_{\Omega}|\nabla w|^{2} d x \\
& \leq \int_{\Omega}[b-\lambda] w^{2} d x+\int_{\Omega}\left[|w|\left|\vec{h}^{*}\right|+|v| e^{-\lambda t}\left|\vec{h}^{*}-\overrightarrow{h_{*}}\right|\right]|\nabla w| d x \\
& \leq \int_{\Omega}[b-\lambda] w^{2} d x+C \int_{\Omega}\left[\left(1+\frac{1}{B}\right)|w|+\frac{e^{-2 \lambda t}}{2 B}|\nabla z|\right]|\nabla w| d x \\
& \leq\left[C+\frac{C}{B^{2}}-\lambda\right] \int_{\Omega} w^{2} d x+\frac{\mu}{2} \int_{\Omega}|\nabla w|^{2} d x+\frac{C}{B^{2}} \int_{\Omega}|\nabla z|^{2} d x .
\end{aligned}
$$

Thus,

$$
\frac{1}{2} \frac{d}{d t} \int_{\Omega} w^{2} d x+\frac{\mu}{2} \int_{\Omega}|\nabla w|^{2} d x \leq\left[C_{1}+\frac{C_{1}}{B^{2}}-\lambda\right] \int_{\Omega} w^{2} d x+\frac{C_{1}}{B^{2}} \int_{\Omega}|\nabla z|^{2} d x .
$$

Integrating this inequality with respect to time, we get

$$
\begin{equation*}
\sup _{t} \int_{\Omega} w^{2} d x+\mu \int_{Q_{T}}|\nabla w|^{2} d x d t \leq 2\left[C_{1}+\frac{C_{1}}{B^{2}}-\lambda\right] \int_{Q_{T}} w^{2} d x d t+\frac{2 C_{1}}{B^{2}} \int_{Q_{T}}|\nabla z|^{2} d x d t . \tag{6.6}
\end{equation*}
$$

Doing the same thing for the equation (6.5), we also get

$$
\begin{aligned}
& -\frac{1}{2} \frac{d}{d t} \int_{\Omega} z^{2} d x+\mu \int_{\Omega}|\nabla z|^{2} d x \\
& \leq \int_{\Omega}[g(x, t, u)-m(x, t)-\lambda] z^{2} d x+\int_{\Omega}\left[\left|\vec{h}^{*}\right||\nabla z|+|d(x, t)|\right]|z| d x \\
& \leq \int_{\Omega}[g(x, t, u)-m(x, t)-\lambda] z^{2} d x+C \int_{\Omega}\left[|\nabla z|+|w| e^{2 \lambda t}\right]|z| d x \\
& \leq\left[C_{2}+C_{3} e^{4 \lambda T}-\lambda\right] \int_{\Omega} z^{2} d x+\frac{\mu}{2} \int_{\Omega}|\nabla z|^{2} d x+C_{4} \int_{\Omega}|w|^{2} d x
\end{aligned}
$$

Thus,

$$
-\frac{1}{2} \frac{d}{d t} \int_{\Omega} z^{2} d x+\frac{\mu}{2} \int_{\Omega}|\nabla z|^{2} d x \leq\left[C_{2}+C_{3} e^{4 \lambda T}-\lambda\right] \int_{\Omega} z^{2} d x+C_{4} \int_{\Omega}|w|^{2} d x
$$

Again, integrating this in time, we get
$\sup _{t>0} \int_{\Omega} z^{2} d x+\mu \int_{Q_{T}}|\nabla z|^{2} d x d t \leq 2\left[C_{2}+C_{3} e^{4 \lambda T}-\lambda\right] \int_{Q_{T}} z^{2} d x d t+2 C_{4} \int_{Q_{T}}|w|^{2} d x d t$.

Note that the constants $C_{1}, C_{2}, C_{3}, C_{4}$ all depend on $\beta, T,|\Omega|, M, d$. We write $C_{k}=C_{k}(T)$ for all $k=1,2,3,4$. Also, note that these constants are decreasing with respect to $T$. From now, we choose $B_{0}>0$ and sufficiently large so that $2 C_{1}(1) / B_{0}^{2}<\mu$. Moreover, we also choose $\lambda$ sufficiently large so that

$$
C_{1}(1)+\mu+C_{4}(1)-\lambda<0, \quad \text { and } \quad C_{2}(1)+C_{3}(1)-\lambda<0 .
$$

Then there exists $T_{0}>0$ and sufficiently small such with $T \leq T_{0}$ and $B \geq B_{0}$, we have

$$
C_{1}(T)+\frac{C_{1}(T)}{B^{2}}+C_{4}(T)-\lambda<0, \quad \text { and } \quad C_{2}(T)+e^{4 \lambda T} C_{3}(T)-\lambda<0
$$

