Successive Differentiation

Find a particular solution to $\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2+2)y = 0$

for $y(0) = 0$, $y'(0) = 1$.

In the previous method we had

$$y = C_0\left(1 - x^2 + \tfrac{1}{4}x^4 + \cdots\right) + C_1\left(x - \tfrac{1}{2}x^3 + \tfrac{3}{40}x^5 + \cdots\right)$$

if we use $y(0) = 0 \Rightarrow$ when $x=0$, $y=0$ we get

$C_0 = 0$.

then $y' = C_1\left(1 - \frac{3}{2}x^2 + \frac{3}{8}x^4 + \cdots\right)$

let $x=0$ & $y'=1 \Rightarrow \underline{1 = C_1}$

So $\boxed{y = x - \frac{1}{2}x^3 + \frac{3}{40}x^5 + \cdots}$ for these initial conditions.

Now for another technique.

We have $\dfrac{d^2y}{dx^2} + x\dfrac{dy}{dx} + (x^2+2)y = 0$

and $\underline{\underline{y(0) = 0}} \Rightarrow$ when $x = 0,\ y = 0$

$\underline{\underline{y'(0) = 1}} \Rightarrow$ when $x = 0,\ \dfrac{dy}{dx} = 1$

Subbing in we find

$$\frac{d^2y}{dx^2} + (0)(1) + (0^2+2)0 = 0$$

or $\underline{\underline{y''(0) = 0}}$.

now rewrite original problem $y'' + xy' + (x^2+2)y = 0$

and differentiate

$$y''' + (xy'' + y') + (x^2+2)y' + 2xy = 0$$

Then using what we know

$$y'''(0) + \big((0)(0) + 1\big) + (0^2+2)(1) + 2(0)(0) = 0$$

$$y'''(0) + 1 + 2 = 0$$

$$y'''(0) = -3$$

diff. again, we have

$$y''' + xy'' + y' + (x^2+2)y' + 2xy = 0$$

$$y^4 + (xy''' + y'') + y'' + (x^2+2)y'' + 2xy' + 2xy' + 2y = 0$$

$$y^4(0) + \cancel{((0)(-3) + 0)} + \cancel{(0)} + \cancel{(0^2+2)(0)} + \cancel{2(0)(1)} + \cancel{2(0)(1)} + \cancel{2(0)} = 0$$

$$\underline{y^4(0) = 0}$$

So we have $y^4 + xy''' + 2y'' + (x^2+2)y'' + 4xy' + 2y = 0$

diff. again

$$y^5 + xy^4 + \underline{y'''} + \underline{2y'''} + (x^2+2)y''' + 2xy'' + 4xy' + 4y' + 2y' = 0$$

$$y^5(0) + \cancel{(0)(0)} + 3(-3) + (0^2+2)(-3) + 2(0)(0) + \cancel{4(0)(+1)}$$
$$+ 4(1) + 2(1) = 0$$

$$y^5(0) - 9 - 6 + 6 = 0 \qquad y^5(0) = 9$$

now remember,

our solution will be in the form $y = \sum_{n=0}^{\infty} C_n (x-x_0)^n$ OR

since we have $x_0 = 0$ here, $y = \sum_{n=0}^{\infty} C_n x^n$ Then

$$y = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + C_4 x^4 + \cdots$$

where $C_i = \frac{f^i(0)}{i!}$ Then

$$y = 0 + 1x + \frac{0}{2!}x^2 + \frac{-3}{3!}x^3 + \frac{0}{4!}x^4 + \frac{9}{5!}x^5 + \cdots$$

$$\underline{y = x - \frac{1}{2}x^3 + \frac{3}{40}x^5 + \cdots}$$ as before!

Homework: $y'' - (x+1)y' + x^2 y = x$

$y(0) = 1$ & $y'(0) = 1$.