Shreeram Abhyankar (July 22, 1930–November 2, 2012)

Shashikant Mulay and Avinash Sathaye, Coordinating Editors

Shreeram Abhyankar

Shreeram Abhyankar was an influential mathematician and an inspiring teacher. His infectious enthusiasm for research and steadfast devotion to teaching have been truly inspirational not only for us, his students, but for all who came in contact with him. His attractive and accessible lectures, delivered in his inimitable style, presented their mathematical content with such clarity that the audience was not just impressed, but sensed an irresistible invitation to try their hand at the topic. Whether it was algebraic geometry or algebra, he always preferred concrete over abstract and was admirably adept at detecting key elementary motivations that opened the doors to a novel mathematical treasure. As a self-proclaimed high school algebraist, he proved the beauty and power of high school algebra through his research while extolling its virtues with missionary zeal in his expositions. He solemnly wrote:

Polynomials and power series
May they forever rule the world.

Most remarkably, he did not erect any egotistical barriers around his personality, never exuded any exclusiveness, and welcomed all students/researchers to his *Gurukul* (guru's extended family). One never felt the weight of his accomplishments in his company. Although not related by blood, we were a part of his family and he was a beloved member of ours; we will miss him dearly for the rest of our lives.

In this article, contributing authors describe the impact of Abhyankar's multifaceted mathematical work and their remembrances of him. Abhyankar's better half, Yvonne, presents a short biographical sketch and a glimpse into their life together. We, the coordinating editors, feel fortunate to have received the contributions and are sincerely grateful to the contributors.

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Yvonne Abhyankar

On November 2, 2012, Ram went to sit at his desk after taking a short nap after breakfast and waking up to answer a call from his brother in Pune. I left the room for a few minutes, and when I came back I found him leaning back in his desk chair and I could not rouse him. The Purdue paramedics came quickly, and he was taken to the hospital, but nothing could be done. He had died of "natural

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causes" according to the death certificate. We had known each other for about fifty-eight years and were married for over fifty-four.

I went to college at Barnard College, which was affiliated with Columbia. As a Barnard student it was possible to take courses at Columbia, so in 1955 I registered for a math course at Columbia. The instructor turned out to be Ram. After some weeks he suggested we go out, and then for a time we went out quite often. Of course, as my friends tell me, this was a highly improper thing to do. Eventually we drifted apart. After some time I heard that Ram had been in a car accident in Maine, so I wrote him a short letter hoping that he would soon be well. Somehow we continued corresponding. He had, by that time, begun teaching at Cornell. I was by then a graduate student at Boston University. Since Ram had studied at Harvard, getting his Ph.D. under Oscar Zariski, he came frequently to Boston to see Zariski, and since I was also there, he visited me. In this way our courtship and lifelong friendship began.

He told me that he was born in Ujjain, but spent most of his life in Gwalior, where his father taught mathematics at a college and later became its principal. Mathematics was the household business, and Ram was surrounded by it since he was a child. Ram fell in love with mathematics as soon as he learned to count and would spend hours on end in the pursuit of further knowledge. Ram became so obsessed with mathematics at a young age that it worried his father; once Ram's father locked up his math books. This disturbed young Ram greatly; with the help of his uncle he would wake at the crack of dawn before anyone else and sneak away and unlock the hidden treasure.

Family was always very important to Ram. Ram was surrounded by a large extended family growing up. He was the second eldest; he had two sisters and three living brothers and many cousins. He told many stories of his life growing up with his siblings.

His father and uncles were able to teach him the foundations of mathematics, but Ram knew that he needed to find a teacher who could teach him more. Ram had studied first in Gwalior, but then he shifted to Bombay and studied at what was then known as the Royal Institute of Science. Since his father was a mathematician, Ram felt it would be better to say that he wanted to go to Bombay to study physics. He referenced the Gateway of India in Bombay as his gateway as well, for he knew that he needed to leave his beloved country to find his true passion. While in Bombay he attended lectures at the new Tata Institute. Among the guest lecturers he mentioned was Professor Stone, who was visiting from Chicago. He also met Masani, who had recently received a Ph.D. from Harvard. These encounters solidified Ram's belief that he had to leave India. In 1950 this was not an easy task, as he could not afford to go without financial help. He managed to get some study grants from India, and thus his journey began. He got free passage on a ship to the US. He fell ill on the ship going to the US and ended up in a seamen's hospital in England, fortunately after socialized medicine had been established there. Ram took some time to recover enough to finish his trip to Harvard.

By the time he reached Cambridge, it was quite late in the semester. Upon arrival Ram immediately went to the mathematics department at Harvard. It was a Saturday morning when he arrived at the department. He enquired whom he could see, and the secretary suggested seeing Professor Oscar Zariski, who happened to be in the department, perhaps getting his mail. Ram often spoke of this first meeting. Zariski apparently asked him various questions about mathematics and at the end asked what courses Ram proposed to take. On seeing the list, which had been suggested by Masani, Zariski proposed that Ram take more advanced courses except for one in projective geometry which Zariski was teaching. Zariski later said it was a beautiful subject but not one to do research in. Later Ram told me that this turned out to be incorrect. Zariski asked Ram what his father did, and Ram reported that he was a college math professor. Then, not knowing that one should not ask personal questions, Ram enquired what Zariski's parents did and was told that his father was dead and that his mother had a cloth shop. Zariski suggested that Ram come to his house and pick up some books since Ram had arrived after the semester had begun so he thought this might be helpful. In this way Ram found the mentor that he had so greatly wanted and his embarkation into algebraic geometry began.

His early days at Harvard were mathematically invigorating but financially challenging, as one grant that he believed he had existed only on paper and the money did not arrive at Harvard. Thus he had some financial difficulties and was obliged to eat all his meals at the college since he could charge them. From his second year onward he got financial aid from Harvard. Ram told many stories of his stay at Harvard and his good fortune in meeting Zariski. He also spoke often of Mrs. Zariski and how kind she had been to him on many occasions. After we were married we often visited the Zariskis in Cambridge, and later they spent time at Purdue.

In 1958 Ram and I decided to get married. Ram came from a culture of arranged marriages, and clearly I did not fit into this scenario. He wrote his father a letter asking permission to marry me, and his father agreed to the marriage. When I enquired what he would have done had his father said no, he said he would have married me anyway. We married on June 5, 1958, which he thought convenient, as it was also my birthday. After getting married we went to Paris for a month. J.-P. Serre arranged the hotel and also told Ram that he did not expect him to do much mathematics during that month, though that had been the original plan, saying that after all he was a Frenchman. He found us a room at a hotel, Montalembert, which was a modest hotel. One of the nice things was that one could have breakfast at any time. Later, when we returned to Paris, as we did often, it had turned into a really elegant, fancy hotel which we could no longer easily afford. At the time we were in Paris on our honeymoon we could afford wonderful meals. After our honeymoon in Paris we went to Gwalior, India, where Ram's parents had lived for many years, and were again married in a Hindu ceremony. It had been seven years since Ram had been in India, and his mother was so excited to see him that she fell down some stairs, fortunately not injuring herself. Ram's parents welcomed me with open arms. Most of his family spoke English, so it was very easy for me to communicate with them. Ram's mother on the other hand did not speak English, and so I decided to learn Marathi.

By the time we were married Ram had a job at Johns Hopkins, and I registered as a student there and got a master's in physics. After some years at Hopkins, Ram accepted a job at Purdue. Here Ram established his career and built his mathematical legacy. Over the years he had twenty-one graduate students and many unofficial students. Ram mostly worked from home, and his collaborators would spend many hours at our house. Over the years he also got to meet faculty in engineering and computer science, and gradually they began to learn mathematics from him as well. During our marriage Ram lectured in many places, was a visiting professor in several universities, and we traveled a lot, first the two of us and later, when we had children, we traveled with them.