It follows from this, (6.6) and (6.7) that $\sup _{t} \int_{\Omega} w^{2} d x+\sup _{t>0} \int_{\Omega} z^{2} d x+\mu \int_{Q_{T}}|\nabla w|^{2} d x d t+\mu \int_{Q_{T}}|\nabla z|^{2} d x \leq \frac{2 C_{1}}{B^{2}} \int_{Q_{T}}|\nabla z|^{2} d x d t$. Also, since $\frac{2 C_{1}(T)}{B^{2}}<\mu$, it follows that

$$
\sup _{t} \int_{\Omega} w^{2} d x+\sup _{t>0} \int_{\Omega} z^{2} d x \leq 0
$$

This implies $u=v, p=q$. Thus, $\vec{h}^{*}=\vec{h}_{*}$.
Next, it is mathematically and biologically interesting to see how the solutions of the optimality system depend on the resource $m$. For this purpose, we let $\vec{h}=\vec{h}(m)$ be the optimal control with the corresponding resource $m$, recalling that we need $B \geq B_{0}$ and $0<T \leq T_{0}$ for the uniqueness of $\vec{h}$. We then obtain the following stability result which is more general than Theorem 6.1:

Theorem 6.2. Let $\beta>0$ be as in Lemma 3.4. There exist $0<T_{1} \leq T_{0}$ and $B_{1} \geq B_{0}$ such that if $B>B_{1}$ and $0<T<T_{1}$, there exists a constant $C=C_{T}>0$ such that the estimate

$$
\left\|\vec{h}\left(m_{1}\right)-\vec{h}\left(m_{2}\right)\right\|_{L^{2}\left(Q_{T}\right)} \leq C\left\|m_{1}-m_{2}\right\|_{L^{2}\left(Q_{T}\right)}
$$

holds for all $m_{1}, m_{2}$ in $L^{\infty}\left(Q_{T}\right)$ with $\left|m_{1}\right|,\left|m_{2}\right| \leq \beta$.
Proof. Let $\vec{h}^{*}=\vec{h}\left(m_{1}\right)$ and $\vec{h}_{*}=\vec{h}\left(m_{2}\right)$ be the two controls corresponding to solutions of the optimal system. As in the proof of Theorem 6.1, we denote

$$
u=u\left(\vec{h}^{*}\right), \quad p=p\left(\vec{h}^{*}\right), \quad v=u\left(\vec{h}_{*}\right), \quad \text { and }, \quad q=q\left(\vec{h}_{*}\right) .
$$

Also, for $\lambda>0$ which will be determined, we set

$$
w(x, t)=e^{-\lambda t}[u(x, t)-v(x, t)], \quad z(x, t)=e^{\lambda t}[p(x, t)-q(x, t)], \quad(x, t) \in Q_{T} .
$$

As in the proof of Theorem 6.1, $w$ solves

$$
\begin{cases}w_{t}-\nabla \cdot\left[\mu \nabla w-w \vec{h}^{*}+v\left(\vec{h}_{*}-\vec{h}^{*}\right) e^{-\lambda t}\right]=[b-\lambda] w+e^{-\lambda t}\left[m_{1}-m_{2}\right], & Q_{T},  \tag{6.8}\\ {\left[\mu \nabla w-w \vec{h}^{*}+v\left(\vec{h}_{*}-\vec{h}^{*}\right) e^{-\lambda t}\right] \cdot \nu=0,} & S_{T}, \\ w(\cdot, 0)=0, & \Omega,\end{cases}
$$

and $z$ also solves

$$
\begin{cases}z_{t}+\mu \Delta z+\vec{h}^{*} \cdot \nabla z=[m-g(x, t, u)+\lambda] z+d(x, t)+e^{\lambda t}\left[m_{1}-m_{2}\right], & Q_{T},  \tag{6.9}\\ \nabla z \cdot \nu=0, & S_{T}, \\ z(\cdot, 0)=0, & \Omega .\end{cases}
$$

Here, $b, d$ are defined as
$b(x, t)=m_{1}(x, t)-g(x, t, u)-v(x, t) \frac{g(x, t, u(x, t))-g(x, t, v(x, t))}{u(x, t)-v(x, t)}$,
$d(x, t)=\left\{\left[\vec{h}_{*}(x, t)-\vec{h}^{*}(x, t)\right] \cdot \nabla q(x, t)+[g(x, t, u(x, t))-g(x, t, v(x, t))] q(x, t)\right\} e^{\lambda t}$.
As in the proof of Theorem 6.1, we can fix $\lambda>0$ sufficiently large and then choose $B_{1} \geq B_{0}$ and $0<T_{1} \leq T_{0}$ such that if $B \geq B_{1}$ and $0<T \leq T_{1}$, it follows that

$$
\|z\|_{V_{2}\left(Q_{T}\right)}^{2}+\|w\|_{V_{2}\left(Q_{T}\right)}^{2} \leq C e^{2 \lambda T}\left\|m_{1}-m_{2}\right\|_{L^{2}\left(Q_{T}\right)}^{2} .
$$

