6/28

#33

pg 395

$y(0) = y'(0) = 1$

$y'' + 2y' + 2y = u(t-2\pi) - u(t-4\pi)$

$\left(s^2 Y(s) - s\,\overset{1}{y(0)} - \overset{1}{y'(0)}\right) + 2\left(sY(s) - \overset{1}{y(0)}\right) + 2Y(s) = \dfrac{e^{-2\pi s}}{s} - \dfrac{e^{-4\pi s}}{s}$

$(s^2 + 2s + 2)Y(s) - s - 3 = \dfrac{e^{-2\pi s} - e^{-4\pi s}}{s}$

$(s^2 + 2s + 2)Y(s) = \dfrac{e^{-2\pi s} - e^{-4\pi s}}{s} + s + 3$

stepped

Normal

$Y(s) = \dfrac{e^{-2\pi s} - e^{-4\pi s}}{s\left((s+1)^2+1\right)} + \dfrac{s+3}{(s+1)^2+1}$ ☺

$\dfrac{1}{s\left((s+1)^2+1\right)} = \dfrac{\overset{1/2}{A}}{s} + \dfrac{\overset{-1/2}{B}(s+1)}{(s+1)^2+1} + \dfrac{1 \cdot \overset{-1/2}{C}}{(s+1)^2+1}$

Let $s = -1$ $\quad -1 = -\frac{1}{2} + \frac{C}{1} \Rightarrow -\frac{1}{2} = \frac{C}{1} \Rightarrow C = -1/2$

Let $s = 1$ $\quad \frac{1}{5} = \frac{1}{2} + \frac{2B}{5} - \frac{1/2}{5}$

$$\frac{1}{5} = \frac{1}{2} + \frac{2B}{5} - \frac{1}{10}$$

$$2 = 5 + 4B - 1$$

$$2 = 4 + 4B$$

$$-2 = 4B \implies B = -\frac{1}{2}$$

$$* \quad \underbrace{\frac{1}{s\left((s+1)^2+1\right)}}_{\text{Stepped}} = \frac{1}{2}\cdot\frac{1}{s} - \frac{1}{2}\frac{(s+1)}{(s+1)^2+1} - \frac{1}{2}\frac{1}{(s+1)^2+1}$$

Normal Part:

$$\frac{\overset{(s+1)+2}{s+3}}{(s+1)^2+1} = \frac{s+1}{(s+1)^2+1} + 2\frac{1}{(s+1)^2+1}$$

$$y(t) = \underbrace{e^{-t}\cos t + 2e^{-t}\sin t}_{\text{Normal}}$$

Stepped $\mathcal{L}^{-1} \implies \underbrace{\frac{1}{2} - \frac{1}{2}e^{-t}\cos t - \frac{1}{2}e^{-t}\sin t}_{\text{But This is stepped (twice)}}$

6/28

$$y(t) = e^{-t}\cos t + 2e^{-t}\sin t + \left(\tfrac{1}{2} - \tfrac{1}{2}e^{-(t-2\pi)}\cos(t-2\pi) - \tfrac{1}{2}e^{-(t-2\pi)}\sin(t-2\pi)\right)u(t-2\pi)$$
$$- \left(\tfrac{1}{2} - \tfrac{1}{2}e^{-(t-4\pi)}\cos(t-4\pi) - \tfrac{1}{2}e^{-(t-4\pi)}\sin(t-4\pi)\right)u(t-4\pi)$$

But $\sin t$ & $\cos t$ are periodic, period 2π

So $\sin(t-2\pi) = \sin(t-4\pi) = \sin t$

$\cos(t-2\pi) = \cos(t-4\pi) = \cos t$

So finally

$$y = e^{-t}\cos t + 2e^{-t}\sin t + \left(\tfrac{1}{2} - \tfrac{1}{2}e^{-(t-2\pi)}\cos t - \tfrac{1}{2}e^{-(t-2\pi)}\sin t\right)u(t-2\pi)$$
$$- \left(\tfrac{1}{2} - \tfrac{1}{2}e^{-(t-4\pi)}\cos t - \tfrac{1}{2}e^{-(t-4\pi)}\sin t\right)u(t-4\pi).$$

6/28

#34

pg 395

$y(0) = y'(0) = 0$

$$y'' + 4y' + 4y = \underline{u(t-\pi)} - u(t-2\pi)$$

$$\left(s^2 Y(s) - s\overset{\phi}{y(0)} - \overset{\phi}{y'(0)}\right) + 4\left(sY(s) - \overset{\phi}{y(0)}\right) + 4Y(s) = \frac{e^{-\pi s}}{s} - \frac{e^{-2\pi s}}{s}$$

$$\left(s^2 + 4s + 4\right)Y(s) = \frac{e^{-\pi s} - e^{-2\pi s}}{s}$$

$$Y(s) = \frac{e^{-\pi s} - e^{-2\pi s}}{s(s+2)^2}$$

$$\frac{1}{s(s+2)^2} = \frac{\overset{1/4}{A}}{s} + \frac{\overset{-1/4}{B}}{s+2} + \frac{\overset{-1/2}{C}}{(s+2)^2}$$

let $s = -1$

$$-1 = -\frac{1}{4} + B - \frac{1}{2}$$

$$-4 = -1 + 4B - 2$$

$$-1 = 4B \quad \Rightarrow \quad B = -\frac{1}{4}$$

So if we find the $\mathcal{L}^{-1}$ of this result we get

$$\frac{1}{4} - \frac{1}{4} e^{-2t} - \frac{1}{2} e^{-2t} t$$

Now remember we have two steps π & 2π

$$Y(t) = \left(\frac{1}{4} - \frac{1}{4} e^{-2(t-\pi)} - \frac{1}{2} e^{-2(t-\pi)} (t-\pi)\right) u(t-\pi) -$$

$$\left(\frac{1}{4} - \frac{1}{4} e^{-2(t-2\pi)} - \frac{1}{2} e^{-2(t-2\pi)}\right) u(t-2\pi).$$

$$\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2+2)y = 0$$

we assume the solution is $y = \sum_{n=0}^{\infty} C_n x^n = C_0 + C_1 x + C_2 x^2 + \ldots$