In 1970 our son, Hari, was born and in 1973 our daughter, Kashi. Ram was determined to have his children speak his language, Marathi, and also know something of the culture of India. For this we spent many years in India. Ram loved having his children around; he said their noise helped him concentrate on his work. After the children grew up and left the house he found it hard to be without them, so we traveled more than ever. When our children married and each had two daughters, Ram very much enjoyed playing with them. Our son and his family visited West Lafayette just the week before Ram died. Hari said he somehow felt a strong urge to come. Kashi and her younger daughter were to come the week after he died, and we both were very much looking forward to their upcoming visit.

Igusa, Abhyankar and Nagata, 1957.

Ram greatly enjoyed teaching and having students, and just as Zariski had invited him to his house, Ram, who generally worked at home, had students coming over very often. This continued until he died very suddenly on November 2, 2012. He was fortunate in that he could continue to teach and have students until he died. He greatly enjoyed doing mathematics and also collaborating with other mathematicians in his research. In fact, the day after he died, a student, who had not heard the sad news, appeared at our house to discuss mathematics with him. Ram never wanted to retire and he had his wish fulfilled.

A number of his former students and friends came to our house when they heard the news, as they were very much a part of his family. A former student, Avinash Sathaye, who was like a son to him, in addition to being a mathematician is also very well versed in Sanskrit, and so helped a great deal by conducting the funeral. Ram was cremated as is the Hindu tradition, and his ashes were scattered in the Wabash River by Hari while Kashi and I looked on. While he remained mentally as acute as ever until he died, in retrospect I realize that he had been physically declining for some time. Walking any distance had become difficult; he needed a wheelchair when we traveled by air. Without being able to do mathematics he would have been miserable. He often said that mathematics was his religion and it was his life. He spent most of his day thinking and breathing mathematics.

Nagata and Abhyankars, 1958.

Chanderjit Bajaj

I greatly valued the opportunity to have Dr. Shreeram Abhyankar as one of my mathematical mentors and collaborators. While I and several others are deeply saddened by his death on November 2, 2012, we are greatly indebted for his many contributions to mathematics (both pure and applied, with the latter sometimes unbeknownst to him) and for the many ways he imparted his love of this great subject to others. His wisdom and continual encouragement to pursue mathematics are most appreciated.

Shortly after I started graduate school at Cornell in 1981 to study theoretical computer science, my advisor, Dr. John Hopcroft, introduced me to Tarski's theorem [37], where algorithmic quantifier elimination yielded a decision procedure for the first order theory of reals. A goal was to explore more efficient algorithms, albeit exponential in the number of alternating quantifications, for the decidability of quantified real algebraic and semialgebraic equations, as several problems in geometric motion planning [36] and geometric optimization [24] were reducible to it. Little did I know at the time that Shreeram Abhyankar, a famous algebraic geometer and chaired professor of mathematics, had been resurrecting and advancing the theory of polynomials/power series [17], factorizations, weighted expansions and birational mappings [14], [13], and the algorithmic formulae (resultants) of elimination theory [18] pioneered by Cayley, Sylvester, popularized by Salmon and later van der Waerden [38], and of course Macaulay [35]. Quantifier elimination would recur several times in my career and also influence the scientific careers of many computer scientists, applied mathematicians, including those interested in geometric finite element modeling, computer-aided curve and surface geometric design, algorithmic robotics and control, all united by a search for efficient characterizations and solutions of systems of algorithmic polynomial equations and power series [30].

When I took up my first faculty job in the computer science department of Purdue University in 1984, my office quite fortuitously turned out to be on the same floor as Abhyankar's office in the mathematical sciences building. At that time the offices of the faculty and postdocs/graduate students of the Purdue departments of mathematics, statistics, and computer science were somewhat commingled, with half or quarter floors sectioned for each. After spending several late nights in my office befriending a few of Abhyankar's students, I was soon led to meet the great man himself. Abhyankar, who did most of his work out of his home office (a couple of blocks away from mathematical sciences), seemed to work tirelessly all day and late nights discussing with his collaborators and lecturing to his students. It was easy for me to join these postdinner sessions, with my own list of questions and problems, attempting to learn the breadth and depth of the intertwined fields of algebra and geometry. It was one of the most intense yet interesting and enjoyable times of my young faculty life, with the day spent teaching computer science to undergraduates and then evenings and nights consumed in learning undergraduate algebraic geometry from Abhyankar interspersed duly with some mythology and work of Indian sages and scholars Bhaskaracharya and Shreedharcharya [1], [3], [7], [8], [11], [39], [21], [12].

One of my initial challenges was to develop computer shape modeling with real algebraic curves and surfaces. Abhyankar taught me the algorithmic work of Newton, Hensel, Tschirnhaus, Weierstrass, and Zariski and how they were theoretical computer scientists in our now agreed-upon distinctions and similarities between computer science, pure, and applied mathematics. This helped develop and implement (in a software package we called GANITH) effective algorithms for computations of the arithmetic genus, curve factorizations, adjoint curve systems, as well as global and local polynomial and rational parameterizations of complex algebraic curves and surfaces, and moreover characterize them to work in nonalgebraically closed fields, finite fields, and in finite precision [19], [20]. The nightly meetings were also an excellent way for me to learn algebraic geometry from a guru, namely, valuations [23], [2], ramifications and local uniformization [4], [3], [8],

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tame coverings and fundamental groups [5], homomorphisms and birational transformations [22], and, in particular, algorithmic desingularization theory [6], [9], [10], and nonzero characteristic, by far his favorite topic. There were of course many other computational challenges, namely, in harnessing compactly bounded portions of real algebraic curves, surfaces to support computational free-form geometric design and analysis (a.k.a. real algebraic finite elements) [25], [29], [26], [31], [32], [33]. The blossoming world of CAGD (computer-aided geometric design) was already replete with a variety of parametric finite elements, including Bernstein-Bezier and B-splines (where the B- stood for basis) [28]. Abhyankar in a series of lectures and articles [16], [15] outlined relevant theorems and techniques that proved useful for several researchers, including me, in developing the real algebraic finite elements with applications.

There was also an implicit trade agreement. I had helped him gravitate from a Windows laptop (that Yvonne used) to a UNIX workstation where he eventually did all his writing, and so he would save his questions for our evening learning sessions, as he learned the UNIX OS as well as $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX, and emails with attachments, etc.

Although some of these computational accomplishments are significant, Abhyankar is recognized for his superlative achievements in mathematics, which I'll let others extol. Suffice it to say that his sixtieth and seventieth birthday conferences [27], [34] had speakers who were the veritable who's who in algebraic geometry. It also soon became apparent to a growing number of computer scientists and engineers that Abhyankar's algorithmic methods in algebra and geometry had the potential to yield efficient algorithms. He received invitations to speak and teach at numerous workshops by computational geometers in Barbados and symbolic computation folks in Italy and Germany. His aforementioned birthday conference speakers also included a veritable mix of renowned computer scientists and engineers. At the BIRS (Banff International Research Station) Workshop on Algebraic Geometry and Geometric Modeling, January 27–February 1, 2013, an evening program was devoted to remembrances of Professor Abhyankar and the influence and impact of his work on algorithmic algebraic geometry with applications to geometric modeling.

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Steven Dale Cutkosky

Abhyankar's interest in resolution of singularities began when he was a graduate student at Harvard in the mid-1950s. This was a subject that his advisor and mentor Zariski was particularly interested in. Zariski had discovered the general definition of nonsingularity over all fields and in mixed characteristic: the local ring of a nonsingular point is a regular local ring [48]. Zariski had also proven resolution of singularities of three-dimensional varieties over an algebraically closed field of characteristic zero [47] and proven local uniformization in any dimension over an arbitrary field of characteristic zero [46], introducing general valuation theory into the subject. Zariski was very interested in the question of resolution of singularities of positive characteristic surfaces and mentioned this to Abhyankar as an important problem which was probably too difficult for a Ph.D. problem. Abhyankar became fascinated with this problem, and after a tremendous effort solved it as his Ph.D. thesis.