This and (6.2) yield

$$
\left\|h^{*}-h_{*}\right\|_{L^{2}\left(Q_{T}\right)} \leq C e^{\lambda T}\left\{\|\nabla z\|_{L^{2}\left(Q_{T}\right)}+\|w\|_{L^{2}\left(Q_{T}\right)}\right\} \leq C_{T}\left\|m_{1}-m_{2}\right\|_{L^{2}\left(Q_{T}\right)} .
$$

The proof is now complete.
7. Numerical Results. We illustrate some numerical results using a variety of function $f$ and source functions $m$.
7.1. Logistic Growth Nonlinearity. For $f(x, t, u)=|u|$, the equation (2.1) becomes

$$
\left\{\begin{array}{cccc}
u_{t}-\nabla \cdot[\mu \nabla u-u \vec{h}] & = & u[m-|u|], & Q_{T},  \tag{7.1}\\
\mu \frac{\partial u}{\partial \nu}-u h \cdot \nu & = & 0, & \mathcal{S}_{T}, \\
u(\cdot, 0) & = & u_{0} \geq 0, & \Omega .
\end{array}\right.
$$

Indeed, from Lemma 3.1, all solutions of (7.1) are non-negative. Hence, the nonlinear part of (7.1) can be written as $u(m-u)$.

We have run several examples for this case, with both time-independent and time-dependent functions $m$. To solve the optimality system, we use an iterative scheme with an explicit finite difference method of order 2 in MATLAB. Starting with an initial guess for the control function and using a forward-backward sweep [17], we approximate first the state, then the adjoint. We then obtain the next approximation to the optimal control by evaluating our optimal control characterization. This is iterated until the optimal state and optimal control converge.

Our first two examples below have time-independent functions $m$. The clear pattern that emerged from all examples that we ran is that at any given time, the optimal control has a form very similar to that of the derivative of $m$, with corrections made near the boundary for the control value to be zero at each end when necessary. In all of the examples that follow, $\mu=.1$ and the final time is taken as $T=.2$.

In our first example, the graph of the resource function,

$$
m(x)=20 x(1-x)+0.1
$$

is given in Figure 1, with the optimal control and state given in Figure 2. As with all of the the time-independent $m$, the shape of the optimal control remains similar for all time, though its scale changes, so we include only a time slice early in the time interval in Figure 3. Changing $B$ appears to have little effect on the shape of the optimal control, but changes its scale. Using the same resource function $m$, and changing $B$ from $B=.05$ to $B=1$, we place the resulting optimal controls at time $t=0.02$ side by side for comparison. In all other examples, $B$ is taken to be $B=.05$.


Figure 1. $m(x)=20 x(1-x)+.1$


Figure 2. Optimal Control and Corresponding State in 1D Over Time


Figure 3. Time Slices of the Optimal Controls in 1D for $B=.05$ and $B=1$

Now we look at another example with time-independent

$$
m(x)=\cos (6 \pi x)+1.1
$$

whose graph is shown in Figure 4, and whose derivative is zero at the boundary. In contrast to the first example, there is no correction needed for the shape of the control to match the derivative of $m$ near the boundary. See the corresponding optimal control and state results in Figures 5 and 6, again showing the control following the derivative of $m$.


Figure 4. $m(x)=\cos (6 \pi x)+1.1$


Figure 5. Optimal Control and Corresponding State in 1D Over Time


Figure 6. Time Slices of the Optimal Control

Now we give results for time-dependent

$$
m(x, t)=(1-t / T)(\cos (2 \pi x)+1)+(t / T)|x-.5|
$$

shown in Figure 7 with $B=0.05$. We include not only the full results for the optimal control and state, but three time slices as well - one early, one midway through the time interval, and one later slice. Figures 8 and 9 allow us to see that there is still some similarity in the shape of the control to the spatial derivative of $m$. In some of the examples we ran, the similarity is not nearly as close as in the $m$ time-independent case, as the control is clearly is influenced at any given time by the form of $m$ in the near future.


Figure 7. $m(x, t)=(1-t / T)(\cos (2 \pi x)+1)+(t / T)|x-.5|$


Figure 8. Optimal Control and Corresponding State in 1D Over Time



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Figure 9. Early, Mid, and Late Time Slices of the Optimal Control

Next is another time-dependent example

$$
m(x, t)=(1-t / T)\left[x^{2}(6-4 x)+.2\right]+.5(t / T)[\cos (6 \pi x)+1]
$$

shown in Figure 10. As illustrated in Figures 11 and 12, this example better illustrates the difference between the time-dependent and time-independent cases.


