Part I: QR method. Consider the matrix

\[ A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & -6 \\ 7 & -8 & 9 \end{bmatrix} \]

We would like to approximate the eigenvalues of \( A \) using the QR method. The main ingredients are

(i) Reduction of a matrix to upper Hessenberg form using similarity transformations.
(ii) QR factorization of a matrix using Householder matrices,
(iii) Deflation

You are not expected to program these individual routines; rather use the MATLAB functions hess, qr or equivalents from other packages such as Mathematica or Maple.

Preprocessing Reduce \( A \) to upper Hessenberg form using the hess function of MATLAB, i.e. let \( H = \text{hess}(A) \). Note that \( A \) and \( H \) are similar. This should be done because it is much more efficient to apply QR to a Hessenberg matrix. Now, you may consider proving that both the basic and shifted QR methods preserve the upper Hessenberg structure.

1) QR without shifts Implement a sequence of QR iterations to \( H \), without shifts, until the the elements in the strictly lower triangular part of \( H \) are less than \( 10^{-8} \) in absolute value. As output, print your approximations to the eigenvalues. Also record the number of iterations. This is a large number given that two of the eigenvalues of \( A \) are close to each other.

2) QR with shifts Implement the QR method with shifts, i.e.

\[ H - sI = QR, \quad H = RQ + sI \]

This should be done as follows:

- use \( s = h_{33} \) until \( h_{32} \) is smaller than \( 10^{-8} \) in absolute value. Note that \( h_{33} \) is the only nonzero element in the last row of \( H \) other than \( h_{33} \) since \( H \) is upper Hessenberg. As \( h_{32} \to 0, h_{33} \) will converge to an eigenvalue of \( H \). Also, \( s \) should have a different value at each QR iteration; if you use the same initial value repeatedly then convergence will be slower.
- Once \( |h_{32}| \leq 10^{-8} \), Deflate the matrix \( H \) by removing the third row and third column. The \( 2 \times 2 \) matrix that’s left has two eigenvalues which are the remaining two eigenvalues of \( H \). (The proof is easy!) This step is important. Since we have to change the shift parameter and without deflation \( h_{32} \) will start to grow back! try it!
- Apply the QR iteration on this \( 2 \times 2 \) matrix with shift \( s = h_{22} \) until \( |h_{21}| < 10^{-8} \). The matrix now is upper triangular (within \( 10^{-8} \) that is) and therefore \( h_{11}, h_{22} \) are approximations to the remaining two eigenvalues of \( H \).

Print your approximations to the eigenvalues and the number of iterations. compare the latter to that required by QR without shifts.

Theory (Optional) Show that the QR iteration preserves the upper Hessenberg structure.
**Part II: Broyden’s method.** Consider the nonlinear system

\[
\begin{align*}
x_1^2 + x_2 &= 37 \\
x_1 - x_2^2 &= 5 \\
x_1 + x_2 + x_3 &= 3.
\end{align*}
\]

Show that this system has exactly two roots.

Use Broyden’s method together with the Sherman-Morrison formula to approximate a root of the system. Start with \(x_0 = 0\) and \(A_0 = J(x_0)\). Stop the iteration when \(\|x_{k-1} - x_k\|_\infty \leq 10^{-14}\). Organize your output as follows

<table>
<thead>
<tr>
<th>(k)</th>
<th>((x(k))_1)</th>
<th>((x(k))_2)</th>
<th>((x(k))_3)</th>
<th>(|F(x_k)|_\infty)</th>
<th>(|x_{k-1} - x_k|_\infty)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>\vdots</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Also, print the last \(A_k\). Is it equal to \(J(x_k)\)?