So substitution problems.

if $\frac{dy}{dx} = f(x,y)$ an we can write $f(x,y)$ in terms of $\frac{y}{x}$'s then this DE is

homogeneous.

That is $\frac{dy}{dx} = f(x,y) = f\left(\frac{y}{x}\right)$.

ex.) $2x^2 \frac{dy}{dx} = x^2 + y^2$

$$\frac{dy}{dx} = \frac{1}{2} + \frac{1}{2}\left(\frac{y}{x}\right)^2$$

Let $v = \frac{y}{x}$ OR $y = vx \Rightarrow \frac{dy}{dx} = v + x\frac{dv}{dx}$

$$v + x\frac{dv}{dx} = \frac{1}{2} + \frac{1}{2}v^2$$

$$x\frac{dv}{dx} = \frac{1}{2}v^2 - v + \frac{1}{2} = \frac{1}{2}(v^2 - 2v + 1)$$

$$x\frac{dv}{dx} = \frac{1}{2}(v-1)^2 \Rightarrow \frac{2}{(v-1)^2}\,dv = \frac{1}{x}\,dx$$

$$2\int \frac{1}{(v-1)^2}\,dv = \int \frac{1}{x}\,dx$$

for $2\int \frac{1}{(v-1)^2}\,dv = 2\int A^{-2}\,dA = 2\,\frac{A^{-1}}{-1} = -\frac{2}{A}$

let $A = v-1$

$dA = dv$

so $2\int \frac{1}{(v-1)^2}\,dv = \frac{-2}{v-1}$

So then $\frac{-2}{v-1} = \frac{\ln x + C}{1} \Rightarrow$

$-2 = (v-1)(\ln x + c)$

$\frac{-2}{\ln x + c} = v-1 \Rightarrow v = 1 - \frac{2}{c+\ln x}$

Then remember $y = vx$ OR $v = \frac{y}{x}$

$\frac{y}{x} = 1 - \frac{2}{c+\ln x}$ so $y = x - \frac{2x}{c+\ln x}$.

ex.) $\dfrac{dy}{dx} = \dfrac{x^2+y^2}{xy} \Rightarrow \dfrac{dy}{dx} = \dfrac{x^2+y^2}{xy}\left(\dfrac{\frac{1}{x^2}}{\frac{1}{x^2}}\right)$

$\dfrac{dy}{dx} = \dfrac{1+\left(\frac{y}{x}\right)^2}{\frac{y}{x}}$ Then let $v = \dfrac{y}{x}$ OR $y = vx$

$\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

$v + x\dfrac{dv}{dx} = \dfrac{1+v^2}{v} \Rightarrow x\dfrac{dv}{dx} = \dfrac{1+v^2}{v} - v$

$= \dfrac{1}{v} + v - v$

So $x\dfrac{dv}{dx} = \dfrac{1}{v}$ Then

$v\,dv = \dfrac{1}{x}dx \Rightarrow \frac{1}{2}v^2 = \ln x + c_1$

$v^2 = 2\ln x + c_2,$ $c_2 = c_1 \cdot 2$

Then $\left(\dfrac{y}{x}\right)^2 = 2\ln x + c_2$

$$\boxed{y^2 = 2x^2\ln x + c_2x^2}$$

So if $\frac{dy}{dx} = f\left(\frac{y}{x}\right)$, the substitution $v = \frac{y}{x}$ results in a separable equation.

Bernoulli Equations

$$\frac{dy}{dx} + P(x)y = Q(x)\, y^n \quad \text{where } n \in \mathbb{Z}$$

not equal to 1 or 0.

To solve use the sub. $v = y^{1-n}$ this will give us in the end a linear DE (Integrating Factor)

ex.) $\frac{dy}{dx} + 2xy = x y^3$ here we key in on the xy^3 term.

So let $\boxed{v = y^{1-3} = y^{-2}}$

Here's the trick: $\frac{dy}{dx} + 2xy = xy^3$

divide by y^3

$$y^{-3} \frac{dy}{dx} + 2x y^{-2} = x$$

$$y^{-3}\frac{dy}{dx} + 2x\underline{\underline{y^{-2}}} = x$$

here is your sub. let $v = y^{-2}$

$$\frac{dv}{dx} = -2y^{-3}\underbrace{\frac{dy}{dx}}_{\text{chain Rule.}}$$

This gives us $y^{-3}\frac{dy}{dx} = -\frac{1}{2}\frac{dv}{dx}$

$$-\frac{1}{2}\frac{dv}{dx} + 2xv = x \Rightarrow \frac{dv}{dx} - 4xv = -2x$$

<u>Linear</u> !

here $P(x) = -4x$

$$\mu(x) = e^{-4\int x\,dx} = e^{-2x^2}$$ Then

$$e^{-2x^2}\frac{dv}{dx} - 4xe^{-2x^2}v = -2xe^{-2x^2}$$

$$\frac{d}{dx}\left(e^{-2x^2}v\right) = -2xe^{-2x^2}$$

$$e^{-2x^2}v = -2\int xe^{-2x^2}dx$$

$$e^{-2x^2} V = -2\int x e^{-2x^2}\,dx$$

let $A = -2x^2$

$dA = -4x\,dx$

$-\frac{1}{4}\,dA = x\,dx$

$$\frac{-2}{-4}\int e^{A}\,dA$$

$$\frac{1}{2}\int e^{A}\,dA = \frac{1}{2}e^{A}$$

Then $e^{-2x^2} V = \frac{1}{2}e^{-2x^2} + C$

$V = \frac{1}{2} + Ce^{2x^2}$, recall $V = y^{-2}$

$$y^{-2} = \frac{1}{2} + Ce^{2x^2} \Rightarrow \boxed{y^2 = \frac{1}{\frac{1}{2} + Ce^{2x^2}}}$$

So $\frac{dy}{dx} + P(x)\,y = Q(x)\,y^n$, $0,1 \neq n \in \mathbb{Z}$

the sub. $V = y^{1-n}$ will result in a linear DE, solve by finding an Int. Factor.