DE with periodic forcing function.

Solve: $y'' + \pi^2 y = Q(t)$, $y(0) = y'(0) = 0$, and

$$Q(t) = \begin{cases} 1, & 0 < t < 1 \\ 0, & 1 \le t < 2 \end{cases} \quad \text{with period } 2.$$

The LT of the left side is

$$\left(s^2 Y(s) - s y(0) - y'(0)\right) + \pi^2 Y(s) = Y(s)\left(s^2 + \pi^2\right)$$

Now write $Q(t)$ in terms of unit step functions.

$$Q(t) = 1\left[u(t) - u(t-1)\right] + 0\left[u(t-1) - u(t-2)\right]$$

$$Q(t) = u(t) - u(t-1) \quad \text{with period } 2.$$

OR $Q_2(t) = u(t) - u(t-1)$ for 1 period

Then the transform for 1 period

$$* \quad \mathcal{L}\{Q_2(t)\} = \frac{1}{s} - \frac{e^{-s}}{s} = \frac{1}{s}\left(1 - e^{-s}\right)$$

By Thm 9, pg 417 $\quad \mathcal{L}\{Q(t)\} = \dfrac{\mathcal{L}\{Q_2(t)\}}{1-e^{-2s}}$

so $\mathcal{L}\{Q(t)\} = \dfrac{\frac{1}{s}\left(1-e^{-s}\right)}{1-e^{-2s}}$, so finally

$$Y(s)\left(s^2+\pi^2\right) = \frac{\frac{1}{s}\left(1-e^{-s}\right)}{1-e^{-2s}}$$

$$Y(s) = \frac{\frac{1}{s}\left(1-e^{-s}\right)}{\left(s^2+\pi^2\right)\left(1-e^{-2s}\right)}$$

The $1-e^{-2s}$ factor in the denominator tells us the solution will be periodic - of period $\underline{2}$.

Then consider the remainder of the expression.

$$\frac{1-e^{-s}}{s\left(s^2+\pi^2\right)} = \frac{1}{s\left(s^2+\pi^2\right)} - \frac{1}{s\left(s^2+\pi^2\right)}e^{-s}$$

So we need $\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+\pi^2)}\right\}$, we will do this two ways this time.

1st Partial fractions

$$\frac{1}{s(s^2+\pi^2)} = \frac{\overset{1/\pi^2}{A}}{s} + \frac{Bs+C\pi}{s^2+\pi^2}$$

let $s=1$

$$\left(\frac{1}{\pi^2+1} = \frac{1}{\pi^2} + \frac{B+C\pi}{\pi^2+1}\right)\pi^2(\pi^2+1)$$

$$\pi^2 = \pi^2+1 + \pi^2(B+C\pi)$$

$$\boxed{-\frac{1}{\pi^2} = B+C\pi}$$

$s=-1$

$$\left(-\frac{1}{\pi^2+1} = -\frac{1}{\pi^2} + \frac{-B+C\pi}{\pi^2+1}\right)\pi^2(\pi^2+1)$$

$$-\pi^2 = -\pi^2-1 + \pi^2(-B+C\pi)$$

$$\boxed{\frac{1}{\pi^2} = -B+C\pi}$$

Solving gives $C=0$ & $B=-\frac{1}{\pi^2}$

$$\frac{1}{s(s^2+\pi^2)} = \frac{1}{\pi^2}\cdot\frac{1}{s} - \frac{1}{\pi^2}\,\frac{s}{s^2+\pi^2}$$

So $$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+\pi^2)}\right\} = \frac{1}{\pi^2}\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \frac{1}{\pi^2}\mathcal{L}^{-1}\left\{\frac{s}{s^2+\pi^2}\right\}$$

$$= \frac{1}{\pi^2} - \frac{1}{\pi^2}\cos(\pi t)$$

2nd way (Slick!!)

Think of $$\frac{1}{s(s^2+\pi^2)} = \frac{1}{\pi}\,\frac{\left(\frac{\pi}{s^2+\pi^2}\right)}{s}$$

Then using #7 in the table

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+\pi^2)}\right\} = \frac{1}{\pi}\int_0^t \sin(\pi v)\,dv$$

$$= -\frac{1}{\pi^2}\cos(\pi v)\Big|_0^t$$

$$= -\frac{1}{\pi^2}\left[\cos(\pi t) - 1\right]$$

$$= \frac{1}{\pi^2}\left(1 - \cos(\pi t)\right)$$

Remember $Y(s) = \dfrac{1-e^{-s}}{s(s^2+\pi^2)(1-e^{-2s})}$

the $1-e^{-2s}$ just tells us the final answer will be periodic with period 2.

And we know $\mathcal{L}^{-1}\left\{\dfrac{1}{s(s^2+\pi^2)}\right\} = \dfrac{1}{\pi^2}\left(1-\cos(\pi t)\right)$

and $\dfrac{1-e^{-s}}{s(s^2+\pi^2)} = \dfrac{1}{s(s^2+\pi^2)} - \dfrac{1}{s(s^2+\pi^2)}e^{-s}$

Putting all this together

$$y(t) = \underline{\frac{1}{\pi^2}\left(1-\cos(\pi t)\right)} - \frac{1}{\pi^2}\left(1-\cos\pi(t-1)\right)\underline{u(t-1)}$$

$$\cos(\pi t-\pi) = -\cos\pi t$$

$$y(t) = \frac{1}{\pi^2}\left(1-\cos\pi t\right)\underline{u(t)} - \frac{1}{\pi^2}\left(1+\cos\pi t\right)u(t-1)$$

with period $\underline{2}$.