4.4 6/19 1

#14 $2z'' + z = 9e^{2t}$

Sp: $y(0) = 1$, $y'(0) = 7$

SoGe $2z'' + z = 0$

Aux: $2r^2 + 1 = 0$

$r^2 = -\frac{1}{2}$

$r = \pm i\sqrt{\frac{1}{2}} = i\frac{1}{\sqrt{2}}$

$r = 0 \pm i\frac{\sqrt{2}}{2}$

$\left\{ \cos\left(\frac{\sqrt{2}}{2}t\right), \sin\left(\frac{\sqrt{2}}{2}t\right) \right\}$

* $e^{\alpha t}\cos(\beta t)$, $e^{\alpha t}\sin(\beta t)$

$Y_p = Ae^{2t}$

$Y_p' = 2Ae^{2t}$

$Y_p'' = 4Ae^{2t}$

$2(4Ae^{2t}) + Ae^{2t} = 9e^{2t}$

$9Ae^{2t} = 9e^{2t}$

$9A = 9$

$A = 1$

$Y = C_1\cos\left(\frac{\sqrt{2}}{2}t\right) + C_2\sin\left(\frac{\sqrt{2}}{2}t\right) + e^{2t}$

#13

$$y'' - y' + 9y = 3\sin(3t)$$

Aux: $r^2 - r + 9 = 0$

$$r = \frac{1 \pm \sqrt{1-36}}{2} \Rightarrow \left\{ e^{\frac{t}{2}}\cos\left(\frac{\sqrt{35}}{2}t\right),\ e^{\frac{t}{2}}\sin\left(\frac{\sqrt{35}}{2}t\right) \right\}$$

$$r = \frac{1 \pm i\sqrt{35}}{2}$$

$$r = \frac{1}{2} \pm i\frac{\sqrt{35}}{2}$$

Solution

$$\boxed{y = C_1 e^{\frac{t}{2}}\cos\left(\frac{\sqrt{35}}{2}t\right) + C_2 e^{\frac{t}{2}}\sin\left(\frac{\sqrt{35}}{2}t\right) + \cos(3t)}$$

$$y_p = A\sin(3t) + B\cos(3t)$$

$$y_p' = 3A\cos(3t) - 3B\sin(3t)$$

$$y_p'' = -9A\sin(3t) - 9B\cos(3t)$$

$$\left(-9A\sin(3t) - 9B\cos(3t)\right) - \left(3A\cos(3t) - 3B\sin(3t)\right)$$

$$+ 9\left(A\sin(3t) + B\cos(3t)\right) = 3\sin(3t)$$

$$\cancel{-9A\sin(3t)} + 3B\sin(3t) + \cancel{9A\sin 3t} = 3\sin(3t)$$

$$\boxed{3B = 3} \Rightarrow \boxed{B = 1}$$

$$\cancel{-9B\cos(3t)} - 3A\cos(3t) + \cancel{9B\cos(3t)} = 0$$

$$-3A = 0$$

$$A = 0$$

#22 $y'' - 2y' + y = 24t^2e^t$

Aux: $r^2 - 2r + 1 = 0$

$(r-1)^2 = 0$

$r=1,\ r=1$

$\{e^t, te^t\}$

$$y_p = t^2(At^2 + Bt + C)e^t = (At^4 + Bt^3 + Ct^2)e^t$$

$$\Longrightarrow \quad y_p' = (At^4 + Bt^3 + Ct^2)e^t + (4At^3 + 3Bt^2 + 2Ct)e^t$$

$$y_p' = (At^4 + (4A+B)t^3 + (3B+C)t^2 + 2Ct)e^t \quad \text{Then}$$

$$y_p'' = (At^4 + (4A+B)t^3 + (3B+C)t^2 + 2Ct)e^t + (4At^3 + 3(4A+B)t^2 + 2(3B+C)t + 2C)e^t$$

$$y_p'' = (At^4 + (8A+B)t^3 + (12A+6B+C)t^2 + (6B+4C)t + 2C)e^t$$

Sub into the DE

$$e^t\left\{(At^4 + (8A+B)t^3 + (12A+6B+C)t^2 + (6B+4C)t + 2C) - 2(At^4 + (4A+B)t^3 + (3B+C)t^2 + 2Ct) + (At^4 + Bt^3 + Ct^2)\right\} = 24t^2e^t$$

equating coefficients gives

For t^4: $A - 2A + A$ cancel

t^3: $8A + B - 8A - 2B + B$ all cancel

t^2: $12A + 6B + C - 6B - 2C + C$ gives

$12At^2e^t = 24t^2e^t \Rightarrow 12A = 24$

$A = 2$

For t: $6B + 4C - 4C = 0$

$6B = 0 \Rightarrow \boxed{B=0}$

For constant term $2C = 0 \Rightarrow \boxed{C=0}$

#22 Continued

we then have the particular solution $y_p = 2t^4 e^t$

Finally the general solution is $y = C_1 e^t + C_2 t e^t + 2t^4 e^t$.