If X is a projective variety over a field k, then a resolution of singularities of X is an algebraic mapping (morphism) $f : Y \to X$ such that Y is projective and nonsingular. The local form of resolution of singularities is local uniformization. If ν is a valuation of the function field of X, then ν determines a unique point of X, called the center of ν on X. The valuation ring of ν dominates the local ring R of the point. A local uniformization of X along ν is a morphism $f : Y \to X$ such that Y is projective and the center of ν on Y is a nonsingular point on Y. If S is the local ring of this point, then we have inclusions of local rings

$$R \to S \to V\nu,$$

where $V\nu$ is the valuation ring of ν and S is a regular local ring. Much of the difficulty of local uniformization comes from the fact that the value groups for general valuations can be very complicated. Some examples are given on pages 99–106 of [49]. For instance, the value group can be the rational numbers.

Throughout the history of resolution, ramification has played a major role. In 1908, Jung [30] showed that if a normal complex surface S is finite over a nonsingular surface and the branch divisor has only simple normal crossing singularities, then S has only Abelian quotient singularities. As a

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consequence, the singularities of S are not difficult to resolve. Abhyankar discovered that this theorem holds if the extension has only tame ramification (which always holds in characteristic zero) but fails if the ramification is not tame. This theorem is often called the Abhyankar-Jung theorem. It appears in Abhyankar's 1955 paper "On the ramification of algebraic functions" [1].

Abhyankar proved local uniformization along a valuation in a two-dimensional algebraic function field over an algebraically closed field of positive characteristic using his development of ramification theory. The proof is published in his 1956 paper "Local uniformization of algebraic surfaces over ground fields of characteristic $p \neq 0$" [2]. Abhyankar uses an ingenious method of descent and ascent to reduce to the case of an extension of function fields of prime degree and where the center of the valuation is a regular point in the smaller field and a singular point above which needs to be resolved. The difficult case is an Artin-Schreier extension. Abhyankar gives a direct proof in this case and deduces resolution of singularities of projective surfaces over an algebraically closed field of positive characteristic.

Abhyankar later gave a proof of embedded resolution of algebraic surfaces in all characteristics. Most of the proof appears in the series of papers [4], [5], [6], [7], and the final part of the proof appears in the 1966 book *Resolution of Singularities of Embedded Algebraic Surfaces* [8]. Using this result, he proved that a resolution of singularities exists for a three-dimensional algebraic variety over an algebraically closed field of characteristic p greater than 5. In [21] Cutkosky gave a simplified proof of this result. Recently, Cossart and Piltant [17] have succeeded in proving that resolutions of singularities exist for algebraic varieties of dimension three over fields of the remaining characteristics 2, 3, and 5. Their proof draws heavily on ideas from Abhyankar's papers. It is still unknown if resolutions of singularities always exist for varieties of dimension greater than or equal to 4 and of positive characteristic.

Abhyankar outlined a proof of resolution for arithmetic surfaces in [10], and Hironaka outlined a simplified proof of resolution of surfaces in [28]. There are very general proofs of resolution of excellent surfaces in [36] and [16]. After taking a suitable generically finite morphism, De Jong [23] has shown that it is possible to find a resolution in positive characteristic. Some recent papers on the problem of resolution in positive characteristic are by Bravo and Villamayor [13], Hauser [26], Hironaka [29], Kawanoue [32], Kawanoue and Matsuki [33], Knaf and Kuhlmann [34], Moh [37], Spivakovsky [41], Teissier [42], and Temkin [43]. A

Abhyankar, Akizuki and Remmert, 1960s.

self-contained introduction to the problem can be found in [19].

In characteristic zero, Abhyankar showed that hypersurfaces of maximal contact always exist for singularities. This allows a reduction of resolution of singularities to one dimension or less and allows an inductive formulation of resolution. He found an explicit construction, which he called a Tschirnhausen transformation in honor of the seventeenth-century mathematician. The transformation is a generalization of the method of completing the square to solve quadratic equations. It is absolutely remarkable that this simple idea makes resolution of singularities possible. In the equation

$$f = z^d + a_1 z^{d-1} + \cdots + ad,$$

where the a_i are polynomials or series in the variables $x_1, \ldots, x_n$ which vanish at the origin to order $\geq i$, make the substitution

$$\overline{z} = z + \frac{a_1}{d!}.$$

Then we have a new equation

$$f = \overline{z}^d + \overline{a}_2 z^{d-2} + \cdots + \overline{a}_d. \tag{1}$$

Abhyankar showed that blowing up the most singular points of $f = 0$ makes the singularity better, except possibly at points which are on the transform of $\overline{z} = 0$. The transform of f continues to have the form (1) at these points. Points where the singularity is not better are always on the transform of $\overline{z} = 0$, no matter how many times you blow up. $\overline{z} = 0$ is called a *hypersurface of maximal contact for* $f = 0$.

Hironaka used the Tschirnhausen transformation as the starting point of his 1964 proof [27] of resolution of singularities of algebraic varieties of any dimension in characteristic zero.

In recent years there have been great simplifications of Hironaka's original proof of resolution in characteristic zero. Some of these important papers are by Bierstone and Millman [12], Encinas and Villamayor [24], Hauser [25], Kollár [35], Villamayor [44], and Włodarczyk [45].

The Tschirnhausen transformation is the major part of Hironaka's proof which does not extend to characteristic $p > 0$. The transformation is not possible if p divides the degree d of f, as p times the identity is zero in characteristic p.

In his 1982 Ph.D. thesis with Abhyankar, Narasimhan [38] gave an example showing that hypersurfaces of maximal contact do not generally exist in positive characteristic.

Abhyankar was also very interested in the structure of birational mappings of algebraic varieties, a problem which is closely related to resolution. A birational mapping of a variety X to a variety Y is an algebraic isomorphism from an open subset U of X to an open subset V of Y. Such a mapping may not extend to a globally defined mapping of X, although there is some largest open subset of X on which the map is defined. When the birational mapping is defined everywhere on X, then we will call it a morphism. If X and Y are nonsingular projective curves, then the only possible birational mapping of X to Y is an isomorphism. The structure of birational morphisms of nonsingular projective surfaces was found by Zariski for surfaces over a perfect field k [47]. This result was known by Castelnuovo within the context in which he worked [14]. Zariski's theorem is that every birational morphism of nonsingular projective surfaces can be factored as a sequence of blowups of points. The blowup of a point can be understood locally as being an algebraic substitution

$$x = x_1, y = x_1 y_1.$$

Abhyankar proved as part of his Ph.D. thesis and in "On the valuations centered in a local domain" [3] that an arbitrary birational extension of two-dimensional regular local rings can be factored by local rings of blowups of maximal ideals. As a consequence, the general statement that birational morphisms of nonsingular projective surfaces over an arbitrary field factor as a sequence of blowups of points follows.

Birational geometry in higher dimensions is much harder. The factorization theorem in dimension two does not extend in a simple way to higher dimensions. In dimension three, besides blowing up points, we can also blow up nonsingular curves. However, there are examples of birational morphisms of nonsingular projective three-folds which cannot be factored by compositions of blowups of points and nonsingular curves. Examples showing that this fails even locally were found and published by Judy Sally [39] (Ph.D. thesis with Kaplansky) and David Shannon [40] (Ph.D. thesis with Abhyankar).

