Given $a\frac{d^2y}{dt^2} + b\frac{dx}{dt} + cy = 0$ & Aux. eqn. $ar^2 + br + c = 0$

if roots are Complex, $r_1, r_2 \in \mathbb{C}$, $r_1 = \alpha + i\beta$ & $r_2 = \alpha - i\beta$ Then

$y_1 = e^{(\alpha+i\beta)t}$ & $y_2 = e^{(\alpha-i\beta)t}$. By Euler's Formula

$$y_1 = e^{(\alpha+i\beta)t} = e^{\alpha t} \cdot e^{i\beta t} = e^{\alpha t}\left(\cos(\beta t) + i\sin(\beta t)\right)$$

$$y_2 = e^{(\alpha-\beta i)t} = e^{\alpha t} \cdot e^{-i\beta t} = e^{\alpha t}\left(\cos(\beta t) - i\sin(\beta t)\right)$$

Now are y_1 & y_2 Linearly Independent?

Consider

$W(y_1, y_2)$?.

$$W(y_1,y_2) = \begin{vmatrix} e^{(\alpha+i\beta)t} & e^{(\alpha-i\beta)t} \\ (\alpha+i\beta)e^{(\alpha+i\beta)t} & (\alpha-i\beta)e^{(\alpha-i\beta)t} \end{vmatrix} =$$

$$(\alpha-i\beta)e^{(\alpha+i\beta)t+(\alpha-i\beta)t} - (\alpha+i\beta)e^{(\alpha+i\beta)t+(\alpha-i\beta)t}$$

$$e^{2\alpha t}\left((\alpha-i\beta)-(\alpha+i\beta)\right) = -2i\beta e^{2\alpha t} \neq 0$$

So $y_1 = e^{(\alpha+i\beta)t} = e^{\alpha t}\left(\cos(\beta t) + i\sin(\beta t)\right)$ and

$y_2 = e^{(\alpha-i\beta)t} = e^{\alpha t}\left(\cos(\beta t) - i\sin(\beta t)\right.$ are L.I.!

So $y_1 = e^{(\alpha+i\beta)t} = e^{\alpha t}\left(\cos(\beta t) + i\sin(\beta t)\right)$ and

$y_2 = e^{(\alpha-i\beta)t} = e^{\alpha t}\left(\cos(\beta t) - i\sin(\beta t)\right)$ are L.I.

Then one can show that $y_1 + y_2$ & $y_1 - y_2$ are also L.I.:

$y_1 + y_2 = 2e^{\alpha t}\cos(\beta t)$ & $y_1 - y_2 = 2i\,e^{\alpha t}\sin(\beta t)$

let $Y_1 = 2e^{\alpha t}\cos(\beta t)$ AND $Y_2 = 2i\,e^{\alpha t}\sin(\beta t)$

if we let $C_1 = 2$ & $C_2 = 2i$ Then we have

$Y_1 = C_1 e^{\alpha t}\cos(\beta t)$ $\quad Y_2 = C_2 e^{\alpha t}\sin(\beta t)$

and so we have

If we have $a\dfrac{d^2y}{dt^2} + b\dfrac{dy}{dt} + cy = 0$, with Aux. eqn $ar^2 + br + c = 0$ with roots $r_1, r_2 \in \mathbb{C}$, $r_1 = \alpha + \beta i$ & $r_2 = \alpha - \beta i$ Then the fundamental solution set may be written as

$\{e^{\alpha t}\cos(\beta t),\ e^{\alpha t}\sin(\beta t)\}$ with general solution

$$y = C_1 e^{\alpha t}\cos(\beta t) + C_2 e^{\alpha t}\sin(\beta t)$$