Pg 193

#2 $y'' + 4y = \underline{\tan(2t)}$

$\int \sec u \, du = \ln|\sec u + \tan u|$

Aux: $r^2 + 4 = 0$

$r = \pm 2i$

$\{ \overset{y_1}{\cos(2t)}, \overset{y_2}{\sin(2t)} \}$

$y_h = C_1 \cos(2t) + C_2 \sin(2t)$

$$W[\cos(2t), \sin(2t)] = \begin{vmatrix} \cos(2t) & \sin(2t) \\ -2\sin(2t) & 2\cos(2t) \end{vmatrix}$$

$$= 2\cos^2(2t) + 2\sin^2(2t) = 2$$

$$V_1 = -\frac{1}{2}\int \sin(2t)\tan(2t)\,dt$$

$$V_1 = -\frac{1}{2}\int \frac{\sin^2(2t)}{\cos(2t)}\,dt$$

$$= -\frac{1}{2}\int \frac{1-\cos^2(2t)}{\cos(2t)}\,dt$$

$$= -\frac{1}{2}\left[\int \sec(2t)\,dt - \int \cos(2t)\,dt\right]$$

$$= -\frac{1}{2}\left[\frac{1}{2}\ln|\sec(2t) + \tan(2t)| - \frac{1}{2}\sin(2t)\right]$$

$$V_2 = \frac{1}{2}\int \cos(2t)\tan(2t)\,dt$$

$$V_2 = \frac{1}{2}\int \sin(2t)\,dt$$

$$V_2 = -\frac{1}{4}\cos(2t)$$

$$Y = Y_h + Y_p$$

$$Y_p = \left(-\frac{1}{4}\ln\left|\sec(2t) + \tan(2t)\right| + \frac{1}{4}\sin(2t)\right)\cos(2t)$$

$$- \frac{1}{4}\cos(2t)\sin(2t)$$

$$+ C_1\cos(2t) + C_2\sin(2t)$$

Cauchy-Euler

$$a t^2 y'' + b t y' + c y = \overset{0}{\cancel{g(t)}}$$

$$y = t^r$$

$$y' = r t^{r-1}$$

$$y'' = r(r-1) t^{r-2}$$

$$a t^2 \left[(r^2 - r) t^{r-2}\right] + b t \left[r t^{r-1}\right] + c t^r = \overset{0}{\cancel{g(t)}}$$

$$(ar^2 - ar) t^r + br\, t^r + c t^r = 0$$

$$t^r \left[ar^2 + (b-a) r + c\right] = 0$$

Aux: $ar^2 + (b-a) r + c = 0$

$\cancel{x}$

e^{rx} t^r

$(e^x)^r$ t^r

$e^x = t$ $x = \ln t$

i.) $r_1, r_2 \in \mathbb{R}$ $\{t^{r_1}, t^{r_2}\}$

ii.) r_1, r_1 $\{t^{r_1}, t^{r_1} \ln t\}$

iii.) $r = \alpha \pm i\beta$ $\{t^{\alpha} \cos(\beta \ln t),\ t^{\alpha} \sin(\beta \ln t)\}$

finally $y_p = v_1 y_1 + v_2 y_2$ where

$$v_1 = -\int \frac{y_2(t)\, g(t)}{W[y_1(t), y_2(t)]}\, dt \quad \&$$

$$v_2 = \int \frac{y_1(t)\, g(t)}{W[y_1(t), y_2(t)]}\, dt .$$

given $\{ y_1(t), y_2(t) \}$ form a fund. soln. set to the homogeneous problem.

Here is another Variation of Parameter Problem

Pg 193

#4 $y'' - 2y' + y = t^{-1}e^{t}$

Aux: $r^2 - 2r + 1 = (r-1)^2 \Rightarrow r = 1, 1$

$\{\overset{y_1}{e^{t}}, \overset{y_2}{te^{t}}\}$ Then

$$w[e^{t}, te^{t}] = \begin{vmatrix} e^{t} & te^{t} \\ e^{t} & (t+1)e^{t} \end{vmatrix} = e^{2t}(t+1) - te^{2t} = \underline{\underline{e^{2t}}}$$

So we have $V_1 = -\int \frac{(te^{t})\left(\frac{e^{t}}{t}\right)}{e^{2t}}\,dt = -\int \frac{e^{2t}}{e^{2t}}\,dt = -\int dt = -t$

$$V_2 = \int \frac{(e^{t})\left(\frac{e^{t}}{t}\right)}{e^{2t}}\,dt = \int \frac{1}{t}\,dt = \ln|t|$$

Then $y_p = -te^{t} + te^{t}\ln|t|$ and the general solution

$$y = c_1e^{t} + c_2te^{t} - te^{t} + te^{t}\ln t$$

$$y = c_1e^{t} + (c_2 - 1)te^{t} + te^{t}\ln t = c_1e^{t} + c_3te^{t} + te^{t}\ln t, \quad \underline{c_3 = c_2 - 1}$$

Extra Cauchy - Euler example

Pg 200

#10 $t^2 y'' + 7t y' - 7y = 0$

here $a=1$ $b=7$ $c=-7$ $\Rightarrow$ $r^2 + (7-1)r - 7 = 0$

$r^2 + 6r - 7 = 0$

$(r+7)(r-1) = 0 \Rightarrow r = -7, 1$

$\therefore$ $y = C_1 t^{-7} + C_2 t$

#12 $t^2 \dfrac{d^2 z}{dt^2} + 5t \dfrac{dz}{dt} + 4z = 0$ $\Rightarrow$ $r^2 + (5-1)r + 4 = 0$

$r^2 + 4r + 4 = 0$

$a=1$ $b=5$ $c=4$

$(r+2)^2 = 0$

$\therefore$ $y = c_1 t^{-2} + C_2 t^{-2} \ln t$.

#14 $t^2 y'' - 3t y' + 4y = 0$ $\Rightarrow$ $r^2 + (-3-1)r + 4 = 0$

$r^2 - 4r + 4 = 0$

$a=1$ $b=-3$ $c=4$

$(r-2)^2 = 0$ $r = 2, 2$

$y = C_1 t^2 + C_2 t^2 \ln t$