Then $y' = \sum_{n=1}^{\infty} n C_n x^{n-1}$ and $y'' = \sum_{n=2}^{\infty} n(n-1) C_n x^{n-2}$

This gives

$$\sum_{n=2}^{\infty} n(n-1) C_n x^{n-2} + x\sum_{n=1}^{\infty} n C_n x^{n-1} + (x^2+2)\sum_{n=0}^{\infty} C_n x^n = 0$$

We now wish for all the exponents of x to be "n" Then we will change the value of the index of the sum to be the same value.

$$\underbrace{\sum_{n=2}^{\infty} n(n-1) C_n x^{n-2}}_{1} + \underbrace{\sum_{n=1}^{\infty} n C_n x^{n}}_{2} + \underbrace{\sum_{n=0}^{\infty} C_n x^{n+2}}_{3} + \underbrace{2\sum_{n=0}^{\infty} C_n x^{n}}_{4} = 0$$

terms 2 & 4 are all right (exponents are "n")

6/28

So look at $\displaystyle\sum_{n=2}^{\infty} n(n-1) C_n x^{n-2}$, just replace n with $n+2$ everywhere

$$\sum_{n+2=2}^{\infty} (n+2)(n+1) C_{n+2} x^n = \sum_{n=0}^{\infty} (n+2)(n+1) C_{n+2} x^n \quad (1)$$

Then for (3)

$\displaystyle\sum_{n=0}^{\infty} C_n x^{n+2}$ replace n with $n-2$ everywhere

$$\sum_{n-2=0}^{\infty} C_{n-2} x^n = \sum_{n=2}^{\infty} C_{n-2} x^n$$

subbing these back in we have

$$\sum_{n=0}^{\infty} (n+2)(n+1) C_{n+2} x^n + \sum_{n=1}^{\infty} n C_n x^n + \sum_{n=2}^{\infty} C_{n-2} x^n + 2\sum_{n=0}^{\infty} C_n x^n = 0$$

*

Now pull off terms so each sum begins at $n=2$.

$$\sum_{n=0}^{\infty} (n+2)(n+1) C_{n+2} x^n = 2C_2 + 6C_3 x + \sum_{n=2}^{\infty} (n+2)(n+1) C_{n+2} x^n$$

$$\sum_{n=1}^{\infty} n C_n x^n = C_1 x + \sum_{n=2}^{\infty} n C_n x^n$$

and

$$2\sum_{n=0}^{\infty} C_n x^n = 2\left(C_0 + C_1 x + \sum_{n=2}^{\infty} C_n x^n\right)$$

rewriting gives

$$2C_2 + 6C_3 x + \sum_{n=2}^{\infty}(n+2)(n+1)C_{n+2}x^n + C_1 x + \sum_{n=2}^{\infty} n C_n x^n$$

$$+ \sum_{n=2}^{\infty} C_{n-2} x^n + 2C_0 + 2C_1 x + 2\sum_{n=2}^{\infty} C_n x^n = 0$$

collect terms and now write as one sum.

$$2C_2 + 2C_0 + (6C_3 + 3C_1)x + \sum_{n=2}^{\infty}\left((n+2)(n+1)C_{n+2} + nC_n + C_{n-2} + 2C_n\right)x^n = 0$$

$\therefore$ $\underline{\underline{C_2 = -C_0}}$ and $C_3 = -\frac{1}{2}C_1$ and

$$C_{n+2} = \frac{-nC_n - C_{n-2} - 2C_n}{(n+2)(n+1)}, \quad n \geq 2$$

6/28

we can use
$$C_{n+2} = \frac{-nC_n - C_{n-2} - 2C_n}{(n+2)(n+1)}, \quad n \geq 2$$

to generate additional constants.

Let $n=2$
$$C_4 = \frac{-2C_2 - C_0 - 2C_2}{12} = \frac{2C_0 - C_0 + 2C_0}{12}$$

$$\therefore \boxed{C_4 = \frac{1}{4}C_0}$$

Let $n=3$
$$C_5 = \frac{-3C_3 - C_1 - 2C_3}{20} = \frac{\frac{3}{2}C_1 - C_1 + C_1}{20}$$

$$C_5 = \frac{3}{40}C_1$$

So lets see what we have

remember our solution is of the form

$$y = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + C_4 x^4 + C_5 x^5 + \ldots$$

then

$$y = C_0 + C_1 x - C_0 x^2 - \frac{1}{2}C_1 x^3 + \frac{1}{4}C_0 x^4 + \frac{3}{40}C_1 x^5 + \ldots$$

OR

$$y = C_0\left(1 - x^2 + \frac{1}{4}x^4 + \ldots\right) + C_1\left(x - \frac{1}{2}x^3 + \frac{3}{40}x^5 + \ldots\right)$$