Cauchy-Euler

$$a_n t^n \frac{d^n y}{dt^n} + a_{n-1} x^{n-1} \frac{d^{n-1} y}{dt^{n-1}} + \cdots + a_1 t \frac{dy}{dt} + a_0 y = f(t)$$

Let $t = e^x$, This substitution reduces the above eqn. to a linear D.E. with constant coefficients.

Consider a 2^{nd} order : $a_2 t^2 \frac{d^2 y}{dt^2} + a t \frac{dy}{dt} + a_0 y = f(t)$

Let $t = e^x$ ($t > 0$ of course!)

$\ln t = x$

$$\boxed{\frac{1}{t} = \frac{dx}{dt}} \quad \text{and} \quad \boxed{\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \frac{1}{t}\frac{dy}{dx}} \quad *$$

The second derivative $\frac{d^2y}{dt^2}$ is pretty neat!

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left[\frac{dy}{dt}\right] = \frac{d}{dt}\left[\frac{1}{t}\frac{dy}{dx}\right] = \frac{1}{t}\frac{d}{dt}\left(\frac{dy}{dx}\right) + \frac{dy}{dx}\frac{d}{dt}\left(\frac{1}{t}\right)$$

in the first term rewrite $\frac{d}{dt} = \frac{d}{dx}\frac{dx}{dt}$ (chain rule)

$$\frac{d^2y}{dt^2} = \frac{1}{t}\overbrace{\frac{d}{dx}\left(\frac{dy}{dx}\right)\frac{dx}{dt}}^{\frac{d}{dt}} - \frac{dy}{dx}\left(\frac{1}{t^2}\right)$$

$$= \frac{1}{t}\left(\frac{d^2y}{dx^2}\frac{1}{t}\right) - \frac{dy}{dx}\left(\frac{1}{t^2}\right)$$

$$\frac{d^2y}{dt^2} = \frac{1}{t^2}\left(\frac{d^2y}{dx^2} - \frac{dy}{dx}\right)$$

To restate:

$$\frac{dy}{dt} = \frac{1}{t}\frac{dy}{dx}$$

$$\frac{d^2y}{dt^2} = \frac{1}{t^2}\left(\frac{d^2y}{dx^2} - \frac{dy}{dx}\right) \Rightarrow t^2\frac{d^2y}{dt^2} = \frac{d^2y}{dx^2} - \frac{dy}{dx}$$

Then if we have

$$a_2 t^2 \frac{d^2y}{dt^2} + a_1 t \frac{dy}{dt} + a_0 y = F(t)$$

$$a_2\left(\frac{d^2y}{dx^2} - \frac{dy}{dx}\right) + a_1 \frac{dy}{dx} + a_0 y = F(e^x)$$

$$a_2 \frac{d^2y}{dx^2} + (a_1 - a_2)\frac{dy}{dx} + a_0 y = F(e^x)$$

which is second order with constant coeffs