7.3 Properties

Translations in s (1st shifting THM)

If $\mathcal{L}\{f(t)\} = F(s)$ exists for $s > \alpha$

Then $\mathcal{L}\{e^{at} f(t)\} = F(s-a)$ if $s > \alpha + a$

Pf: $\mathcal{L}\{e^{at} f(t)\} = \int_0^\infty e^{-st} f(t) e^{at}\, dt$

$$= \int_0^\infty e^{-(s-a)t} f(t)\, dt = F(s-a)$$

where $F(s) = \int_0^\infty e^{-st} f(t)\, dt.$

here just replace (shift) s with $s-a$.

ex.) $\mathcal{L}\{e^{at} \sin(bt)\} = \dfrac{b}{(s-a)^2 + b^2}$

Recall $\mathcal{L}\{\sin(bt)\} = \dfrac{b}{s^2 + b^2}$

LT of the Derivative * (Very important)

$$\mathcal{L}\{f'(t)\} = s\,\mathcal{L}\{f(t)\} - f(0).$$

Pf: $$\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt$$

$$= \lim_{A\to\infty} \int_0^A e^{-st} f'(t)\,dt$$

Now consider $\int e^{-st} f'(t)\,dt$ IBP

let $u = e^{-st}$ $\quad dv = f'(t)\,dt$

$du = -s\,e^{-st}\,dt$ $\quad v = f(t)\,dt$

$$\int e^{-st} f'(t)\,dt = e^{-st} f(t) + s\int e^{-st} f(t)\,dt$$

So then

$$\mathcal{L}\{f'(t)\} = \lim_{A\to\infty}\left[e^{-st} f(t)\Big|_0^A \right] + s\int_0^\infty e^{-st} f(t)\,dt$$

$$= \lim_{A\to\infty}\left[e^{-st} f(t)\Big|_0^A \right] + s\int_0^\infty e^{-st} f(t)\,dt$$

$$= \lim_{A\to\infty}\left[e^{-sA} f(A) - f(0)\right] + s\,\mathcal{L}\{f(t)\}$$

Now what about $\lim_{A\to\infty} e^{-sA} f(A)$?

Recall $f(t)$ is of exponential order $\Longrightarrow$

there is an M such that eventually

$$\left| e^{-sA} f(A)\right| \le e^{-sA}\left(Me^{\alpha T}\right) = Me^{-(s-\alpha)T}$$

So for $s > \alpha$

$$0 \le \lim_{A\to\infty}\left| e^{-sA} f(A)\right| \le \lim_{A\to\infty} \underbrace{Me^{-(s-\alpha)T}}_{0} = 0$$

This ~~tells~~ us via the Squeeze Thm

That $\lim_{A\to\infty} e^{-sA} f(A) = 0$

So finally

$$\boxed{\mathcal{L}\{f'(t)\} = s\,\mathcal{L}\{f(t)\} - f(0)} \ast$$

To get $\mathcal{L}\{f''(t)\}$ or higher just apply the above.

ex.) $\mathcal{L}\{f''(t)\} = s\,\mathcal{L}\{f'(t)\} - f'(0)$ Then

$$s\left[s\,\mathcal{L}\{f(t)\} - f(0)\right] - f'(0)$$

$$\mathcal{L}\{f''(t)\} = s^2\,\mathcal{L}\{f(t)\} - s f(0) - f'(0)$$

Now look at things from another angle.

Suppose we have $\mathcal{L}\{f(t)\} = F(s)$

we have just seen $\mathcal{L}\{f'(t)\} = s\,\mathcal{L}\{f(t)\} - f(0)$

what if $\mathcal{L}\{f(t)\} = F(s)$ and we find $\frac{dF(s)}{ds}$

is there some expression in t, $G(t)$ st

$$\mathcal{L}\{G(t)\} = \frac{dF(s)}{ds}$$

, This question gives us this:

let $\mathcal{L}\{f(t)\} = F(s)$, $f(t)$ pw cont. on $[0,\infty)$ and of exponential order α Then

$$(-1)^n \frac{d^n F(s)}{ds^n} = \mathcal{L}\{t^n f(t)\}.$$