This led Abhyankar to reformulate the problem locally as a conjecture (Section 8 of [9]): Given a valuation ν dominating a birational extension of regular local rings $R \to S$, does there exist a regular local ring T which is dominated by ν and is obtained from both R and S by sequences of local blowups of regular prime ideals along ν? This local factorization problem was solved in dimension three for the case of valuations of maximal rank by Christensen in his thesis with Abhyankar [15]. In this case, the value group is $\mathbb{Z}^d$ where $d = 3$ is the dimension of the variety, and the problem is readily translated into a (difficult) problem in combinatorics. Karu [31] has given a proof of local factorization along a maximal rank valuation in all dimensions and characteristic zero using toric geometry. A proof of the maximal rank case is given using determinantal identities by Cutkosky and Srinivasan in [22]. Cutkosky proved local monomialization in characteristic zero along an arbitrary valuation in [18], which proves Abhyankar's local conjecture (in characteristic zero).

Abhyankar (Section 8 of [9]) and earlier Hironaka (in [27]) conjectured that, in dimensions greater than or equal to three, a birational morphism of nonsingular projective varieties $f : X \to Y$ can be factored by first performing a sequence of blowups of nonsingular subvarieties above X and then performing a sequence of inverse blowups of nonsingular subvarieties above Y (blowdowns) to reach Y. This conjecture is still open, even for the case of morphisms of toric varieties, although there has been a lot of progress in characteristic zero. It was shown by Abramovich, Karu, Matsuki and Włodarczyk [11] that it is possible to construct a finite sequence of blowups and blowdowns with nonsingular centers to obtain a factorization. Their proof uses methods from geometric invariant theory and Mori theory. Cutkosky has given a completely different proof of this theorem in dimension three as a consequence of his monomialization theorem for maps of projective threefolds [20].

Abhyankar's conjectures on factorization in positive and mixed characteristic remain completely open.

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David Harbater

One day in 1989 I was surprised to receive a letter in the mail from Prof. Shreeram Abhyankar. Although I had known of his work since I was a graduate student, we had not met, and in those days before email was commonplace, an unsolicited letter from a well-known senior mathematician was quite unexpected. The contents were a surprise as well: he wrote that after a long hiatus in his work on algebraic fundamental groups, he wished to resume research in that direction, and he was asking if I could provide a summary of recent developments on that topic.

That letter led to many exchanges, including a number of visits of mine to West Lafayette and of his to Philadelphia, where we discussed (and sometimes argued about) mathematics for long hours. Our discussions during my visits to Purdue took place, not in his office in the mathematics department, but in the office/seminar room that was set up in his house, where grad students and others would gather for long mathematical sessions in front of his large blackboard. The discussions were lively to say the least, and at times even a bit unnerving, as he sought to see the underlying essence beneath the surface presentation of the mathematics.

Besides the mathematical discussions during my visits, he would also tell me tales of his "guru" (and my mathematical grandfather), Oscar Zariski, as well as about his other interests, such as the characters in Indian lore. There were often crowds of mathematicians who would come to the house in the evening, and all would graciously be fed Indian food by his wife and loyal companion, Yvonne.

Before I met Ram I had been aware of two seemingly distinct threads in his research. One concerned the resolution of singularities on algebraic varieties, especially in finite and mixed characteristic. The other concerned the study of branched covers of varieties and, in particular, the formulation of "Abhyankar's Conjecture" on the fundamental group of affine curves in characteristic p. But as I learned from him during our many talks, these two threads were in fact intimately intertwined, and both grew out of his Ph.D. thesis.

For the thesis Zariski had proposed that he study resolution of singularities of algebraic surfaces in characteristic p. Zariski had proven resolution over algebraically closed fields of characteristic zero, and he had suggested that a proof in arbitrary characteristic could be obtained by translating into algebra an even earlier analytic argument of H. W. E. Jung that concerned complex surfaces. As Ram liked to tell the story, there were two parts to his thesis: the successful part and the "failure" part. What he meant was that he discovered that Jung's method breaks down in characteristic p [1], and as a result he had to use another approach to prove resolution of singularities in that context [2]. But the failure of Jung's method to generalize did not prove to be a dead end. Instead, it led to Ram's work on branched coverings of varieties and to his introduction of the idea of an algebraic fundamental group that can be viewed as analogous to the fundamental group in topology [3].

To obtain resolution of singularities, what was needed was to prove "local uniformization," which asserted that the given variety could be written as a union of finitely many Zariski open sets, each of which is the image of a birational morphism from a smooth variety. In Jung's strategy for surfaces, the idea was to express the given variety as a branched cover of a linear space and then to blow up so that the branch locus has only ordinary double points. This relied on the fact that the variety is smooth at any point lying over a smooth point of the branch locus, with cyclic inertia group, and that over an ordinary double point of the branch locus the inertia group is a product of two cyclic groups.

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This indeed holds over an arbitrary algebraically closed field of characteristic zero.

But, as Abhyankar found in the "failure" part of his thesis, these properties of branched covers do not hold in characteristic p. To use his phrase, in characteristic p there can be "local splitting of a simple branch variety by itself" [3, Sect. 1], meaning that the inertia group can increase over a smooth point of the branch locus, as distinct components of the ramification locus meet at a point on the cover lying over that smooth point. In addition, as he found, inertia groups over an ordinary double point of the branch locus can be quite complicated, even being nonsolvable. Rather than simply abandoning this approach to resolution in favor of his successful desingularization strategy, he initiated a serious study of covers in characteristic p. This included not only surfaces and varieties of higher dimension [4] but also of curves. What he observed was that, by taking a slice of a cover of the plane that has nonsolvable inertia, he could obtain an unramified cover of the affine line with nonsolvable Galois group. Although it was well known by Artin-Schreier theory that the affine line in characteristic p has nontrivial unramified covers with Galois group a p-group, the proliferation of covers with much more complicated Galois groups was a real surprise.

This work also led to his defining a "fundamental group" for varieties that would make sense even in characteristic p (and later, in mixed characteristic). For a variety over the complex numbers, the finite quotients of the (topological) fundamental group are the Galois groups of finite Galois (or "normal") covering spaces, and these quotients form an inverse system. Abhyankar defined the algebraic fundamental group to be the collection of Galois groups of finite unramified covers of the given variety. Of course this is a set, not a single group. But he added that "eventually one may have to consider the Galois group…of the compositum" of the function fields of the finite unramified covers [3, 4.2]. This is equivalent to taking the inverse limit of the finite groups in the corresponding inverse system (which is what Grothendieck later did in his work on étale fundamental groups in SGA1).

This work also led him to make his conjecture stating which finite groups can be Galois groups over a curve in characteristic p of a given genus and with a given number of punctures. What he proposed is a type of "maximal conjecture," asserting that anything that cannot be ruled out must occur (a sort of Murphy's Law). Namely, he said that a group will occur as a Galois group of some unramified cover of a given affine curve if and only if its maximal prime-to-p quotient can occur over a characteristic zero curve of the same genus with the same number of punctures [3, 4.2]. (The groups that occur in characteristic zero are well known by topology.) This condition is seen to be necessary once one knows that the prime-to-p Galois groups are the same in characteristics 0 and p. But it was a nonobvious leap to conjecture, based on the examples he had found, that it is also sufficient.

Years later, after his conjecture was proven [9], [8], his "maximal" philosophy led him to formulate possible analogous conjectures in related situations. These included higher-dimensional varieties over an algebraically closed field of characteristic p, affine curves over finite fields, and local fundamental groups in the higher-dimensional situation. In some cases, further investigation led to additional obstructions to the Galois groups that can occur in those situations, leading to difficulties in formulating the correct "maximal conjecture," and these problems remain wide open. Two other problems that also remain wide open are determining which groups can be inertia groups over infinity for a given Galois group over the affine line (the maximal conjecture in this setting was Ram's "inertia conjecture") and determining the structure of the fundamental group of the affine line as a profinite group (on which he and Serre had some amusing exchanges). Also, motivated by the inexplicit nature of the proof of his conjecture, he worked on obtaining explicit realizations of interesting groups as Galois groups over the affine line in characteristic p, which often provided realizations as well in the case of the affine line over the field of p elements. These realizations appeared in a number of papers, particularly in his "nice equations" series in the 1990s (beginning with [6]).