Figure 10. $m(x, t)=(1-t / T)\left[x^{2}(6-4 x)+.2\right]+.5(t / T)[\cos (6 \pi x)+1]$


Figure 11. Optimal Control and Corresponding State in 1D Over Time



Figure 12. Early, Mid, and Late Time Slices of the Optimal Control
7.2. Some Other Nonlinearities. We are also interested in the behavior of the optimal control solutions for different type nonlinearities (see [14]). In this section, we give two examples of two different nonlinearities $f$. Our first example is the following nonlinearity

$$
\begin{equation*}
f(x, t, u)=|u|+\frac{1}{1+|u|} \tag{7.2}
\end{equation*}
$$

The resulting optimal control and state, shown in Figures 13 and 14, for this situation are actually strikingly similar to the results we obtain with the same $m$ as in Figure 1 and $B=0.05$ for $f=|u|$ given in Figure 2 and the first graph of Figure 3. The form of the resulting controls for the two different functions $f$ is very similar, though the scale is slightly different.


Figure 13. Optimal Control and Corresponding State in 1D Over Time


Figure 14. Time Slices of the Optimal Control and Optimal State

We now look at the results with the nonlinearity,

$$
\begin{equation*}
f(x, t, u)=|u|+\frac{|u|}{1+u^{2}} . \tag{7.3}
\end{equation*}
$$

Here we exclude the figures, as we again have very similar results to what was obtained for the previous function $f$ and for $f=|u|$.

Moreover, we obtained similar results for each case of $m$ that we ran, whether $f=|u|$, or $f=|u|+\frac{1}{1+|u|}$ or $f=|u|+\frac{|u|}{1+|u|^{2}}$. These results might be further indication that, generally speaking, the shape of the optimal control really is driven by the form of $m$, with only small variation due to the form of $f$, however further investigation clearly must be done before any definite conclusion may be drawn. For example, would it be possible to construct an $f$ such that the shape of the resulting control would vary strongly from that of $\nabla m$ ?
8. Conclusions. We have investigated the problem of determining the optimal advection direction for a population with nonlinear growth and diffusive movement in a heterogeneous resource environment with zero flux across the boundary, where optimality is defined by maximizing the total population size integrated over the time domain and minimizing the cost of energy expended in directed movement.

The existence and characterization of an optimal control are established and, moreover, we show the optimal control is unique for a small enough time horizon $T$ and large enough cost $B$. Necessary conditions are provided for the characterization of the optimal control.

Numerical results are presented for one-dimensional habitats, and we illustrate our investigation into the dependence of our optimal control on the form of $m$, on whether $m$ is time dependent or independent, and on the form of $f$. Our research suggests that an optimal strategy for directed advection (away from the boundary) is to follow the form of the gradient of $m$, or in other words for the population to go toward regions of better resources. This appears to be fairly robust, and the form of $f$ appears to have little influence on the form of the optimal control, relative to the influence of $m$. We do see variation in some instances in the case when $m$ is time-dependent, as the coming changes in the resource allocation is reflected in the current form of the optimal advection, and we see a kind of combination of the gradient of $m$ at the current time and near future time. This seems to suggest that in the case of a changing resource environment, it is optimal to have a strategy that combines moving toward better resources as they are allocated at the current time while also incorporating influence from predicting changes in the distribution of resources that will come in the near future. Of course, unless the time dependence in $m$ is seasonally driven, the population may not be able to predict $m$ values in the future. Finally, while the form of $f$ does not appear to have much effect on the form of the optimal control according to our particular investigations, further work must be done to conclusively determine the effect of $f$ on the optimal control in general.

Relating to [7] about advection along $\ln m$, when doing the numerical investigations we also looked at the resulting optimal control given that the initial guess for the control was the gradient of $\ln m$, in which case the result was that the control inevitably moved away from this initial guess and toward the gradient of $m$.

Many interesting questions remain open for future investigation. How does the choice of boundary conditions affect the optimal control? For instance, past work by Belgacem and Cosner shows that persistence of the species is enhanced by advection along $\nabla m$ in the case of no-flux boundary conditions, but in the case of Dirichlet boundary conditions this same type of movement can be harmful or beneficial depending on other parameters. It is possible that in the case of Dirichlet boundary conditions we may get a very different optimal advection pattern. Another question that arises is whether or not it is possible to construct an $f$ that results in an optimal advection which is very dissimilar from $\nabla m$, or if the optimal advection always follow the shape of $\nabla m$ away from the boundary regardless of the choice of $f$. Obviously, it would be intriguing to obtain an analytical characterization of the optimal control in which the dependence on $\nabla m$ is explicit, but this seems
quite difficult to achieve. It may be interesting to look at changing the objective functional to understand different strategies for populations with differing goals.

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