Ram liked to say that what he did was just "high school mathematics." But this would have to have been a *very* good high school indeed to include his work on resolution of singularities on arithmetic surfaces [5] in the curriculum! His point, though, was that one could get far by thinking concretely, e.g., in terms of polynomials, and that rushing to use abstractions such as categories and functors (which he called "university mathematics") could well distract one from the deeper issues.

Still, there is a bit of a paradox here. As I mentioned above, his work on the algebraic fundamental group led to the development and presentation of this topic in Grothendieck's SGA1, whose capstone result [7, XIII, Corollaire 2.12] was similar in spirit to Abhyankar's results in his 1950s papers. The proof of Abhyankar's Conjecture on Galois groups over curves relied heavily on results in "university mathematics," though those results were analogs of the GAGA theorem of his friend J.-P. Serre. Ram's work on arithmetic surfaces would surely be viewed by most mathematicians today as being part of the abstract algebraic geometry of schemes. His efforts to find covers of the affine line given by polynomials whose coefficients are just 0 or 1 (as in his nice equations series) could be viewed as studying fundamental groups of curves over the highly speculative object $\mathbb{F}1$. And his study of the structure of the fundamental group of affine curves in characteristic p, including how it depends on the curve and its base field, is related to Grothendieck's anabelian conjecture.

Regardless of one's own views on "high school mathematics" and "university mathematics," Shreeram Abhyankar's work has had a major impact on mathematics and the mathematical community and continues to influence the direction of research in algebra and algebraic geometry. He certainly had a profound influence on my own mathematical career, for which I am grateful.

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William Heinzer and David Shannon

Abhyankar's research career spanned the period from 1954 through 2012. A great deal of his work during this period of nearly sixty years has connections to commutative algebra. At the

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Ed Davis and the Abhyankars.

suggestion of his advisor, Oscar Zariski, Ram studied the papers of Krull and Chevalley, in particular [20] and [22]. Ram also often mentioned his friendship with I. S. Cohen and the influence Cohen's work [23] had on his career. Ram's 1956 paper on simultaneous resolution [4] is dedicated to the memory of Cohen.

Ram's work in commutative algebra was almost always motivated by concepts in algebraic geometry. For example, his work on valuations was motivated by his interest in local uniformization and resolution. Indeed, these concepts can be formulated in algebraic terms, but Ram often stressed that the motivation and intuition came from algebraic geometry. However, while Ram saw the bulk of his research as a blend of commutative algebra and algebraic geometry, he also would resist being compartmentalized into a particular area of mathematics. For example, in describing his work on developing a simple proof of desingularization in characteristic zero, he observed that "once again this reflects the fundamental unity of all Mathematics from Control Theory to Complex Analysis to Algebra to Algebraic Geometry" [13, p. 285].

Ram extolled the value of "elementary" algebra. He observed in his book [10] that his method of desingularization may be termed the method of Shreedharacharya of completing the square. Ram was especially adept at what is traditionally called theory of equations. Ram interpreted this to mean "simplifying expressions, factoring polynomials, making substitutions, and solving equations." Topics such as quadratic transformations, resultants, Newton polygons, and power series expansions were recurring interests. Ram especially valued algorithmic mathematics.

Nevertheless, Ram was also at home in "abstract" commutative algebra. This is exemplified in his book [15]. In a review of this book in *Math*

Reviews [25], P. Schenzel characterizes the book as "a unique, original exposition full of valuable insights," and recommends the book "to anybody who is willing to see the fascinating, concrete as well as abstract development of algebra during the last centuries."

Ram's work in commutative algebra may be roughly divided into the following three periods:

(1) **1954–1967.** The first period is focused on the problems from algebraic geometry of local uniformization and resolution of singularities. To attack these problems in the algebraic fashion begun by Zariski, Abhyankar needed to extend results of Zariski and others on valuations and local domains. In particular, he needed to consider the case where the ground field has nonzero characteristic. Ram's early paper [3] on valuations centered in a local domain is one of his papers most cited by algebraists. In this paper he first generalizes results of Mac Lane and Schilling [21] on the classification of zero-dimensional valuations of a function field to arbitrary valuations centered on a local domain. He then generalizes Zariski's factorization theorem on birational transformations between algebraic surfaces to abstract regular two-dimensional local domains. Ram also proves in [3] the much-used result (see, for example, [24]), that states that if (R, M) is a regular local domain of dimension n and w is a prime divisor centered on M, then the residue field of w is a purely transcendental extension of a finitely generated extension of R/M.

    In addition to the above-mentioned paper, key papers from this first period include his two thesis papers, [1] and [2], and papers on simultaneous resolution and coverings and fundamental groups of algebraic varieties. Ram was very involved in the question of local uniformization and resolution of singularities over fields of positive characteristic. A little later he considered the arithmetic (mixed characteristic) case where a local ring (R, M) has characteristic zero and the residue field R/M has positive characteristic. During this period Ram made extensive use of a result he proved for the removal of ramification by means of base field extension. This result is now known as Abhyankar's Lemma.

    This first period includes his book on resolution of embedded algebraic surfaces [5] and his address at the 1966 International Congress in Moscow [7]. Ram's 1966 book [5], reprinted and expanded in [13], develops an enormous amount of commutative algebra to deal with concepts such as blowing up ideals and local quadratic and monoidal transformations.

(2) **1967–2005**. This second period saw a broadening of Ram's research interests that included several topics in commutative algebra as well as topics in group theory combinatorics and computer science. He also occasionally returned to resolution, as in his paper from the 1981 Arcata conference on the desingularization of plane curves [10]. This second period is also characterized by much collaboration with students and colleagues in commutative algebra as well as in other areas.

    We list several commutative algebra research topics and collaborations of this period:

    (a) A much-quoted paper from this period is Ram's proof in [6] of an upper bound for the embedding dimension $m(R)$ of a Cohen-Macaulay local ring (R, M) in terms of the dimension $d(R)$ and multiplicity $e(R)$ of R, namely, $m(R) \leq d(R) + e(R) - 1$.

    (b) In a paper with Eakin and Heinzer [8], Ram considers the general question: If A and B are integral domains and the polynomial rings in n variables over A and B are isomorphic, how are A and B related? It is shown in [8] that A and B are then isomorphic if A is of transcendence degree one over a field or if A is a Dedekind domain containing a field of characteristic 0.

    (c) The epimorphism theorem of Abhyankar and Moh [9] is a major result from this period. This asserts: Let $u(t)$ and $v(t)$ be two polynomials in one variable t such that t can be expressed as a polynomial in $u(t)$ and $v(t)$. Let degree of $u(t) = n > n' =$ degree of $v(t)$, and assume that either n or n' is not divisible by the characteristic. Then n' divides n. An alternative formulation states: Any two epimorphisms of the polynomial ring in two variables onto the polynomial ring in one variable whose degrees are nondivisible by the characteristic differ from each other by an automorphism of the polynomial ring in two variables.

    (d) The epimorphism theorem naturally led to a consideration of the Jacobian question that asks: If $y_1, \ldots, y_n$ are polynomials in $x_1, \ldots, x_n$ with coefficients in a field k of characteristic zero and the determinant of the associated Jacobian matrix is a nonzero constant, does it follow that $x_1, \ldots, x_n$ are polynomials in $y_1, \ldots, y_n$? Ram's work on the Jacobin problem in the early 1970s is presented in [12]. He obtains an affirmative answer in the case where the field extension $k(x_1, \ldots, x_n)$ over $k(y_1, \ldots, y_n)$ is Galois. For $n = 2$ he also obtains an affirmative answer in cases including one place at infinity or two characteristic pairs.

(e) In a sequence of five papers with Heinzer [11], Ram investigates integral closure and ramification of prime ideals in infinite algebraic field extensions.

(f) In the paper with Heinzer and Sathaye [14], Ram investigates Bertini-type properties of pencils, such as their singular set and irreducible set.

(3) **2005–2012**. The third period is marked by a renewed interest by Ram in the Jacobian problem. In [19] Ram notes that only part of his work of the 1970s on the Jacobian problem had been published. In a sequence of three papers ([16], [17], and [18]) he meticulously resharpens the techniques (approximate roots, polynomial and power series expansions with meromorphic coefficients, Newton polygons) to push the Jacobian theorem past the result of two characteristic pairs.

However, after listening to Artal, Bodin, and Luengo during 2008–2009, he became fascinated with developing an algebraic theory of dicritical divisors with the potential of application to the Jacobian problem. The result was several papers (some in collaboration with Artal, some with Luengo, and some with Heinzer) in which a comprehensive algebraic theory of dicriticals is developed. Important tools used are Rees rings and a revamped version of Zariski's theory of complete ideals of a two-dimensional regular local ring and new results on quadratic transformations and Newton polygons.

Ram's recent work on dicritical divisors reflects a "rejuvenation" of research Ram did earlier in his career. For example, explicit results on valuations from [3], on resolution from [10], and on quadratic transformations are used. He was actively continuing this research up to his passing in November 2012.

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Thompson and Abhyankar, 1995.

Shashikant Mulay

In the spring of 1982, while I was a graduate student under Abhyankar's guidance, Vinay Deodhar of Indiana University lectured at Abhyankar's home in West Lafayette; the topic of his lectures was Kazdan-Lusztig polynomials and singularities of Schubert varieties. This marked the beginning of Abhyankar's decade-long engagement with Schubert varieties and other related determinantal loci.

For the smooth variety $FL(n)$ of full-flags on an n-dimensional vector space over a field k, consider Schubert varieties X_σ (indexed by the permutations of $\{1, 2, \ldots, n\}$) corresponding to a (fixed) Bruhat-decomposition of $FL(n)$. Studying singularities of an X_σ is equivalent to studying singularities of the affine variety $V_\sigma := X_\sigma \cap W$, where W is the *opposite big cell* for the Bruhat-decomposition. Defining equations of V_σ in the affine coordinate ring of W were the first thing Abhyankar wanted to know. To test the waters, he computed the (radical) ideal I_τ defining V_τ for certain transpositions τ and discovered I_τ to be the ideal $I(2, \mathcal{L})$ of 2×2 minors in a *ladder*-shaped region $\mathcal{L}$ of a matrix having indeterminate entries. Proving primality of $I(2, \mathcal{L})$ was the key part of his argument. This discovery led him to pose the question: Is the ideal generated by the $p \times p$ minors of a ladder (with indeterminate entries) a prime ideal? He felt that, in general, the (radical) ideal I_σ defining V_σ is likely to have some sort of "determinantal" description. Naturally then, he wished to update his knowledge of determinantal ideals. A rectangle being the simplest ladder, it was imperative to understand that case first. So Abhyankar studied the Standard Monomial Theory, initially from [12] and later from [13]. Until he had his own restructuring of what he was trying to learn, Abhyankar seldom felt that he had come to a satisfactory understanding. Here he set out to re-prove the linear independence of Standard Monomials by dimension counting. By the spring of 1983 he had accomplished this and en route obtained new results. He found an explicit expression for the Hilbert function of the ideal of $p \times p$ minors of a matrix with indeterminate entries. More generally, he computed explicit expressions for the Hilbert functions of the usual homogeneous coordinate rings of a class of Schubert subvarieties in a Grassmannian (see [1]). The determinantal ideals, whose primality he established in [1], had the property that their Hilbert polynomials coincided with their Hilbert functions for all nonnegative integers. He called such ideals *Hilbertian*. Subsequently, he retained a keen interest in investigating the Hilbertian property of determinantal ideals.

For the chosen algebra/algebraic geometry topic of his study, Abhyankar insisted on tracing the evolution of the core notions to their historic roots. He had a firm conviction that the inspiration for the main theorems of the topic and the machinery developed for their verification both originate from some concrete high school algebraic idea. This philosophical standpoint allowed him to gain an original and unique perspective of the subject at hand. However, it must be noted that he regarded his historical quest as a personal journey to those splendors of the past that attracted him, not necessarily a comprehensive or scholarly expedition.

From his student days, Abhyankar was acquainted with Hermann Weyl's classic text [30], especially the first two chapters and the last chapter. Abhyankar held the opinion that Weyl had not well-motivated the invariant theoretic part of [30]. Thus, in pursuit of the sought-after motivation, Abhyankar embarked on an in-depth study of nineteenth-century invariant theory. Starting from 1984 in Pune (India) and later in West Lafayette, Abhyankar and the participants of his seminars studied Grace and Young [17], Elliot [14], and Turnbull [27]. Very quickly Abhyankar became an expert in the symbolic (German) method of treating invariants. Once motivated, he had an uncanny ability to master a new area of learning in a relatively short period of time. The tableaux appearing in the last chapter of [17] qualified as Young tableaux in Abhyankar's eyes, and the proof of Peano's theorem presented there, furnished the kind of motivation he was looking for.

As much as he was impressed by Young's work in invariant theory, Abhyankar was even more impressed by the work of Young's student, Turnbull. After a careful study of the invariant theoretic part of [27], Abhyankar proved the Second Fundamental Theorem associated to Turnbull's First Fundamental Theorem for multilinear covariants.

When he explained his proof, I found it quite difficult to cope with Turnbull's "dot" notation, which Abhyankar had so fondly adopted. But Abhyankar claimed that had it not been for the clever and suggestive notation of Turnbull, he would not have discovered the theorem that he did. As it turned out, a less compact but easier proof was found in Weitzenbök's book [29].

Concurrently, Abhyankar kept working on enumerative techniques in the Standard Monomial Theory, with the aim of proving primality of the ladder determinantal ideals. In the latter half of 1984, Abhyankar's student Himanee Narasimhan succeeded in proving a more general result by the method of initial forms (see [26]); she showed the primality of the ideal of $p \times p$ minors of a two-sided ladder with indeterminate entries. A few months later, having sharpened his counting methods, Abhyankar provided (see [2]) a different proof of the same result. Going beyond, in [2] he proves the primality of the ideals generated by certain sets of mixed-sized minors of a matrix with indeterminate entries. The enumerative nature of Abhyankar's proofs made it possible, at a later stage, for Abhyankar and his student Devadatta Kulkarni to establish the Hilbertianness of these determinantal ideals (see [9]). Over time, the project of gaining a better understanding of the coefficients of the Hilbert polynomials of such determinantal ideals has yielded many interesting results (see [16], [19], [20], [22], [28]) as well as raised many open questions.

In 1986 I was able to show (see [23]) that given a (one-sided) ladder $\mathcal{L}$ with indeterminate entries, for all sufficiently large n there exists $\sigma \in S_n$ such that $I_\sigma = I(p, \mathcal{L})$, the ideal generated by the $p \times p$ minors of $\mathcal{L}$. Furthermore, in [23] it is proved that, for each $\sigma \in S_n$, V_σ has a defining ideal of the form $J(\pi, L) := I(p_1, \mathcal{L}_1) + \cdots + I(p_r, \mathcal{L}_r) + \Lambda$, where $L : \mathcal{L}_1 \subset \cdots \subset \mathcal{L}_r$ is a chain of ladder-shaped regions inside an $(n-1) \times (n-1)$ matrix $Z := [Z_{ij}]$ with indeterminate entries, $\pi : p_1 < \cdots < p_r$ is a chain of positive integers, and Λ is the ideal generated by $\{Z_{i(j+1)} - \delta_{ij} \mid 1 \le i \le j \le n-2\}$ (here δ_{ij} is the Kronecker-delta). A result in [23] proves that a proper ideal of type $I(p, \mathcal{L}) + \Lambda$ defines a V_σ, where σ is explicitly constructed from the shape parameters of $\mathcal{L}$ and the value of p. When I explained my proof to Abhyankar, he suggested that a similar description for the equations of the tangent cone to V_σ at the point X_ι should be found; he stressed his requirement that such a description should facilitate computation of the local Hilbert function of V_σ at X_ι. Leaving aside some special cases, description of such *initial ideals* and the associated local Hilbert functions remains unknown. For arbitrary L and π as above, $J(\pi, L)$ need not define any V_σ; in particular, it need not be a prime ideal.

So there arises the question of determining the minimal primes of $J(\pi, L)$ or at least their number. In 1993 I could settle the first nontrivial case: given a matrix $\mathcal{M}$ with indeterminate entries, a submatrix $\mathcal{N}$ of $\mathcal{M}$, and positive integers $q < p$, a description of the minimal primes of $I(p, \mathcal{M}) + I(q, \mathcal{N})$ is provided in [24]. This at once led to examples of X_σ whose singular locus is not equidimensional. In fact, given an integer d, there exists a Schubert variety X_σ whose singular locus has only two irreducible components and their dimensions differ by at least d (see [25]). A guiding research problem in this area asked whether the singular locus of an X_σ is necessarily equidimensional. Towards determining the number of minimal primes of $J(\pi, L)$, Masato Kobayashi has conjectured $\lfloor n^2/4 \rfloor$ as the upper bound on the number of irreducible components of $X_\alpha \cap X_\beta$ for $\alpha, \beta \in S_n$; in [18] he shows that for each n, this is the least possible upper bound.

Ever since Abhyankar learned the "straightening law" for monomials in minors, he dreamed of finding a "straightening formula." Given a bitableau T and a standard bitableau S, he posed the problem of determining the coefficient of S in the straightened expression of T (see section 19 of [3]). With this goal in mind, he studied the straightening process in penetrating detail by employing variants of the Robinson-Schensted-Knuth algorithm. Although his dream is yet to be realized, he was able to produce what he calls a "bijective proof" of straightening (see [10], [11]). In another direction, standard monomial theory has been generalized to a higher-dimensional setting by Abhyankar and his students Sudhir Ghorpade and Sanjeevani Joshi (see [4], [5], [6], [7], [8]). In dimensions greater than 2, the standard monomials in multiminors of a multidimensional matrix are linearly independent but fail to form a standard basis.

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David Mumford

I first met Ram when I was a graduate student several years behind him. We were both studying under Oscar Zariski. Ram was already famous for writing quite long and difficult reports. Oscar warned me when I asked for a copy of a draft of Ram's thesis that I might find it hard going, which turned out to be quite true. Long it might be, but this thesis led to his proof of resolution of singularities for 3-folds in characteristic p, a result that was a wonderful tour de force.

We were good friends, but our paths diverged. Grothendieck came to visit, and I found his new setting for algebraic geometry very congenial. Ram did not. Ram had an independent streak that always led him to look at problems from his unique point of view, often different from that of the crowd. A small example concerns the words *analytic geometry*, which stood at that time for the high school study of conic sections and their equations. Ram said nonsense, algebraic geometry is the study of the geometry of algebraic equations, so analytic geometry must be the study of the geometry of all real or complex analytic equations. His opinion won the day, and this has become common usage. Not so in the case of the value of Grothendieck's schemes, on which he fought the consensus all his life. Our tastes in math were always different, but I admired (and admire) his traditional algebraic approach, which he pursued with such great skill and insight. Few could match him in polynomial calculations, for example, in the construction of extensions of curves and number fields with given Galois group.

I had a wonderful time visiting Ram and his father in Pune at the time he was running a small institute there. Ram was a strong Maharashtrian patriot, proud to feel he was following in the footsteps of the eminent mathematician Bhaskaracharya, as well as those of Shivaji, who stopped the Muslim juggernaut with his famous "tiger claws." Bhaskara's great work in algebra, the *Bijaganita*, had been translated into Marathi by Ram's father. Not knowing that Colebrooke had done this in the nineteenth century, I suggested to Ram he translate it into English, where there is a bigger audience,

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but he didn't feel such a project was worthy of the effort. Bhaskaracharya's math fit in well with Ram's own inclinations: pursuing straight polynomial algebra with formulas, a tradition which I now know goes back in India at least to Brahmagupta around 600 CE. I dined with Father too, although he confessed it was not at all proper for him to do this with a Westerner.

Ram and I also had several long conversations about Indian religion. We were both admirers of Zimmer's books on the religion, symbols, and art of India and found the story of Narada and Vishnu especially powerful. This is food for the soul, and I am sure it served Ram well. He was a distinctive and important figure in the mathematical pantheon who will be greatly missed.

Avinash Sathaye

Abhyankar's early mathematical life was occupied by deep problems concerning resolution of singularities and local and global properties of algebraic as well as analytic varieties, especially in positive characteristic or even in mixed characteristic.

In the late 1960s he made a conscious move towards what is described as "affine geometry" these days. His objective was to offer problems which are accessible to graduate students yet are significant. During his own student days, he was led to believe that most important issues about curves were settled, and so he had consciously turned towards theory of surfaces and higher-dimensional varieties.

However, when he turned his attention towards the subject of curves to attract students, he soon discovered that many interesting problems are still unresolved. Abhyankar's foray into affine geometry is said to have been initiated during a car trip to the University of Kentucky in the late 1960s. Many interesting problems were discussed during this trip. These topics stayed as the focus of Abhyankar's seminars, as well as his lectures around the world, during the 1970s.

Unlike his earlier work, he mostly stayed with the characteristic zero, partly because the positive characteristic often has only counterexamples to offer.

Suppose that $A \supset B$ are rings. We shall say that A is a polynomial ring in m variables over B if $A \equiv B[X_1, \ldots, X_m]$ for indeterminates $X_1, \ldots, X_m$ over B. We express this as $A = B^{[m]}$. Now suppose $A = B^{[1]}$.

Then three questions naturally arise: (i) Given an element $u \in A$, what are the conditions for u to be a variable, i.e., $A = B[u]$? (ii) How unique is B? Is it unique as a subring ? (iii) If it is not unique as a subring, then is it unique up to isomorphism (a cancellation problem)?

Abhyankar and Aroca (Hon. Doctorate, France), 1998.

These questions are at the heart of understanding the polynomial rings that form the basis of affine geometry, i.e., the study of homomorphic images of polynomial rings over various chosen base rings.

Here are the results and questions developed by Abhyankar.

(1) **The Epimorphism Theorem:** In the paper [42] Abhyankar, Heinzer, and Eakin analyzed the above questions in great detail and also formulated their extensions to many variables. For more details see [Details, 2b].[1]

(2) In the 1973 paper [41] Abhyankar and Moh developed a fundamental structure theorem for affine curves with one place at infinity. In it they introduced a fundamental new idea of "approximate roots" which served both as inspiration and tool for many results in affine geometry.

    The main thrust of the paper was the celebrated "Abhyankar-Moh" epimorphism theorem, which was formally published in 1975 [38]. Abhyankar continued to refine and elaborate the structure of planar one place curves, establishing (with B. Singh) the finiteness of their embeddings in the plane [35] and giving simpler explanations of the new techniques in [36], [34]. The Abhyankar-Moh epimorphism theorem established that if $A = k[X, Y]$ is a polynomial ring in two variables over *a characteristic zero field* k, then $F(X, Y) \in A$ is a variable (i.e., $A = k[F][G]$

[1] *[Details] refers to the section of the current article authored by W. Heinzer and D. Shannon.*

for some $G \in A$) if and only if $A/(F) = k^{[1]}$. A proof of this result over complex numbers using techniques of complex analysis was also developed by Suzuki in 1974. The theorem fails in positive characteristic and even the weaker question, Does $A/(F) = k^{[1]}$ imply that $A/(F+c) = k^{[1]}$ for all $c \in k$? is still unresolved!

(3) **Hypersurfaces as variables:** A more general version of the epimorphism theorem, known as the Abhyankar-Sathaye conjecture, asks if "$A = k^{[m]}$ and $A/(F) = k^{[m-1]}$ imply $A = k[F]^{[m-1]}$." Except for several cases for $m = 3$ and isolated cases for higher m, the problem stays unresolved. Numerous alternative proofs using different machinery have been constructed for the Abhyankar-Moh theorem.

(4) **Lines in Space:** Another natural extension was to study a line in three space, i.e., an ideal $I \subset A = k[X,Y,Z]$ such that $A/I = k^{[1]}$. Abhyankar asked the natural question whether I represents an axis; i.e., does I have two generators F, G such that $A = k[F,G,H]$ for some H? This question raised many interesting issues. At that time it was not even clear if I was necessarily two-generated. Abhyankar made an extensive study of the number of generators of ideals of curves in three space ([43], [40], [39]). He produced an explicit construction for three generators for the ideal of any nonsingular space curve, and M. P. Murthy and J. Towber found a way to use it for the first proof of the three-dimensional Serre conjecture in 1974 (later extended to all dimensions independently by Suslin and Vaserstein using very different techniques). It settled the question about two generators for the ideals of lines in space in the affirmative. The question about lines in three space representing an axis is still unsettled over an algebraically closed field, but there are examples of lines in real three space with knots, which cannot represent an axis over reals, since the fundamental group of their complement would be nontrivial.

(5) **The Jacobian Problem:** Abhyankar talked extensively about the two variable Jacobian Problem, namely "given $F, G \in k[X,Y]$ such that their Jacobian $J_{(X,Y)}(F,G)$ is a nonzero constant, is it true that $k[F,G] = k[X,Y]$; i.e., do they form a matched pair of variables?" While this is easily seen to be false in positive characteristic, the problem is still very open in characteristic zero. It also has the notorious distinction of having multiple incorrect published proofs. Abhyankar was thrust into thinking about this problem, partly because two of his own students made unsuccessful attempts and he faced the task of finding holes in their proofs. He made several private calculations, but only the T.I.F.R. lecture notes contain the details of his results, mostly using the Abhyankar-Moh theory developed for the epimorphism theorem.

After the 1970s he did not actively pursue the problem until in the 1990s he yielded to numerous requests to publish his unpublished notes and results. He published a series of papers: [26], [23], [22], [20], [15], [14], [13], [12], [11], [10], [9].

For more details see [Details, 2(d), 3].

During the 1980s and 1990s Abhyankar was occupied with other fields of algebraic geometry, including computational geometry, combinatorics, group theory, etc. In the 1990s he was getting all fired up to devote his full energy and immense concentration to the Jacobian Problem; however, he got diverted into the problem of "dicritical divisors," which appeared to be potentially useful for the Jacobian Problem. We describe this next.

Abhyankar was introduced to the theory of dicritical divisors by I. Luengo and E. Artal Bartolo. Originally, Abhyankar was a reluctant listener, complaining about fancy and unclear terminology. But patience on all sides paid off, and Abhyankar managed to not only understand the subject but to make it completely algebraic and transparent. He, Luengo, Artal, and Heinzer managed to develop an extensive new theory. Abhyankar would sometimes joke that the fancy terminology was like sand introduced into a pearl oyster: it irritates but ends up producing a pearl!

Given a regular local ring R of dimension 2, consider a nonconstant z in its quotient field. In Abhyankar's formulation, the dicritical divisors of z relative to R are simply all the divisors V of R for which z is residually transcendental with respect to R.

With Abhyankar's fundamental knowledge of quadratic transformation and mastery of the technique of inductive proofs, he started formulating and answering questions about the nature of dicritical divisors, the structure of the tree formed by them, and the fundamental problem of how to construct a suitable z with a preassigned tree (with multiplicities) of dicritical divisors. He reworked and enhanced the theory of complete ideals pioneered by Zariski and even generalized it to higher dimensions. Unlike other treatments, his theory worked well even in mixed characteristic, which he considered a test for a good theorem. He produced a series of papers, some by him and some with Heinzer, Luengo, and Artal. Towards the end he was in the process of building concrete formulas for construction of the so-called Zariski ideals, i.e., ideals in the base R with the given dicritical tree. See the section on dicritical divisors in the references.

Abhyankar was clearly in his element while working on the dicritical divisors. He used to see the truth of the theorems way before writing down their formulation and used to say that, like well-formed poetry, these theorems are not only true but beautiful. He had an ulterior motivate for working on dicritical divisors; namely, he saw a potentially new attack on the Jacobian Problem through this new theory. A hint of his thoughts is in [9].

Alas, that challenge is now left for those who learned from him!

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Dinesh Thakur

The first real research mathematician that I came across was Professor Shreeram Abhyankar back in 1976, and then under quite peculiar circumstances.

As an overreaction to Abhyankar's verbal arguments—vigorous and probably even provocative, as per his patent style—with other professors in the University of Pune, the vice chancellor had banned Abhyankar from the premises of the university. My uncle, P. L. Deshpande, who was a famous author, wrote a fantastic parody of these ridiculous circumstances in the literary supplement of the Sunday newspaper in Marathi (the mother tongue both Abhyankar and I shared). They did not know each other back then, and the column did not mention any names; however, after reading this, Abhyankar was amused, and together with his student Shashikant Mulay came to my uncle's house to chat with him. Fortunately, I was there, just a schoolboy at that time, but already quite interested in mathematics.

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Provoked by these circumstances and also recognizing the need for a new, independent research institute in India, Abhyankar then founded "Bhaskaracharya Pratishthan." In May 1977 I attended a summer school under the auspices of this new institute, and I remember being quite impressed by Abhyankar's ability to penetrate to the heart of the matter directly.

Later, during my graduate years, whenever he came to give a seminar he would take me for dinner and we would talk for hours. He loved to gather an audience, tell anecdotes, and whenever possible, stir up a heated debate. Even for mathematical questions, he seldom gave straightforward answers. Instead, he preferred long rambling discourses, often unfocused and tangential, where he elaborated on related things and mixing mathematics with other things. He would not necessarily follow a logical or linear path, but believed in a "repeat, meditate, and meaning will be revealed to you" philosophy of old Indian masters speaking in sutra/mantra. But these "mathematical ramblings" were full of interesting insights and spontaneous, original viewpoints. In his book, *Lectures on Algebra*, one single lecture is four hundred pages! For him, research developed through discussion; he always preferred to visit, call up, and discuss mathematics in person.

He genuinely enjoyed arguments and was proud that he could argue from any stance. Often he would quickly size up the viewpoint of the listener and then take the opposite side. It was a cultivated, competitive sport for him, and he was ready to fight the battle from both sides! Though he often went to extremes, he would often be speaking "deep truths" (it is the hallmark of any deep truth that its negation is also a deep truth: Bohr).

Once he started working on something, he became energized and completely devoted himself to his work. I remember the day before his son's wedding he became so engrossed in our mathematical discussions at his house that he totally ignored the wedding party and even got angry when the relatives tried to involve him. Finally, to avoid trouble in his household, I had to escape under some pretense!

Fortunately he was able to lead a very productive life right to the end. His dedication and energy in his research and teaching will always be an inspiration to many of us.